A CONDITIONAL REGULARITY RESULT FOR P-HARMONIC FLOWS

KRYSTIAN KAZANIECKI, MICHAŁ ŁASICA, KATARZYNA EWA MAZOWIECKA, AND PAWEŁ STRZELECKI

Abstract. We prove an ε-regularity result for a wide class of parabolic systems

$$u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = B(\,\cdot\,, u, \nabla u)$$

with the right hand side B growing critically, like $|\nabla u|^p$. It is assumed *a priori* that the solution $u(t,\cdot)$ is uniformly small in the space of functions of bounded mean oscillation. The crucial tool is provided by a sharp nonlinear version of the Gagliardo–Nirenberg inequality which has been used earlier in the elliptic context by T. Rivière and the last named author.

1. Introduction

In this note, we study the ε-regularity of solutions of the system of nonlinear parabolic equations,

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = B(\,\cdot\,, u, \nabla u)\,, \tag{1.1}$$

for a vector $u = (u^1, \dots u^N)$, given a vector $B = (B^1, \dots, B^N)$ with

$$u^k \colon (0,T] \times \Omega \to \mathbb{R}, \quad B^k \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nm} \to \mathbb{R}, \qquad 1 \le k \le N,$$

where Ω stands for an open domain in $\mathbb{R}^m$. We restrict ourselves to the case $p > 2$. We assume that the functions B^k, prescribing the nonlinearity of the right hand side, satisfy the growth condition

$$|B^k(\,\cdot\,, u, \nabla u)| \le \Lambda |\nabla u|^p. \tag{1.2}$$

The study of regularity of weak solutions to equations involving the parabolic p-Laplace operator $u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ was initiated by DiBenedetto and Friedman [7] (see also DiBenedetto's book [5] and the recent memoirs of Duzaar, Mingione and Steffen [8] and of Bögelein, Duzaar and Mingione [1]). Known explicit solutions to (1.1) with $B^k = 0$ show that even in this case no regularity higher than $C^{1,\alpha}$ can be expected. Such regularity was indeed shown in [7] for (1.1) with B^k satisfying at most

$$|B^k(\,\cdot\,, u, \nabla u)| \le \Lambda |\nabla u|^{p-1}. \tag{1.3}$$

However, the more restrictive growth assumption (1.3) is typically *not* satisfied in numerous inhomogeneous flows of geometric origin. The p-harmonic heat flow with values in a Riemannian manifold $\mathcal{N}$ (which is assumed to be isometrically embedded in some $\mathbb{R}^d$) is a prototype of (1.1); in this case

$$B(\,\cdot\,, u, \nabla u) = |\nabla u|^{p-2} A(u)(\nabla u, \nabla u), \tag{1.4}$$

where $A(u)(\cdot)$ is the second fundamental form of $\mathcal{N}$ at $u(\cdot)$. The interest in such systems arose in connection with the *homotopy problem for (p-)harmonic maps*,

i. e. the problem of finding a (p-)harmonic map homotopic to a given map between smooth compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$.[1] In this context, with $m = 2$, $p = 2$, the flow of harmonic maps was investigated by Eells and Sampson [9], who proved the existence of global regular solutions under the assumption that $\mathcal{N}$ has non-positive sectional curvature (and solved the homotopy problem in this case).

Without this assumption one cannot expect global regularity. In the case $\mathcal{N} = \mathbb{S}^m$, $m > 3$ examples of evolutions with finite time blow-up were constructed (see [4], also [2] and references therein). Later, Struwe [15, 16] and Struwe in collaboration with Chen [3] considered weak solutions to the harmonic heat flow with values in arbitrary $\mathcal{N}$ and were able to show existence and bounds on the set of singularities. The crucial tool in their work (in case $m > 2$) was Struwe's monotonicity formula [16, Proposition 3.3.], which is only available if $p = 2$.

To the best of our knowledge, there is no general existence result for the flow of p-harmonic maps into an arbitrary compact manifold for $p > 2$. Partial results in this direction include the work of Misawa [13] which generalizes the classic work of Eells and Sampson ($p = 2$) obtaining global regular solutions for $\mathcal{N}$ of nonpositive sectional curvature, and the paper [10] by Hungerbühler, where the existence of weak solutions is shown for $\mathcal{N}$ being a homogeneous space. In the latter case, the proof exploits symmetry of the image and no additional regularity is obtained. Finally, Hungerbühler [11] obtained the existence of global weak solutions (regular except a finite set of times) in the conformally invariant case $p = m$. The proof is based on a local estimate on energy concentration together with a conditional a priori estimate that controls the norms of ∇u in higher L^q spaces allowing to bound ∇u in L^∞ via Moser iteration. Those estimates hold provided the m-energy is appropriately small,

$$\sup_{t,x} \int_{B_R(x)} |\nabla u(t,x)|^m dx < \varepsilon. \tag{1.5}$$

In the present paper, we obtain conditional estimates of a similar form, but instead of smallness of the energy (1.5), we assume smallness of local BMO norm of the solution (which, in light of the embedding $W^{1,m}$ into BMO in dimension m, is a weaker assumption). In particular, our method works for any $p > 2$ and (formally) we only need to control a norm of the solution and not of its derivatives. On the other hand, we have no proof of existence; therefore we do need the assumption that the solution actually exists in $L^p((0,T], W^{2,p}(\Omega))$.

Before we state our results, let us introduce some notation for properly scaled cylinders. For a point $(t_0, x_0) \in (0,T) \times \Omega$ we write

$$\begin{aligned} B_R = B_R(x_0) &= \{|x - x_0| < R\}, \\ Q_R = Q_R(x_0, t_0) &= B_R \times (t_0 - R^p, t_0), \\ Q_R(\sigma_1, \sigma_2) &= B_{R-\sigma_1 R} \times (t_0 - (1-\sigma_2)R^p, t_0) \quad \text{for } \sigma_i \in (0,1). \end{aligned} \tag{1.6}$$

Theorem 1. *Assume that $u \in L^p((0,T], W^{2,p}(\Omega))$, where Ω is an open domain in $\mathbb{R}^m$, is a weak solution to (1.1). Let $q > p$, $\Omega' \subset\subset \Omega$, $\delta > 0$. There exist a positive number $\varepsilon_0 = \varepsilon_0(\Omega, N, p, q, \Lambda)$ such that if*

$$\|u\|_{L^\infty((0,T], BMO(\Omega))} < \varepsilon_0 \tag{1.7}$$

[1] Strictly speaking, in this case the p-Laplace operator has to be substituted with appropriate p-Laplace-Beltrami operator on $\mathcal{M}$.

then there holds

$$\|\nabla u\|_{L^q((\delta,T]\times\Omega')} \le C,$$

where $C = C(\Omega, N, p, q, \Lambda, \operatorname{dist}(\Omega', \partial\Omega), \delta)$.

Theorem 2. *Assume that* $u \in L^p((0,T], W^{2,p}(\Omega))$*, where* Ω *is an open domain in* $\mathbb{R}^m$*, is a weak solution to* (1.1)*, satisfying the smallness assumption* (1.7)*. Let* $\vartheta_0 > \frac{p+m}{2}$, $m \ge 2$*. There exists a constant* $C = C(\Omega, N, p, \Lambda, \vartheta_0, R)$ *such that*

$$\|\nabla u\|_{L^\infty\left(Q_{R/2}\right)} \le C\left(1 + \int_{Q_R} |\nabla u|^{2\vartheta_0}\right)^{\frac{1}{\vartheta_0 - \frac{p+m}{2}}}.$$

Together with the work of DiBenedetto and Friedman, these two theorems imply the following.

Corollary 3. *Assume that* $u \in L^p((0,T], W^{2,p}(\Omega))$ *is a weak solution to* (1.1)*. There exists a constant* $\varepsilon_0 = \varepsilon_0(\Omega, m, p, \Lambda)$ *such that if the condition*

$$\|u\|_{L^\infty((0,T],BMO(\Omega))} < \varepsilon_0$$

is satisfied, then $u \in C^{1,\alpha}_{\mathrm{loc}}((0,T] \times \Omega)$.

Comparing these results with Kuusi and Mingione [12], we see that Corollary 3 is, on one hand, stronger than e.g. [12, Thm 1.3] which gives a borderline version of the L^∞–boundedness of Du from [5, Chapter VIII]. Even for solutions $u \in L^\infty((0,T], BMO \cap W^{2,p}(\Omega))$ the right hand side of (1.1) is formally only in L^2, and not in the Lorentz space $L^{m+2,1}$, as it is assumed in [12, Thm 1.3]. On the other hand, the smallness assumption (1.7) is pretty strong and seems to be restrictive (yet, in a sense, necessary: even in the elliptic case, e.g. for harmonic maps into Riemannian manifolds, an assumption of this kind is needed for regularity; it is well known that near to an isolated singularity of a harmonic map $w\colon \mathbb{B}^3 \to \mathbb{S}^2$ the BMO norm of w is *not* small).

Our main technical tool is an interpolation inequality of Gagliardo-Nirenberg type, discovered by Rivière and the last named author of the present paper, see [14]. Let $\psi \in C^\infty_c(\mathbb{R}^m)$ be fixed. Using $\mathcal{H}^1 - BMO$ duality, the authors of [14] proved the existence of a constant $C = C(m)$ such that

$$\int_{\mathbb{R}^m} \psi^{s+2}|\nabla u|^{s+2} \le Cs^2\|u\|^2_{BMO(\mathbb{R}^m)}\left\{\int_{\mathbb{R}^m} \psi^{s+2}|\nabla u|^{s-2}|\nabla^2 u|^2 + \|\nabla\psi\|^2_{L^\infty(\mathbb{R}^m)}\int_{\mathbb{R}^m}\psi^s|\nabla u|^s\right\} \tag{1.8}$$

for any function $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^m) \cap BMO(\mathbb{R}^m)$ for which the right hand side is finite. A version of (1.8) in time-dependent setting follows immediately. Let now $\psi \in C^\infty_c([0,T] \times \mathbb{R}^m)$. Integrating (1.8) over time yields

$$\int_0^T\int_{\mathbb{R}^m} \psi^{s+2}|\nabla u|^{s+2} \le Cs^2\|u\|^2_{L^\infty([0,T],BMO(\mathbb{R}^m))} \cdot\left\{\int_0^T\int_{\mathbb{R}^m} \psi^{s+2}|\nabla u|^{s-2}|\nabla^2 u|^2 + \|\nabla\psi\|^2_{L^\infty([0,T]\times\mathbb{R}^m)}\int_0^T\int_{\mathbb{R}^m}\psi^s|\nabla u|^s\right\} \tag{1.9}$$

for any $u \in L^\infty([0,T], W^{2,1}_{\mathrm{loc}}(\mathbb{R}^m)) \cap L^\infty([0,T], BMO(\mathbb{R}^m))$. This inequality allows us to control the right hand side of (1.1) provided that the smallness condition from

Theorem 1 is satisfied. This part of our work is in fact a parabolic version of [14]. Having obtained such conditional bounds on sufficiently high local L^q norms of the gradient, one may then proceed with Moser iteration similarly as in [11], obtaining Theorem 2.

Notation. We denote by $BMO(\Omega)$ the space of functions on a given domain Ω of *bounded mean oscillation*, with the seminorm

$$\|f\|_{BMO(\Omega)} := \sup_Q \left(⨍_Q |f(y) - f_Q| dy \right) < \infty,$$

the supremum being taken over all cubes Q in Ω, where f_Q denotes the average of f on Q, i.e. $⨍_Q f dx = |Q|^{-1} \int_Q f dx$, $|Q|$ being the Lebesgue measure of Q. Given $t \in (0,T]$, we write Ω_t for the cylindrical domain $(0,t] \times \Omega$.

2. Caccioppoli inequality

In order to obtain L^q estimates in the homogeneous case (i. e. the right hand side of (1.1) equal 0) DiBenedetto and Friedman [6] tested the system with $\zeta^2|\nabla u|^{2\alpha}\nabla u$, where ζ is a suitably chosen smooth cutoff function. The same standard test functions were used by Hungerbühler in the conformally invariant case and by Rivière and the last author in the elliptic case.

We modify their derivation obtaining the following Caccioppoli inequality for derivatives of solutions of (1.1).

Lemma 4. *Assume that $u \in L^p((0,T], W^{2,p}(\Omega))$ is a weak solution of (1.1). Let $\zeta \in C_c^\infty((0,T]\times\Omega)$ and set $w := |\nabla u|^2$. There exists a constant $C_1 = C_1(m, N, p, \Lambda)$ such that for each $\alpha \geq 0$ we have*

$$\begin{aligned} \frac{1}{2+2\alpha} \operatorname*{ess\,sup}_{t\in(0,T]} \int_\Omega \zeta(t,\cdot)^2 w^{1+\alpha} + \frac{p-2+\alpha}{8} \iint_{\Omega_T} \zeta^2 w^{\frac{p}{2}-2+\alpha} |\nabla w|^2 \\ + \frac{1}{2} \iint_{\Omega_T} \zeta^2 w^{\frac{p}{2}-1+\alpha} |\nabla^2 u|^2 \leq \left(\frac{2(p-1)^2}{p-2+2\alpha} + \frac{1}{2} \right) \iint_{\Omega_T} |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha} \\ + \frac{1}{1+\alpha} \iint_{\Omega_T} \zeta\zeta_t w^{1+\alpha} + C_1(p+\alpha) \iint_{\Omega_T} \zeta^2 w^{\frac{p}{2}+1+\alpha}, \end{aligned} \tag{2.1}$$

provided the right hand side is finite.

Proof. Differentiating both sides of (1.1) with respect to x_j, we see that for $t \in (0,T]$ and each matrix of test functions φ^{ij} compactly supported in $(0,T] \times \Omega$ (indices i, j are summed),

$$\begin{aligned} \iint_{\Omega_t} \frac{\partial}{\partial x_j} \left(\frac{\partial u^i}{\partial t} \right) \varphi^{ij} \\ + \iint_{\Omega_t} \left[|\nabla u|^{p-2} \nabla \left(\frac{\partial u^i}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left(|\nabla u|^{p-2} \right) \nabla u^i \right] \cdot \nabla \varphi^{ij} \\ = - \iint_{\Omega_t} \frac{\partial \varphi^{ij}}{\partial x_j} B^i(x, u, \nabla u). \end{aligned} \tag{2.2}$$

Now, we plug in $\varphi^{ij} = \zeta^2 |\nabla u|^{2\alpha} u^i_{x_j}$, where $\alpha \geq 0$ and $\zeta \in C_c^\infty((0,T] \times \Omega)$ is nonnegative.

We estimate the left and right hand side of (2.2) separately.

Left hand side of (2.2). A routine but somewhat tedious computation leads to the following three equalities:

$$\iint_{\Omega_t} \frac{\partial}{\partial x_j}\left(\frac{\partial u^i}{\partial t}\right)\varphi^{ij} = \frac{1}{1+\alpha}\iint_{\Omega_t}\frac{\partial}{\partial t}\left(w^{1+\alpha}\right)\varphi^{ij} = \frac{1}{2+2\alpha}\int_\Omega w^{1+\alpha}\zeta^2(t,\cdot) - \frac{1}{1+\alpha}\iint_{\Omega_t} w^{1+\alpha}\zeta\frac{\partial\zeta}{\partial t}\,; \tag{2.3}$$

$$\begin{aligned}\iint_{\Omega_t}|\nabla u|^{p-2}\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\cdot\nabla\varphi^{ij} &= \iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}+\alpha}\left|\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\right|^2 \\ &+\frac{\alpha}{2}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-1+\alpha}|\nabla w|^2 + \iint_{\Omega_t}\zeta(\nabla\zeta\cdot\nabla w)w^{\frac{p-2}{2}+\alpha} \\ &=: I_1+I_2+I_3\,;\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\iint_{\Omega_t}\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\right)\nabla u^i\cdot\nabla\varphi^{ij} \\ &= \frac{p-2}{4}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-1+\alpha}|\nabla w|^2 \\ &\quad+\frac{(p-2)\alpha}{2}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-2+\alpha}\sum_i\left(\nabla w\cdot\nabla u^i\right)^2 \\ &\quad+(p-2)\iint_{\Omega_t}\zeta w^{\frac{p-2}{2}-1+\alpha}\sum_i(\nabla\zeta\cdot\nabla u^i)(\nabla w\cdot\nabla u^i) \\ &=: I_4+I_5+I_6.\end{aligned} \tag{2.5}$$

Using Cauchy's inequality with ε: $ab\le\frac{\varepsilon^2a^2}{2}+\frac{b^2}{2\varepsilon^2}$, we estimate

$$\begin{aligned}|I_3|+|I_6| &\le (p-1)\int_t^T\int_\Omega\zeta|\nabla\zeta|\,|\nabla w|w^{\frac{p-2}{2}+\alpha} \\ &\le\frac{(p-1)\varepsilon^2}{2}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-1+\alpha}|\nabla w|^2+\frac{p-1}{2\varepsilon^2}\iint_{\Omega_t}|\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,,\end{aligned} \tag{2.6}$$

splitting the integrand so that the term with $|\nabla w|^2$ can be absorbed in I_2+I_4. Choosing ε^2 so that $(p-1)\varepsilon^2/2=(p-2+2\alpha)/8$, and combining (2.4), (2.5) and (2.6), we obtain finally

$$\begin{aligned}\text{left hand side of (2.2)} &\ge \frac{1}{2+2\alpha}\int_\Omega w^{1+\alpha}\zeta^2(t,\cdot)-\frac{1}{1+\alpha}\iint_{\Omega_t}w^{1+\alpha}\zeta\frac{\partial\zeta}{\partial t} \\ &+\frac{p-2+2\alpha}{8}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-1+\alpha}|\nabla w|^2 \\ &+\int_t^T\int_\Omega\zeta^2 w^{\frac{p-2}{2}+\alpha}|\nabla u^i_{x_j}|^2 \\ &+\frac{(p-2)\alpha}{2}\iint_{\Omega_t}\zeta^2 w^{\frac{p-2}{2}-2+\alpha}\sum_i\left(\nabla w\cdot\nabla u^i\right)^2 \\ &-\frac{2(p-1)^2}{p-2+2\alpha}\iint_{\Omega_t}|\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}.\end{aligned} \tag{2.7}$$

Right hand side of (2.2). Using the growth condition $|B(x,u,\nabla u)| \le \Lambda |\nabla u|^p$, we write

$$\left| \iint_{\Omega_t} \frac{\partial \varphi^{ij}}{\partial x_j} B^i(x,u,\nabla u) \right| \le C(J_1 + J_2 + J_3)\,, \tag{2.8}$$

where the constant $C = C(m,N,\Lambda)$ and

$$J_1 = \iint_{\Omega_t} \zeta^2 w^{\frac{p}{2}+\alpha} \big|\nabla u^i_{x_j}\big| \qquad J_2 = \alpha \iint_{\Omega_t} \zeta^2 w^{\frac{p-1}{2}+\alpha}\, |\nabla w| \tag{2.9}$$
$$J_3 = \iint_{\Omega_t} \zeta |\nabla \zeta| w^{\frac{p+1}{2}+\alpha}.$$

Set

$$J_0 \colon = \iint_{\Omega_t} \zeta^2 w^{\frac{p}{2}+1+\alpha}\,.$$

To absorb all terms that contain second order derivatives of u, we again apply the Cauchy–Schwarz inequality in a familiar way and obtain

$$\begin{aligned} J_1 &\le \frac{\varepsilon_1^2}{2} \iint_{\Omega_t} \zeta^2 w^{\frac{p-2}{2}+\alpha} \big|\nabla u^i_{x_j}\big|^2 + \frac{1}{2\varepsilon_1^2} J_0\,, \\ J_2 &\le \frac{\alpha \varepsilon_2^2}{2} \iint_{\Omega_t} \zeta^2 w^{\frac{p}{2}-2+\alpha}\, |\nabla w|^2 + \frac{\alpha}{2\varepsilon_2^2} J_0\,. \end{aligned}$$

Finally,

$$J_3 \le \frac{C}{2} J_0 + \frac{1}{2C} \iint_{\Omega_t} |\nabla \zeta|^2 w^{\frac{p}{2}+\alpha}\,. \tag{2.10}$$

Making appropriate choices of $\varepsilon_1, \varepsilon_2 > 0$, we combine the estimates of J_1, J_2, J_3 with (2.7) and, taking supremum of both sides over $t \in (0,T]$, complete the proof of the lemma. □

3. Gradient estimates

This section contains proofs of Theorems 1 and 2. The high integrability of solution to (1.1) that we seek will eventually follow upon the iterations of the Caccioppoli inequality (2.1) combined with a parabolic version of the Sobolev inequality,

$$\iint_{(\delta,T]\times\Omega'} |w|^{\frac{p+2\alpha}{2}+\frac{2}{m}(1+\alpha)} \le C(m) \operatorname*{ess\,sup}_{t\in(\delta,T]} \left(\int_{\Omega'} |w(t,\cdot)|^{1+\alpha} \right)^{\frac{2}{m}}$$
$$\cdot \left(\iint_{(\delta,T]\times\Omega'} \big|\nabla w^{\frac{p+2\alpha}{4}}\big|^2 + c_{\Omega'} \iint_{(\delta,T]\times\Omega'} |w|^{\frac{p+2\alpha}{2}} \right), \tag{3.1}$$

which holds for any $\delta > 0$, any smooth bounded $\Omega' \Subset \Omega$, and will be applied to $w = |w| = |\nabla u|^2$. Recall that inequality (3.1) is obtained by an application of

Hölder inequality and Sobolev embedding[2] $W^{1,2} \hookrightarrow L^{\frac{2m}{m-2}}$ in dimension m,

$$\begin{aligned}\int_{\Omega'} |w|^{\frac{p+2\alpha}{2}+\frac{2}{m}(1+\alpha)} &\le \left(\int_{\Omega'} |w|^{1+\alpha}\right)^{\frac{2}{m}} \left(\int_{\Omega'} |w|^{\frac{p+2\alpha}{4}\cdot\frac{2m}{m-2}}\right)^{\frac{m-2}{2m}\cdot 2} \\ &\le C(m)\left(\int_{\Omega'} |w|^{1+\alpha}\right)^{\frac{2}{m}} \left(\int_{\Omega'} \left|\nabla w^{\frac{p+2\alpha}{4}}\right|^2 + c_{\Omega'}\int_{\Omega'} |w|^{\frac{p+2\alpha}{2}}\right),\end{aligned}$$

and integrating the result over the time interval $(\delta, T]$. We also note that if Ω' is a cube or a ball, then the above inequalities hold with

$$c_{\Omega'} = (\operatorname{diam}\Omega')^{-2}\,. \tag{3.2}$$

However, it is easy to see that if the exponent α is small, then the combined inequalities (Caccioppoli and 'parabolic' Sobolev) do not yield any increase in integrability of w. This is caused precisely by the critical term (3.3), with $|\nabla u|^{p+2+2\alpha} = w^{\frac{p}{2}+1+\alpha}$. Therefore, we need first to control ∇u in a sufficiently high L^q space. This is why some smallness assumption is required for our assertions to hold. We exploit such an assumption through a bootstrap procedure involving a finite number of applications of the interpolation inequality (1.9) to control the 'bad' term

$$C_1(p+\alpha)\iint_{\Omega_T} \zeta^2 w^{\frac{p}{2}+1+\alpha} \tag{3.3}$$

on the right hand side of the Caccioppoli inequality.

Once the threshold level of $q = p+m$ is exceeded, one indeed gets an increase of integrability of ∇u only from the equation, and may use Moser iteration to obtain the assertion of Theorem 2. We set forth the details below, giving a parabolic version of the 'elliptic' argument from [14].

3.1. **L^q estimate for ∇u.** We now explain how to iterate the Caccioppoli inequality, using the Gagliardo–Nirenberg inequality at each step. Let C_1 and C_2 denote the constants from Caccioppoli inequality (2.1) and the interpolation inequality (1.9), respectively. Fix a sufficiently large number $\alpha_{\max}$ that shall be specified later.

We need the following smallness condition:

$$C_1 C_2 (p+2\alpha)^3 \|u\|^2_{L^\infty((0,T],BMO(\Omega))} \le \frac{1}{2}, \tag{3.4}$$

for every $\alpha \in [0, \alpha_{\max}]$. We choose two nonnegative functions $\zeta, \psi \in C_c^\infty(\Omega_T)$ so that $\psi \equiv 1$ on $(\delta, T]\times\Omega'$ and

$$\zeta^2 = \psi^{p+2+2\alpha}, \quad 0 \le \zeta \le 1, \quad |\nabla\zeta| \le 2\operatorname{dist}(\Omega', \partial\Omega)^{-1}, \quad |\zeta_t| \le 2\delta^{-1}. \tag{3.5}$$

We set $s = p+2\alpha$ in the interpolation inequality (1.9) and use it to estimate the bad term (3.3).

[2] We tacitly assume here that $m > 2$; for $m = 2$ the exponent $2m/(m-2)$ can be replaced by any $s \in (2, \infty)$.

Due to smallness condition (3.4) we get

$$\begin{aligned}\frac{1}{2+2\alpha}\operatorname*{ess\,sup}_{t\in(0,T]}\int_\Omega \zeta(t,\cdot)^2 w^{1+\alpha} + \frac{1}{2}\iint_{\Omega_T}\zeta^2 w^{\frac{p}{2}-1+\alpha}|\nabla^2 u|^2 \\ + \frac{p-2+\alpha}{8}\iint_{\Omega_T}\zeta^2 w^{\frac{p}{2}-2+\alpha}|\nabla w|^2 \\ \le \left(\frac{2(p-1)^2}{p-2+2\alpha}+\frac{1}{2}\right)\iint_{\Omega_T}|\nabla\zeta|^2 w^{\frac{p}{2}+\alpha} + \frac{1}{1+\alpha}\iint_{\Omega_T}\zeta\zeta_t w^{1+\alpha} \\ + \frac{1}{2}\|\nabla\psi\|^2_{L^\infty(\Omega_T)}\iint_{\Omega_T}\psi^{p+2\alpha}w^{\frac{p}{2}+\alpha} + \frac{1}{2}\iint_{\Omega_T}\zeta^2 w^{\frac{p}{2}-1+\alpha}|\nabla^2 u|^2.\end{aligned} \tag{3.6}$$

The last term in the right-hand side cancels with the matching term in the left hand side. Invoking the properties of ζ and ψ, cf. (3.5), and remembering that

$$\left(\frac{p+2\alpha}{4}\right)^2 w^{\frac{p}{2}-2+\alpha}|\nabla w|^2 = \left|\nabla w^{\frac{p+2\alpha}{4}}\right|^2,$$

we obtain from (3.6), upon multiplication by $2(1+\alpha)$,

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in(\delta,T]}\int_{\Omega'} w^{1+\alpha} + \iint_{(\delta,T]\times\Omega'}\left|\nabla w^{\frac{p+2\alpha}{4}}\right|^2 \\ \le \gamma(1+\alpha)\left(\iint_{\Omega_T} w^{\frac{p}{2}+\alpha} + \iint_{\Omega_T} w^{1+\alpha}\right),\end{aligned} \tag{3.7}$$

where γ stands for a generic constant which may depend on Ω, T, m, p, δ, and Ω'. Inequality (3.7), combined with the parabolic Sobolev embedding (3.1), leads to the estimate

$$\begin{aligned}\left(\iint_{(\delta,T]\times\Omega'}|\nabla u|^{p+2\alpha+\frac{4}{m}(1+\alpha)}\right)^{\frac{1}{\kappa}} \\ \le \gamma(1+\alpha)\left(\iint_{\Omega_T}|\nabla u|^{p+2\alpha} + \iint_{\Omega_T}|\nabla u|^{2+2\alpha}\right),\end{aligned} \tag{3.8}$$

where $\kappa = 1+\frac{2}{m}$ (and γ could change). Now, applying Hölder inequality to replace $2+2\alpha$ by $p+2\alpha$ in the second exponent of the right hand side, we obtain, adjusting the constant γ again,

$$\begin{aligned}\left(\iint_{(\delta,T]\times\Omega'}|\nabla u|^{p+2\alpha+\frac{4}{m}(1+\alpha)}\right)^{\frac{1}{\kappa}} \\ \le \gamma(1+\alpha)\left(\iint_{\Omega_T}|\nabla u|^{p+2\alpha} + \left(\iint_{\Omega_T}|\nabla u|^{p+2\alpha}\right)^{\frac{2+2\alpha}{p+2\alpha}}\right).\end{aligned} \tag{3.9}$$

We iterate this inequality finitely many times, starting from $\alpha = 0$. After each step we obtain higher local integrability of ∇u (on a smaller domain). To achieve the desired goal, we choose $\alpha_{\max}$ large enough to get $\nabla u \in L^q_{\mathrm{loc}}(\Omega_T)$ for the given value of q. Since the time instant $\delta > 0$ above can be arbitrary, Theorem 1 follows.

3.2. L^∞ **estimates for** ∇u**.** In this section we obtain boundedness of the gradient of the solution to (1.1). To prove the estimates on the gradient of the solution we mimic the approach from [11, Lemma 8, p. 608-611], adjusting it to match the case $p > 2$, $m \ge 2$. In the Caccioppoli inequality (2.1) we use a cutoff function ζ such that

$$\begin{aligned} \zeta &= 1 \quad \text{on } Q_R(\sigma_1, \sigma_2)\,, \\ \zeta &= 0 \quad \text{in a neighborhood of the parabolic boundary of } Q_R\,, \end{aligned}$$

and

$$0 \le \zeta \le 1, \quad |\nabla \zeta| \le \frac{2}{\sigma_1 R}, \quad |\zeta_t| \le \frac{2}{\sigma_2 R^p}. \tag{3.10}$$

Throughout this Section γ denotes a generic constant independent of α. Combining the Caccioppoli inequality (2.1) with (3.1), we obtain

$$\begin{aligned} &\iint_{Q_R(\sigma_1,\sigma_2)} w^{\frac{p+2\alpha}{2}+\frac{2}{m}(1+\alpha)} \le \\ &\gamma\Bigg(\left(\frac{1+\alpha}{\sigma_1^2 R^2} + \frac{1}{R^2(1-\sigma_1)^2}\right) \iint_{Q_R} w^{\frac{p+2\alpha}{2}} + \frac{1}{\sigma_2 R^p} \iint_{Q_R} w^{1+\alpha} \\ &\quad + (1+\alpha^2) \iint_{Q_R} w^{\frac{p+2+2\alpha}{2}} \Bigg)^{1+\frac{2}{m}}. \end{aligned} \tag{3.11}$$

We introduce the notation

$$\begin{aligned} \alpha_\nu &= \alpha, \qquad R_\nu = \frac{R_0}{2}\left(1+\frac{1}{2^\nu}\right), \qquad Q_\nu = Q_{R_\nu}, \\ \sigma_1 R_\nu &= \frac{R_0}{2^{\nu+2}}, \qquad \sigma_2 R_\nu^p = \frac{R_0^p}{2^{\nu+p}}, \qquad \kappa = 1 + \frac{2}{m}\,. \end{aligned}$$

A calculation shows

$$Q_{\nu+1} \subseteq Q_{R_\nu}(\sigma_1, \sigma_2).$$

For each $\nu \in \mathbb{N}$ we apply Young's inequality to (3.11) and obtain

$$\begin{aligned} &\iint_{Q_{\nu+1}} w^{\frac{p+2\alpha_\nu}{2}+\frac{2}{m}(1+\alpha_\nu)} \le \\ &\gamma \left(\frac{4^\nu}{R_0^p}|Q_\nu| + \left(1+\alpha_\nu^2 + \frac{4^\nu}{R_0^p}\right) \iint_{Q_\nu} w^{\frac{p+2+2\alpha_\nu}{2}} \right)^{\kappa}, \end{aligned} \tag{3.12}$$

Next we define

$$\begin{aligned} \vartheta_\nu &= \frac{p+2+2\alpha_\nu}{2}, \\ \vartheta_{\nu+1} &= \frac{p+2\alpha_\nu}{2} + \frac{2}{m}(1+\alpha_\nu)\,. \end{aligned}$$

We see that sequence ϑ_ν satisfies the recurrence

$$\vartheta_{\nu+1} = \kappa \vartheta_\nu - \left(1 + \frac{p}{m}\right).$$

Thus,

$$\vartheta_\nu = \kappa^\nu \left(\vartheta_0 - \frac{p+m}{2}\right) + \frac{p+m}{2}.$$

Hence, $\vartheta_\nu \to \infty$ as $\nu \to \infty$ provided that we start with $\vartheta_0 > \frac{p+m}{2}$.

We check that $\alpha_\nu = \kappa^\nu \left(\vartheta_0 - \frac{p+m}{2}\right) + \frac{m}{2} - 1$. We rewrite inequality (3.12) in the form

$$\iint_{Q_{\nu+1}} w^{\vartheta_{\nu+1}} \leq \gamma 8^{\nu\kappa} \left(1 + \iint_{Q_\nu} w^{\vartheta_\nu}\right)^\kappa \tag{3.13}$$

We also set

$$I_\nu = \int_{Q_\nu} w^{\vartheta_\nu},$$

since $m \geq 2$ we have $\kappa \leq 2$ and $(1+x)^\kappa \leq 4(1+x^\kappa)$ for each $x \geq 0$. Hence, inequality (3.13) implies the recursive relation

$$I_{\nu+1} \leq 4 \cdot 64^\nu \gamma \left(1 + I_\nu^\kappa\right) \tag{3.14}$$

for every $\nu \in \mathbb{N}$. It can be proved by induction that

$$I_\nu \leq L^{b_\nu} \left(1 + I_0^{\kappa^\nu}\right)$$

with $L = 4\gamma + 64$ and a convenient choice of exponents b_ν satisfying

$$b_{\nu+1} = \kappa b_\nu + 2\kappa + \nu, \quad b_0 = 0.$$

From elementary calculations one derives the explicit formula for b_ν which turns out to be

$$b_\nu = \kappa^\nu \left(2 + m + \frac{m^2}{4}\right) - \left(2 + m + \frac{m^2}{4} + \frac{\nu m}{2}\right).$$

We immediately obtain the limits

$$\frac{\kappa^\nu}{\vartheta_\nu} \to A = \frac{1}{\vartheta_0 - \frac{p+m}{2}},$$

$$\frac{b_\nu}{\vartheta_\nu} \to B = \frac{2 + m + \frac{m^2}{4}}{\vartheta_0 - \frac{p+m}{2}}.$$

Hence from (3.14) we have following estimate

$$\|\nabla u\|_{L^\infty(Q_{R_0/2})} = \sup_{\nu \in \mathbb{N}} I_\nu^{1/\vartheta_\nu} \leq L^B (1 + I_0^A).$$

This proves Theorem 2.

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Krystian Kazaniecki
Institute of Mathematics, University of Warsaw
Banacha 2, 02–097 Warszawa, Polska
E-mail address: K.Kazaniecki@mimuw.edu.pl

Michał Łasica
Institute of Applied Mathematics and Mechanics, University of Warsaw
Banacha 2, 02–097 Warszawa, Polska
E-mail address: M.Lasica@mimuw.edu.pl

Katarzyna Ewa Mazowiecka
Institute of Mathematics, University of Warsaw
Banacha 2, 02–097 Warszawa, Polska
E-mail address: K.Mazowiecka@mimuw.edu.pl

Paweł Strzelecki
Institute of Mathematics, University of Warsaw
Banacha 2, 02–097 Warszawa, Polska
E-mail address: P.Strzelecki@mimuw.edu.pl