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A new approach to interior regularity of elliptic systems with quadratic Jacobian structure in dimension two

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Abstract. We prove a new regularity result for systems of nonlinear elliptic equations with quadratic Jacobian type nonlinearity in dimension two. Our proof is based on an adaptation of John Lewis' method which has not been used for such systems so far.

1. Introduction

Rivière [16] proved the following remarkable result.

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open set. If $\Omega^i_j \in L^2(D, \mathbb{R}^2)$, $\Omega^i_j = -\Omega^j_i$, $i, j = 1, 2, \ldots, m$ and $u = (u^1, u^2, \ldots, u^m) \in W^{1,2}(D, \mathbb{R}^m)$ solves the system of equations*

$$-\Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j, \quad i = 1, 2, \ldots, m, \tag{1.1}$$

then u is continuous.

This result solves a conjecture of Heinz about regularity of solutions to the prescribed bounded mean curvature equation and a conjecture of Hildebrandt about regularity of all critical points of continuously differentiable elliptic conformally invariant Lagrangians in dimension two. In particular it provides a new proof of Hélein's theorem [10,12] about regularity of two dimensional harmonic mappings into arbitrary compact manifolds.

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An important example is provided by the equation of prescribed mean curvature (H-surface equation):

$$\Delta u = 2H(u)u_{x_1} \wedge u_{x_2}, \tag{1.2}$$

where $u \in W^{1,2}(D, \mathbb{R}^3)$, $D \subset \mathbb{R}^2$ and $H \in L^\infty(\mathbb{R}^3)$ is a Borel function. Heinz conjectured that under these assumptions u is continuous. Let $\nabla^\perp = (-\partial_y, \partial_x)$. It is easy to see that (1.2) can be rewritten in the form

$$-\Delta u^i = \sum_{j=1}^{3} \Omega^i_j \cdot \nabla u^j, \quad i = 1, 2, 3,$$

where

$$\Omega = (\Omega^i_j)_{i,j=1,2,3} = H(u) \begin{bmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{bmatrix}$$

and therefore the Heinz conjecture directly follows from Rivière's theorem.

The antisymmetry condition $\Omega^j_i = -\Omega^i_j$ is crucial in Theorem 1.1 because a well known example of Frehse [8] (cf. [16]) shows that without this condition solutions to the system (1.1) may be discontinuous.

Our aim is to generalize Rivière's theorem to the case in which we lack the antisymmetric structure, but on the other hand the functions Ω^i_j are more regular than in Theorem 1.1. Our main result reads as follows.

Theorem 1.2. *Let $D \subset \mathbb{R}^2$ be an open set. Let $h_{jk} = (h^i_{jk})_{i=1,\dots,m} \in L^\infty \cap W^{1,2}(D, \mathbb{R}^m)$, $j, k = 1, \dots, m$, and let $v \in W^{1,2}(D, \mathbb{R}^m)$. If $u \in W^{1,2}(D, \mathbb{R}^m)$ is a solution to the system*

$$-\Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j, \quad i = 1, 2, \dots, m, \tag{1.3}$$

where

$$\Omega^i_j = \sum_{k=1}^{m} h^i_{jk} \nabla^\perp v^k,$$

then u is locally Hölder continuous.

In this theorem the Laplacian can easily be replaced by a more general elliptic operator in divergence form, but since the main difficulty lies in the structure of the right hand side, we prefer to consider the Laplace operator for the sake of simplicity.

The theorem (with the proof almost unchanged) is true even for a more general right hand side

$$\Omega^i_j = \sum_{k=1}^{m} h^i_{jk} \nabla^\perp v^k_i,$$

where $h^i_{jk} \in L^\infty \cap W^{1,2}(D)$, $v^k_i \in W^{1,2}(D)$, $i, j, k = 1, \dots, m$.

Theorem 1.2 cannot be deduced from that of Rivière, because the functions Ω^i_j do not necessarily satisfy $\Omega^i_j = -\Omega^j_i$. On the other hand the condition $h_{jk} \in L^\infty \cap W^{1,2}$ is a very strong one. This is a price we have to pay for the lack of the antisymmetry.

Observe that our system can be written as

$$-\Delta u = \sum_{j,k=1}^{m} h_{jk}\, dv^k \wedge du^j,$$

i.e.

$$-\Delta u^i = \sum_{j,k=1}^{m} h^i_{jk}\, dv^k \wedge du^j, \quad i = 1, 2, \ldots, m. \tag{1.4}$$

Here $dv^k \wedge du^j = v^k_{x_1} u^j_{x_2} - v^k_{x_2} u^j_{x_1}$. In particular if $v = u$ and

$$H_{jk} = (H^i_{jk})_{i=1\ldots,m} : \mathbb{R}^m \to \mathbb{R}^m, \quad 1 \le j < k \le m,$$

is a family of bounded Lipschitz functions, then $h_{jk} = -H_{jk} \circ u \in L^\infty \cap W^{1,2}$ and hence the theorem gives local Hölder continuity for the solutions to the system which generalizes the H-surface equation

$$-\Delta u = \sum_{1 \le j < k \le m} H_{jk}(u)\, du^j \wedge du^k. \tag{1.5}$$

Once we know that u is continuous, Gehring's lemma gives higher integrability of $|\nabla u|$ and then a routine bootstrap argument implies that solutions to (1.4) belong to $C^{2,\alpha}_{\mathrm{loc}}(D)$ for all $0 < \alpha < 1$.

In the case of the H-surface Eq. (1.2), this regularity result for solutions to (1.5) gives the following theorem which is, however, weaker than that of Rivière.

Corollary 1.3. (Bethuel [2]) *Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a bounded Lipschitz function. Assume that $u \in W^{1,2}(D, \mathbb{R}^3)$ is a weak solution of the H-surface Eq. (1.2). Then, $u \in C^{2,\alpha}_{\mathrm{loc}}(D)$ for every $\alpha < 1$.*

Two different proofs of Bethuel's theorem presented in [2,17] can be adapted to cover Theorem 1.2, but the main novelty in our paper is a new method of the proof. The common feature of all proofs is a heavy use of delicate analytic tools: the duality of Hardy space and BMO (inspired by Coifman et al. [5]), L^p estimates for Hodge decomposition and its variants, interpolation in Lorentz spaces, etc. Our approach is more elementary. It still employs the the duality of Hardy space and BMO, but even that can be replaced by a more elementary argument (we will comment on it later on).

All known proofs seem to be purely two-dimensional (including Rivière's result). That is, they all break down when one tries to adapt them to the case of higher-dimensional H-systems,

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = H(u) u_{x_1} \wedge u_{x_2} \wedge \cdots \wedge u_{x_n}, \tag{1.6}$$

where $u \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$ for some domain $\Omega \subset \mathbb{R}^n$, or to the system of n-harmonic maps into compact manifolds,

$$-\mathrm{div}\,(|\nabla u|^{n-2}\nabla u) \perp T_{u(x)}N \quad \text{a.e.}, \qquad u(x) \in N \text{ a.e.}$$

One of our motivations was to give one more argument, fairly general, and to see whether it can be generalized to obtain full regularity of $W^{1,n}$ weak solutions of H-system (1.6) for $n > 2$.

The main difficulty in proving regularity of the solutions to the system (1.3) stems from the fact that the right hand side of (1.3) is only in L^1 and we cannot use u as a test function. Instead, we follow an idea of Lewis [13] (cf. [6,7,14,15,18]) and we build a test function that coincides with u on the set where the maximal function of the gradient does not exceed t and which has Lipschitz constant equal to Ct, see [1]. This method is combined here with the proof given in [17].

The notation is mostly standard. The integral average over a ball will be denoted by

$$u_B = \fint_B u\,dx = \frac{1}{|B|}\int\limits_B u\,dx$$

and C will denote a general constant that can change its value in a single string of estimates. By an absolute constant we mean a constant that does not depend on any parameter involved. The symbol B will be used to denote a ball.

2. Proof of Theorem 1.2

Some of the steps of the proof are similar to analogous steps in [17] and they will be sketched only.

Lemma 2.1. *Assume that $u \in W^{1,2}(D, \mathbb{R}^m)$ is a weak solution of the system (1.3). There exist numbers $r_0 > 0$, $\varepsilon \in (0, \frac{1}{2})$ and $\lambda \in (0, 1)$ such that for all $a \in D$ and all radii $r < \min(r_0, \frac{1}{4}\mathrm{dist}\,(a, \partial D))$ the following decay inequality holds:*

$$M_{2-\varepsilon}(a, r) \le \lambda M_{2-\varepsilon}(a, 4r), \tag{2.1}$$

where

$$M_{2-\varepsilon}(a, r) := \sup \frac{1}{\varrho^{\varepsilon}} \int\limits_{B(z,\varrho)} |\nabla u|^{2-\varepsilon}\,dx,$$

the supremum being taken over all z, ϱ such that $B(z, \varrho) \subset B(a, r)$.

If $B(a, R) \Subset D$ and $R < r_0$, then for $0 < r < R$ iterations of estimate (2.1) and the Hölder inequality lead to

$$\int\limits_{B(a,r)} |\nabla u|^{2-\varepsilon}\,dx \le r^{\gamma+\varepsilon} R^{-\gamma} \lambda^{-1} \pi^{\varepsilon/2} \left(\int\limits_{B(a,R)} |\nabla u|^2\,dx \right)^{(2-\varepsilon)/2}$$
$$\le C r^{\gamma+\varepsilon} \|\nabla u\|_2^{2-\varepsilon},$$

where $\gamma = \log_4(1/\lambda) > 0$. Thus by the Dirichlet Growth Theorem u is locally Hölder continuous with the exponent $\gamma/(2-\varepsilon)$.

The proof of Lemma 2.1 has two separate stages. First, we test system (1.3) with functions that are good Lipschitz approximations of u (i.e. they agree with u on the set where the maximal function of the gradient of u is not too large). This yields an estimate for the integral of $|\nabla u|^2$ on, roughly speaking, sets of the form $\{x : M|\nabla u|(x) \le t\}$.

The second stage is to average this estimate w.r.t. t, with weight equal to $t^{-1-\varepsilon}$, and to obtain an *averaged Caccioppoli inequality*. Then, we show that any function u satisfying this averaged Caccioppoli inequality must also satisfy (2.1). In this last step, is not at all important that u solves (1.2).

3. Proof of Lemma 2.1

Fix a and $r > 0$ such that $B_r \equiv B(a,r) \subset B_{4r} = B(a,4r) \Subset D$. The choice of r_0, ε and λ shall be specified later on.

It suffices to prove that

$$\frac{1}{r^{\varepsilon}} \int_{B(a,r)} |\nabla u|^{2-\varepsilon} \le \lambda M_{2-\varepsilon}(a,4r). \tag{3.1}$$

Indeed, for $B(z,\varrho) \subset B(a,r)$, (3.1) gives

$$\frac{1}{\varrho^{\varepsilon}} \int_{B(z,\varrho)} |\nabla u|^{2-\varepsilon} \le \lambda M_{2-\varepsilon}(z,4\varrho) \le \lambda M_{2-\varepsilon}(a,4r)$$

and hence (2.1) follows after taking supremum over all $B(z,\varrho) \subset B(a,r)$. If

$$\int_{B_{2r}} |\nabla u|^{2-\varepsilon} > 8 \int_{B_r} |\nabla u|^{2-\varepsilon},$$

then

$$\frac{1}{r^{\varepsilon}} \int_{B_r} |\nabla u|^{2-\varepsilon} < \frac{2^{\varepsilon}}{8} \frac{1}{(2r)^{\varepsilon}} \int_{B_{2r}} |\nabla u|^{2-\varepsilon} \le \frac{1}{4} M_{2-\varepsilon}(a,2r)$$

and hence (3.1) follows with $\lambda = 1/4$. Therefore we can assume that

$$\int_{B_{2r}} |\nabla u|^{2-\varepsilon} \le 8 \int_{B_r} |\nabla u|^{2-\varepsilon}. \tag{3.2}$$

We will frequently use the following well known lemma.

Lemma 3.1. *If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$|u(x) - u(y)| \leq C|x - y|(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)) \; a.e.$$

and

$$|u(x) - u_B| \leq Cr\mathcal{M}|\nabla u|(x) \quad \text{for a.e. } x \in B,$$

where r is the radius of the ball B and $\mathcal{M}|\nabla u|$ is the Hardy–Littlewood maximal function of $|\nabla u|$.

For the proof see, for example [1,7,9,13,14].

Step 1: Choice of test functions. Fix $t > 0$ and a cutoff function $\varphi \in C_0^\infty(B_{2r})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on B_r and $|\nabla\varphi| \leq C/r$.

Set

$$g(x) = |\varphi(x)|\,|\nabla u(x)| + |u(x) - u_{B_{2r}}|\,|\nabla\varphi(x)|\,.$$

We define $g \equiv 0$ in $\mathbb{R}^2 \backslash B_{2r}$. Let

$$F_t : \;= \{x \in B_{2r} : \mathcal{M}g(x) \leq t\}$$

and $\tilde{u}(x) = \varphi(x)\big(u(x) - u_{B_{2r}}\big)$. We claim that $\tilde{u}$ is Lipschitz continuous with constant Ct on $(\mathbb{R}^2 \backslash B_{2r}) \cup F_t$.

Case 1. Let $x, y \in F_t$. Then, since $|\nabla\tilde{u}| \leq g$, we have

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C|x - y|\big(\mathcal{M}|\nabla\tilde{u}|(x) + \mathcal{M}|\nabla\tilde{u}|(y)\big) \leq Ct|x - y|$$

by definition of F_t.

Case 2. Assume that $x \in F_t$, $y \in \mathbb{R}^2 \backslash B_{2r}$. Let $\varrho : \;= 2\mathrm{dist}\,(x, \partial B_{2r})$. Since $\tilde{u}$ equals zero on a large part of the ball $B(x, \varrho)$, Poincaré inequality yields

$$|\tilde{u}_{B(x,\varrho)}| \leq C\varrho \fint_{B(x,\varrho)} |\nabla\tilde{u}| \leq C\varrho\mathcal{M}g(x) \leq C|x - y|t.$$

Therefore

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| = |\tilde{u}(x)| &\leq |\tilde{u}(x) - \tilde{u}_{B(x,\varrho)}| + |\tilde{u}_{B(x,\varrho)}| \\ &\leq C\varrho\mathcal{M}|\nabla\tilde{u}|(x) + Ct|x - y| \\ &\leq Ct|x - y|. \end{aligned}$$

This proves the claim. We now extend $\tilde{u} : F_t \cup (\mathbb{R}^2 \backslash B_{2r}) \to \mathbb{R}^m$ to a Lipschitz continuous function $u_t : \mathbb{R}^2 \to \mathbb{R}^m$ such that $\mathrm{Lip}\,(u_t) \leq Ct$, i.e. $|\nabla u_t| \leq Ct$ on $\mathbb{R}^2$, and $u_t \equiv \tilde{u}$ in $F_t \cup (\mathbb{R}^2 \backslash B_{2r})$—so that, in particular, $u_t \equiv 0$ off B_{2r}.

Step 2. We use u_t as a test function for system (1.4) which is equivalent to (1.3). This gives

$$\int_{F_t} \nabla u \cdot \nabla u_t \, dx \le Ct \int_{B_{2r}\setminus F_t} |\nabla u| \, dx + \left| \sum_{j,k=1}^{m} \int_{\mathbb{R}^2} h_{jk} \cdot u_t \, dv^k \wedge du^j \right|.$$

and next

$$\int_{F_t} |\nabla u|^2 \varphi \, dx \le \int_{F_t} |\nabla u| \, |\nabla \varphi| \, |u - u_{B_{2r}}| \, dx + Ct \int_{B_{2r}\setminus F_t} |\nabla u| \, dx + |I_t|, \tag{3.3}$$

where

$$I_t : = \sum_{j,k=1}^{m} \int_{\mathbb{R}^2} h_{jk} \cdot u_t \, dv^k \wedge du^j. \tag{3.4}$$

Inequality (3.3) holds for all $t > 0$. To obtain estimates for solutions to (1.3) involving local norms of $|\nabla u|$ in Morrey spaces, we multiply (3.3) by $t^{-1-\varepsilon}$ and integrate with respect to $t \in (t_0, \infty)$, for an appropriately chosen number t_0. Before doing that, however, we record a crucial estimate for I_t.

Step 3: Estimating the critical nonlinearity. We claim that

$$|I_t| \le K_\varepsilon \cdot \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)}, \tag{3.5}$$

where

$$K_\varepsilon : = C(\varepsilon) M_{2-\varepsilon}(a, 4r)^{1/(2-\varepsilon)} r^{\varepsilon/(2+\varepsilon)} (\|h\|_\infty + \|\nabla h\|_2) \|\nabla v\|_{L^2(B_{2r})}, \tag{3.6}$$

and the constant $C(\varepsilon)$ *does depend* on ε.

This estimate follows from the duality of Hardy space $\mathcal{H}^1$ and the space BMO of functions of bounded mean oscillation. Here are some details. I_t is the sum of expressions I^i_{jk}, where

$$I^i_{jk} = \int_{\mathbb{R}^2} h^i_{jk} \, u^i_t \, dv^k \wedge du^j.$$

Let $\zeta_1 \equiv 1$ on B_{2r}, $|\nabla\zeta_1| \le 2/r$, $\zeta_1 \equiv 0$ off B_{3r}. Formal integration by parts gives

$$\begin{aligned} I^i_{jk} &= \int_{\mathbb{R}^2} h^i_{jk} u^i_t \, dv^k \wedge d(\zeta_1(u^j - u^i_{B_{2r}})) \\ &= -\int_{\mathbb{R}^2} \zeta_1(u^j - u^j_{B_{2r}}) \, dv^k \wedge d[h^i_{jk} u^i_t]. \end{aligned} \tag{3.7}$$

Observe, however, that the integral on the right hand side does not necessarily exist, because the Jacobian $dv^k \wedge d[h^i_{jk} u^i_t]$ is integrable only and $\zeta_1(u^j - u^j_{B_{2r}})$ is not necessarily bounded. However, it follows from the Poincaré inequality that $\zeta_1(u^j - u^j_{B_{2r}})$ belongs to BMO, and by the theorem of Coifman, Lions, Meyer and Semmes, see [5], $dv^k \wedge d[h^i_{jk} u^i_t]$ belongs to the Hardy space $\mathcal{H}^1$. Hence we interpret the right hand side of (3.7) as a duality between $\mathcal{H}^1$ and BMO. More precisely, if we replace $\zeta_1(u^j - u^j_{B_{2r}})$ by a compactly supported smooth function, then the integration by parts in (3.7) is fully justified. Then we approximate $\zeta_1(u^j - u^j_{B_{2r}})$ by such functions in the $W^{1,2}$ norm and passing to the limit yields (3.7). Now we have

$$\begin{aligned} |I^i_{jk}| &\le \left| \int_{\mathbb{R}^2} \zeta_1(u^j - u^j_{B_{2r}}) dv^k \wedge d[h^i_{jk} u^i_t] \right| \\ &\le C \|\zeta_1(u^j - u^j_{B_{2r}})\|_{\mathrm{BMO}} \, \|\nabla v^k\|_{L^2(B_{2r})} \, \|\nabla[h^i_{jk} u^i_t]\|_{L^2(B_{2r})} \\ &\le C M_{2-\varepsilon}(a, 4r)^{1/(2-\varepsilon)} \, \|\nabla v^k\|_{L^2(B_{2r})} \, \|\nabla[h^i_{jk} u^i_t]\|_{L^2(B_{2r})} \, . \end{aligned}$$

The first inequality follows from Fefferman's duality theorem and the result of Coifman et al. [5]. The second inequality is an elementary estimate of the BMO norm of $\zeta_1(u^j - u^j_{B_{2r}})$; see [17] for details.

Since we estimate the BMO norm in terms of the Morrey norm of the gradient, the above inequality can be proved in a more elementary way bypassing Fefferman's theorem, see [3,4] and (P. Hajłasz, P. Strzelecki, X. Zhong, in preparation).

Further,

$$\begin{aligned} \|\nabla[h^i_{jk} u^i_t]\|_{L^2(B_{2r})} &\le \|h\|_\infty \|\nabla u_t\|_{L^2(B_{2r})} + \|\nabla h\|_2 \|u_t\|_{L^\infty(B_{2r})} \\ &\le C(\varepsilon) r^{\varepsilon/(2+\varepsilon)} (\|h\|_\infty + \|\nabla h\|_2) \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)} . \end{aligned}$$

To obtain the last line, we apply the Hölder inequality to deal with $\|\nabla u_t\|_{L^2}$, and the Sobolev imbedding theorem to deal with $\|u_t\|_{L^\infty}$ (this explains why $C(\varepsilon)$ does depend on ε: $W^{1,2}(\mathbb{R}^2)$ is *not* imbedded in L^∞ and therefore $C(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$). The point here is that u_t is a priori more regular than u is.

Step 4: Averaging. We now rewrite (3.3) as

$$\int_{F_t} |\nabla u|^2 \varphi \, dx \le \int_{F_t} |\nabla u| \, |\nabla \varphi| \, |u - u_{B_{2r}}| \, dx + Ct \int_{B_{2r} \setminus F_t} |\nabla u| \, dx + CK_\varepsilon \cdot \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)}, \tag{3.8}$$

multiply both sides of (3.8) by $t^{-1-\varepsilon}$ and integrate w.r.t. $t \in (t_0, \infty)$, setting

$$t_0 := \delta \left(\fint_{B_r} |\nabla u|^{2-\varepsilon} \, dx \right)^{1/(2-\varepsilon)}. \tag{3.9}$$

Here, δ is a small absolute constant (and hence independent of ε) that will be chosen later. We obtain an *averaged Caccioppoli inequality* of the form

$$J_1 \le C_1 (J_2 + J_3 + J_4), \tag{3.10}$$

where C_1 is an absolute constant and

$$J_1 = \int_{t_0}^{\infty} t^{-1-\varepsilon} \int_{F_t} |\nabla u|^2 \varphi \, dx \, dt, \tag{3.11}$$

$$J_2 = \int_{t_0}^{\infty} t^{-1-\varepsilon} \int_{F_t} |\nabla u| \, |\nabla \varphi| \, |u - u_{B_{2r}}| \, dx \, dt, \tag{3.12}$$

$$J_3 = \int_{t_0}^{\infty} t^{-\varepsilon} \int_{B_{2r} \setminus F_t} |\nabla u| \, dx \, dt, \tag{3.13}$$

$$J_4 = K_\varepsilon \int_{t_0}^{\infty} t^{-1-\varepsilon} \left(\int_{B_{2r}} |\nabla u_t|^{2+\varepsilon} \, dx \right)^{1/(2+\varepsilon)} dt. \tag{3.14}$$

Step 5: Estimates of J_1–J_4. Tedious but elementary estimates of J_1–J_4 (involving only the Fubini theorem, Hölder, Young and Poincaré inequalities, and the Hardy–

Littlewood maximal theorem) yield the following inequalities:

$$J_1 \geq \frac{C_2}{\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, \tag{3.15}$$

$$\begin{aligned} J_2 &\leq \frac{C_3}{\varepsilon} \left(\int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx \right)^{\frac{1}{2-\varepsilon}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \right)^{\frac{1-\varepsilon}{2-\varepsilon}} \\ &\leq \frac{C_4}{\varepsilon} \int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx + \frac{C_2}{4C_1\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, \end{aligned} \tag{3.16}$$

$$J_3 \leq C_5 \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx\,, \tag{3.17}$$

$$J_4 \leq C_6(\varepsilon) r^{\varepsilon} (\|h\|_\infty + \|\nabla h\|_2) \|\nabla v\|_{L^2(B_{2r})} M_{2-\varepsilon}(a, 4r)\,, \tag{3.18}$$

where the constants C_1, C_2, C_3, C_4, C_5 are absolute and hence they do not depend on ε, whereas $C_6 = C_6(\varepsilon)$ does.

The details of these estimates are given in the next Section. Here we just show how to conclude the proof of Lemma 2.1, assuming these estimates.

Step 6: Conclusion. Inserting the above estimates into (3.10), we obtain

$$\begin{aligned} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \ &\leq C_7 \int_{B_{2r}\setminus B_r} |\nabla u|^{2-\varepsilon}\, dx + \frac{1}{4} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \\ &\quad + C_8 \varepsilon \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \\ &\quad + C_9(\varepsilon) r^{\varepsilon} (\|h\|_\infty + \|\nabla h\|_2) \|\nabla v\|_{L^2(B_{2r})} M_{2-\varepsilon}(a, 4r). \end{aligned} \tag{3.19}$$

Now we add $C_7 \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx$ to both sides to "fill the hole" on the right hand side and after elementary calculations we arrive at

$$\begin{aligned} \frac{1}{r^{\varepsilon}} \int_{B_r} |\nabla u|^{2-\varepsilon} &\leq \frac{C_7 2^{\varepsilon}}{C_7+1} \frac{1}{(2r)^{\varepsilon}} \int_{B_{2r}} |\nabla u|^{2-\varepsilon} + \frac{\frac{1}{4} + C_8\varepsilon}{C_7+1} \frac{1}{r^{\varepsilon}} \int_{B_r} |\nabla u|^{2-\varepsilon} \\ &\quad + \frac{C_9(\varepsilon)(\|h\|_\infty + \|\nabla h\|_2)\|\nabla v\|_{L^2(B_{2r})}}{C_7+1} M_{2-\varepsilon}(a, 4r) \\ &\leq \frac{C_7 2^{\varepsilon} + \frac{1}{4} + C_8\varepsilon + C_9(\varepsilon)(\|h\|_\infty + \|\nabla h\|_2)\|\nabla v\|_{L^2(B_{2r})}}{C_7+1} M_{2-\varepsilon}(a, 4r). \end{aligned}$$

We now fix ε so small that

$$C_7 2^{\varepsilon} + \frac{1}{4} + C_8\varepsilon < C_7 + \frac{1}{2}$$

and then $r_0 = r_0(\varepsilon)$ so small that

$$C_9(\varepsilon)(\|h\|_\infty + \|\nabla h\|_2)\|\nabla v\|_{L^2(B_{2r})} < \frac{1}{4}$$

for all points $a \in D$ and all radii $r < r_0(\varepsilon)$. Now (3.1) follows with $\lambda = (C_7 + 3/4)/(C_7+1)$. This completes the proof of the lemma and hence that of the theorem. □

4. Averaged Caccioppoli inequality: proofs of (3.15)–(3.18)

In this Section we provide details of Step 5 of the proof from the previous Section. Numerous estimates are based on the inequalities

$$\begin{aligned} \int_{B_{2r}} (\mathcal{M}g)^{2-\varepsilon}\,dx &\le C \int_{B_{2r}} g^{2-\varepsilon}\,dx \le C \int_{B_{2r}} |\nabla u|^{2-\varepsilon}\,dx \\ &\le C \int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\,. \end{aligned} \tag{4.1}$$

All constant are absolute and hence independent of ε. The first estimate follows from Hardy–Littlewood maximal theorem, the second one—from Poincaré inequality. The last one is just the assumption (3.2).

Estimate of J_1. Recall that $F_t = \{x \in B(a, 2r) \colon \mathcal{M}g(x) \le t\}$. Since $\varphi \equiv 1$ on B_r, Fubini's theorem yields

$$\begin{aligned} J_1 &= \int_{t_0}^{\infty} t^{-1-\varepsilon} \int_{F_t} |\nabla u|^2 \varphi\,dx\,dt \ge \int_{B_r \cap \{\mathcal{M}g > t_0\}} |\nabla u|^2 \varphi \int_{\mathcal{M}g(x)}^{\infty} t^{-1-\varepsilon}\,dt\,dx \\ &= \frac{1}{\varepsilon} \int_{B_r} |\nabla u|^2 (\mathcal{M}g)^{-\varepsilon}\,dx - \frac{1}{\varepsilon} \int_{B_r \cap \{\mathcal{M}g \le t_0\}} |\nabla u|^2 (\mathcal{M}g)^{-\varepsilon}\,dx \\ &=: J_{11} - J_{12}\,. \end{aligned}$$

We apply Hölder's inequality and (4.1) to estimate J_{11}. We have

$$\begin{aligned} \int_{B_r} |\nabla u|^{2-\varepsilon}\,dx &\overset{(H)}{\le} \left(\int_{B_r} |\nabla u|^2 (\mathcal{M}g)^{-\varepsilon}\,dx \right)^{\frac{2-\varepsilon}{2}} \left(\int_{B_r} (\mathcal{M}g)^{2-\varepsilon}\,dx \right)^{\frac{\varepsilon}{2}} \\ &\overset{(4.1)}{\le} C \left(\int_{B_r} |\nabla u|^2 (\mathcal{M}g)^{-\varepsilon}\,dx \right)^{\frac{2-\varepsilon}{2}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\,dx \right)^{\frac{\varepsilon}{2}}. \end{aligned}$$

(with some constant C that is independent from ε). Thus,

$$J_{11} \geq \frac{C_0}{\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx.$$

To estimate J_{12} we note that $|\nabla u| \leq g \leq \mathcal{M}g$ in B_r. Hence,

$$|J_{12}| \leq \frac{1}{\varepsilon} t_0^{2-\varepsilon} |B_r| = \frac{1}{\varepsilon} \delta^{2-\varepsilon} \int_{B_r} |\nabla u|^{2-\varepsilon}\, dx.$$

Choosing $\delta = \min(\frac{1}{2}, C_0/2)$, we obtain $\delta^{2-\varepsilon} < \delta \leq C_0/2$. Combining the estimates of J_{11} and J_{12}, we finish the proof of (3.15).

Estimate of J_2. Using Fubini's theorem, we have

$$\begin{aligned}
J_2 &\leq \int_0^\infty t^{-1-\varepsilon} \int_{F_t} |\nabla u|\, |\nabla \varphi|\, |u - u_{B_{2r}}|\, dx\, dt \\
&= \int_{B_{2r}} |\nabla u|\, |\nabla \varphi|\, |u - u_{B_{2r}}| \int_{\mathcal{M}g(x)}^\infty t^{-1-\varepsilon}\, dt\, dx \\
&= \frac{1}{\varepsilon} \int_{B_{2r}} |\nabla u|\, |\nabla \varphi|\, |u - u_{B_{2r}}|\, (\mathcal{M}g)^{-\varepsilon}\, dx \\
&\leq \frac{1}{\varepsilon} \int_{B_{2r}} |\nabla u|\, |\nabla \varphi|^{1-\varepsilon}\, |u - u_{B_{2r}}|^{1-\varepsilon}\, dx \\
&\leq \frac{C}{\varepsilon} \left(\int_{B_{2r} \setminus B_r} |\nabla u|^{2-\varepsilon}\, dx \right)^{\frac{1}{2-\varepsilon}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\, dx \right)^{\frac{1-\varepsilon}{2-\varepsilon}}.
\end{aligned}$$

[Note that $|\nabla \varphi|\, |u - u_{B_{2r}}| \leq g \leq \mathcal{M}g$. In the last line, we apply Hölder and Poincaré inequalities combined with assumption (3.2).] By a standard application of Young's inequality, (3.16) follows.

Estimate of J_3. Since $t < \mathcal{M}g(x)$ in the complement of F_t, we obtain

$$\begin{aligned}
J_3 &\leq \int_0^\infty t^{-\varepsilon} \int_{B_{2r} \setminus F_t} |\nabla u|\, dx\, dt \\
&= \frac{1}{1-\varepsilon} \int_{B_{2r}} |\nabla u|\, (\mathcal{M}g)^{1-\varepsilon}\, dx
\end{aligned}$$

$$\overset{(H)}{\leq} \frac{1}{1-\varepsilon}\left(\int\limits_{B_{2r}} |\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{1}{2-\varepsilon}} \left(\int\limits_{B_{2r}} (\mathcal{M}g)^{2-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{2-\varepsilon}}$$

$$\leq C\int\limits_{B_r} |\nabla u|^{2-\varepsilon}\,dx\,.$$

To obtain the last line, one applies inequalities (4.1) and the fact that $1/(1-\varepsilon) < 2$.

Estimate of J_4**.** This is the heart of the matter. We split

$$\begin{aligned}
J_4 &= K_\varepsilon \int\limits_{t_0}^{\infty} t^{-1-\varepsilon}\left(\int\limits_{B_{2r}} |\nabla u_t|^{2+\varepsilon}\,dx\right)^{1/(2+\varepsilon)} dt\\
&\leq K_\varepsilon \int\limits_{t_0}^{\infty} t^{-1-\varepsilon}\left(\int\limits_{F_t} |\nabla \tilde{u}|^{2+\varepsilon}\,dx\right)^{1/(2+\varepsilon)} dt\\
&\quad + K_\varepsilon \int\limits_{t_0}^{\infty} t^{-1-\varepsilon}\cdot Ct\cdot |B_{2r}\backslash F_t|^{1/(2+\varepsilon)}\,dt\\
&=: K_\varepsilon(J_{41}+J_{42}).
\end{aligned}$$

We used the fact that $\nabla u_t = \nabla\tilde{u}$ in F_t and $|\nabla u_t| \leq Ct$ everywhere, in particular in $B_{2r}\backslash F_t$. To estimate J_{41} observe that $|\nabla\tilde{u}| \leq g \leq \mathcal{M}g \leq t$ in F_t and hence $|\nabla\tilde{u}|^{2+\varepsilon} \leq t^{2\varepsilon}|\nabla\tilde{u}|^{2-\varepsilon}$. Moreover, the Poincaré inequality gives

$$\left(\int\limits_{F_t} |\nabla\tilde{u}|^{2-\varepsilon}\right)^{\frac{1}{2+\varepsilon}} \leq C\left(\int\limits_{B_{2r}} |\nabla u|^{2-\varepsilon}\right)^{\frac{1}{2+\varepsilon}}.$$

Hence

$$\begin{aligned}
J_{41} &\leq \int\limits_{t_0}^{\infty} t^{-1-\varepsilon} t^{\frac{2\varepsilon}{2+\varepsilon}}\left(\int\limits_{F_t} |\nabla\tilde{u}|^{2-\varepsilon}\,dx\right)^{\frac{1}{2+\varepsilon}} dt\\
&\leq C(\varepsilon) t_0^{-\varepsilon+\frac{2\varepsilon}{2+\varepsilon}}\left(\int\limits_{B_{2r}} |\nabla u|^{2-\varepsilon}\right)^{\frac{1}{2+\varepsilon}}\\
&\leq C(\varepsilon) r^{\frac{2\varepsilon^2}{4-\varepsilon^2}}\left(\int\limits_{B_r} |\nabla u|^{2-\varepsilon}\right)^{\frac{1-\varepsilon}{2-\varepsilon}}.
\end{aligned}$$

The last constant depends also on δ, but δ is an absolute constant, so there is no need to write dependence on δ explicitly.

To estimate J_{42} first observe that Cavalieri's principle and (4.1) give

$$(2-\varepsilon)\int_0^\infty t^{1-\varepsilon}|B_{2r}\setminus F_t|\,dt = \int_{B_{2r}} (\mathcal{M}g)^{2-\varepsilon}\,dx \le C\int_{B_r} |\nabla u|^{2-\varepsilon}dx.$$

Hence

$$\begin{aligned} J_{42} &\le C\int_{t_0}^\infty t^{-\varepsilon}|B_{2r}\setminus F_t|^{\frac{1}{2+\varepsilon}}\,dt \\ &\le C\left(\int_{t_0}^\infty \left(t^{-\varepsilon-\frac{1-\varepsilon}{2+\varepsilon}}\right)^{\frac{2+\varepsilon}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{2+\varepsilon}} \left(\int_{t_0}^\infty t^{1-\varepsilon}|B_{2r}\setminus F_t|\,dt\right)^{\frac{1}{2+\varepsilon}} \\ &\le C(\varepsilon)\left(t_0^{1+(-\varepsilon-\frac{1-\varepsilon}{2+\varepsilon})\frac{2+\varepsilon}{1+\varepsilon}}\right)^{\frac{1+\varepsilon}{2+\varepsilon}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\,dx\right)^{\frac{1}{2+\varepsilon}} \\ &= C(\varepsilon) r^{\frac{2\varepsilon^2}{4-\varepsilon^2}} \left(\int_{B_r} |\nabla u|^{2-\varepsilon}\right)^{\frac{1-\varepsilon}{2-\varepsilon}}. \end{aligned}$$

Now (3.18) follows from the definition of K_ε. This completes the whole proof. □

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