Adv. Calc. Var. **1** (2008), 205–222 DOI 10.1515 / ACV.2008.008

On a nonlinear fourth order elliptic system with critical growth in first order derivatives

Paweł Strzelecki and Anna Zatorska-Goldstein

Communicated by F. Duzaar

Abstract. We prove that all bounded weak solutions of the fourth order system

$$\Delta^2 u = Q(x, u, \nabla u), \qquad u \in W^{2,2},$$

where the nonlinearity grows critically with the gradient of a solution, i.e., $|Q(x, u, \nabla u)| \lesssim |\nabla u|^4$, are regular once an appropriate smallness condition (expressed in terms of Morrey norms and guaranteeing that u is small in BMO) is satisfied. The result holds in every dimension.

Keywords. Elliptic systems, regularity, BMO.

AMS classification. 35J60, 35J65, 42B35.

1 Introduction

In this paper, we consider the regularity of bounded weak solutions $u \in W^{2,2}(\Omega, \mathbb{R}^N)$ of the fourth order nonlinear elliptic system

$$\Delta^2 u = Q(x, u, \nabla u), \tag{1.1}$$

where $Q : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ is of class C^∞ and satisfies, for some constant Λ, the growth condition

$$|Q(x, u, \nabla u)| \le \Lambda |\nabla u|^4. \tag{1.2}$$

No other assumptions about Q – in particular *no special structure assumptions* – are needed. It follows from (1.2) and the classical Gagliardo–Nirenberg inequality that for each map $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ the nonlinear term $Q(x, u, \nabla u)$ is (only) of class L^1, and therefore the following definition makes sense.

Definition 1.1. A mapping $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ is a weak solution of (1.1) if and only if the identity

$$\int \Delta u \Delta \varphi \, dx = \int \varphi \, Q(x, u, \nabla u) \, dx \tag{1.3}$$

is satisfied for each $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

This work was supported by a MNiSzW grant no 1 PO3A 005 29.

Our main result is the following.

Theorem 1.2. *Assume that* $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ *is a weak solution of* (1.1). *Denote* $M = \|u\|_\infty$. *There exists a constant* $\varepsilon_0 = \varepsilon_0(n, N, \Lambda, M)$ *such that if* $|\nabla u| \in L^{4,n-4}(\Omega)$ *and*

$$\sup_{x\in\Omega,\varrho>0} \varrho^{4-n} \int_{\Omega\cap B(x,\varrho)} |\nabla u(\xi)|^4 \, d\xi < \varepsilon_0^4 \tag{1.4}$$

then $u \in C^\infty(\Omega)$.

Remark. This is of course a local result. We do not claim that u is smooth up to $\partial\Omega$ (and do not in fact know under what conditions such a result would hold).

Note that the smallness condition (1.4) implies that the solution u has a small norm in the space BMO of functions of bounded mean oscillation. This will play an absolutely crucial role in the sequel. The boundedness of u alone would not be sufficient, as the eternal example of $u_0(x) = x/|x|$ shows: in dimensions $n > 4$ we have $u_0 \in W^{2,2} \cap L^\infty$ and

$$\Delta^2 u_0 = c(n)|\nabla u_0|^4 u_0$$

with $c(n) = -1 + 2(4-n)/(n-1)$. This system satisfies the growth condition (1.2) but u_0 is not small in BMO.

Theorem 1.2 might seem surprising at first glance. It is well known that the regularity theory of nonlinear elliptic systems with right hand side that grows critically with the derivatives of solutions is a delicate topic. Weak solutions are sometimes regular, under appropriate growth, smallness *and structure* assumptions.

Let us however note that the right hand side of (1.1) is critical – a priori just L^1 – *but it does not contain the second order derivatives of the solution.* This is the key to Theorem 1.2: due to this fact, we may use a sharp version of Gagliardo–Nirenberg inequalities[1], see Lemma 2.6 in Section 2 for a precise statement, and interpolate between BMO and $W^{2,2}$ to show that for all solutions satisfying the smallness conditions (1.4) the critical term is small and can be absorbed. This is why the precise structure of the nonlinearity is not important. To see this in a better perspective, let us discuss a handful of well-known examples and results.

For second order systems, a classical example by Frehse [4] shows that even in dimension $n = 2$ bounded weak solutions $u \in W^{1,2}(D^2, \mathbb{R}^2)$ of systems of the form $\Delta u = F(u, \nabla u)$ with $|F| \lesssim |\nabla u|^2$ may have singularities. On the other hand, harmonic maps $u \in W^{1,2}(D^2, \mathcal{N})$ from a disk into an arbitrary compact Riemannian manifold $\mathcal{N}$ are smooth, see Hélein [7], [8]. However, this celebrated result is linked in an intricate way to hidden symmetries of the harmonic map system

$$-\Delta u = A(u)(\nabla u, \nabla u), \tag{1.5}$$

[1] Roughly speaking, in the sharp version of Gagliardo–Nirenberg inequalities one may replace the supremum norm of a function by its BMO norm.

where A stands for the second fundamental form of the isometric embedding $\mathcal{N} \subset \mathbb{R}^d$. If $\mathcal{N}$ is a round sphere, or more generally a homogeneous space, then (1.5) is equivalent to a system of (first order) conservation laws in divergence form, see the discussion in [7, Ch. 3]. In this case the right hand side of (1.5) turns out to be not just integrable, but also an element of the Hardy space.

For general targets $\mathcal{N}$ the situation is more complicated. Roughly speaking, one has to see that the nonlinearity possesses some Jacobian-like structure and combine this observation with the duality of Hardy space and BMO. The imbedding $W^{1,2}(\mathbb{R}^2) \subset VMO$ is used to show that, locally, critical terms coming from the right hand side can be absorbed. Thus, both structure of the right hand side and appropriately understood smallness of solutions are vital ingredients of the proof.

Extending Hélein's work to higher dimensions, Evans [3] (for targets that are round spheres) and then Bethuel [1] (for arbitrary compact targets) have proved that the $(n-2)$-dimensional Hausdorff measure of the singular set of every stationary harmonic map is equal to zero. We do not want to discuss all definitions and formal statements of their results here. Let us just say that the key step in their work is the following statement: one can find a positive number $\varepsilon_0 = \varepsilon_0(n, \mathcal{N})$ such that if $u : \mathbb{R}^n \supset \Omega \to \mathcal{N}$, $u \in W^{1,2}$, is a weak solution of (1.5) with $\|u\|_{\mathrm{BMO}} < \varepsilon_0$, then u is of class C^∞.

Very recently, Rivière [14], solving a conjecture of Hildebrandt on critical points of conformally invariant variational functionals in dimension 2, has proved the following result: if $u \in W^{1,2}(D^2, \mathbb{R}^N)$ solves the system

$$\Delta u^i = \sum_{j=1}^{N} \Omega_{ij} \nabla u^j, \qquad i = 1, \dots, N, \tag{1.6}$$

where $\Omega_{ij} \in L^2(D^2, \mathbb{R}^2)$ are vector fields satisfying the antisymmetry condition $\Omega_{ij} = -\Omega_{ji}$, $i, j = 1, \dots, N$, then u is continuous. His work yields, as a byproduct, a new proof of Hélein's result. Again, the structure assumption $\Omega_{ij} = -\Omega_{ji}$ is of crucial importance. In a sense, it is used to build a change of variables in the image after which the right hand side of (1.6) becomes a linear constant coefficient combination of Jacobian determinants, see [14] for more details.

Rivière and Struwe [15] have proved that in fact weak solutions of (1.6) are continuous in any dimension if one assumes that

$$\sup_{x,r} r^{2-n} \int_{B(x,r)} |\nabla u(y)|^2 \, dy \le M < \infty .$$

Similar results are known for fourth order systems, in particular for biharmonic maps into Riemannian manifolds, see e.g. [2, 9, 11, 17, 19, 20] and references therein. In particular, Lamm and Rivière [9] have recently obtained a general result similar to [14] for fourth order system in dimension four. They formulate quite general symmetry conditions which the right hand side of a fourth order nonlinear system should satisfy in order to guarantee that all weak solutions of class $W^{2,2}$ are continuous. One particular example of a system that satisfies these conditions is given by (1.1), see [9, Remark 1.6]. For this particular system in dimension 4, regularity of weak solutions was proved by Wang, as a byproduct of his work on regularity of biharmonic maps. (His proof for $n = 4$, using Lorentz spaces, is different from the one presented in this paper for all dimensions $n \geq 4$.)

Indeed, it was our main motivation to provide an explicit example of a situation where the structure of critical nonlinearity does not matter at all and only the growth conditions and a smallness assumption (which allows to control the mean oscillation of a solution) are satisfied. We believe that this is, in fact, a more general phenomenon, deserving further investigation. Namely, consider a nonlinear elliptic system of order $2k$, of the type

$$\Delta^k u = Q(x, u, \nabla u, \ldots, \nabla^{k-1} u), \tag{1.7}$$

where $k \geq 2$ and the solution u is of class $W^{k,2} \cap L^\infty(\Omega, \mathbb{R}^N)$ for some domain $\Omega \subset \mathbb{R}^n$, $n \geq 2k$. Assume that the growth conditions

$$|Q(x, u, \nabla u)| \lesssim \sum_{j=1}^{k-1} |\nabla^j u|^{2k/j}$$

imposed on Q imply, via standard Gagliardo–Nirenberg interpolation inequalities, that for all $u \in W^{k,2} \cap L^\infty$ the right hand side is of class L^1. This means that the right hand side is critical *but* does not contain the highest derivatives of u. We conjecture that if u satisfies an appropriate smallness condition, say e.g.

$$\sup_{x,r} r^{2k-n} \int_{B(x,r)\cap\Omega} |\nabla u(y)|^{2k} \, dy < \varepsilon_0,$$

then u is C^∞.

The readers which are familiar with the role of cancellation phenomena in the theory of harmonic maps, H-systems etc. should bear in mind that cancellation also plays a hidden role in Theorem 1.2 and in the conjecture stated above. Indeed, the sharp Gagliardo–Nirenberg inequalities are based not on size conditions for the function, but on cancellation – see the proofs in [10] or [18]; the latter one, translated to the particular case needed in our proof of Theorem 1.2, uses the fact that for $u \in W^{2,2}(\mathbb{R}^n)$ the 4-laplacian $\operatorname{div}(|\nabla u|^2 \nabla u)$ is not only integrable, but also belongs to the Hardy space.

Let us finally say a few words about the structure of proof of Theorem 1.2. First, we show that $|D^2 u| + |\nabla u|^2$ is integrable with an exponent larger than 2. Here, sharp Gagliardo–Nirenberg inequalities (see Section 2 below) play a crucial role. Next, we derive Morrey decay estimates for first and second derivatives, comparing u with harmonic and biharmonic functions. At this stage, to overcome the lack of maximum principle for fourth order systems, we combine sharp Gagliardo–Nirenberg inequalities with a result of Muckenhoupt and Wheeden on Riesz potentials and fractional maximal functions (and we use explicit estimates of the Green function for the biharmonic operator with zero boundary data on the unit ball). Some technical difficulties are caused by the fact that the reverse Hölder inequalities we obtain are valid for the sum $|D^2u| + |\nabla u|^2$, and not for first and second derivatives *separately*.

The rest of the paper is organized as follows. In Section 2, we gather all necessary technical tools and auxiliary facts. In Section 3, we prove higher integrability of $|D^2u| + |\nabla u|^2$. In Section 4, we prove Morrey estimates and conclude the whole proof.

2 Notation and tools

Barred integrals denote averages, i.e.

$$\fint_{B(x,\varrho)} u\, dy = \frac{1}{|B(x,\varrho)|}\int_{B(x,\varrho)} u\, dy,$$

where $B(x,\varrho)$ stands for an open ball with a center $x \in \mathbb{R}^n$ and a radius ϱ. From time to time we write B_ϱ instead of $B(x,\varrho)$ and $(u)_{B_\varrho}$ instead of $\fint_{B_\varrho} u\, dy$. By C we will denote a general constant which is allowed to depend on n, N and the growth constant Λ. We remark that constants in the proofs may change from line to line.

Remark 2.1. Let $f, g \geq 0$. We write $f \lesssim g$, iff there is $C > 0$ depending only on n, N, Λ such that

$$f(x) \leq C\, g(x).$$

We write $f \sim g$ iff there are $C_1, C_2 > 0$ such that

$$C_1 g(x) \leq f(x) \leq C_2 g(x).$$

We recall first a couple of well-known analytic results. Gehring–Giaquinta–Modica lemma on self improving property of reverse Hölder inequalities and Campanato characterization of Hölder continuous functions can be found e.g. in the book of Giaquinta [5].

Lemma 2.2. *Let Ω be an open set in $\mathbb{R}^n$, $1 < p < \infty$ and $g \in L^p_{\mathrm{loc}}(\Omega)$ be a nonnegative function such that for any ball B_ϱ with $B_{4\varrho} \Subset \Omega$ there holds*

$$\fint_{B_\varrho} g^p\, dx \leq b\left(\fint_{B_{4\varrho}} g\, dx\right)^p + \kappa \fint_{B_{4\varrho}} g^p\, dx$$

where $b > 1$ and $0 < \kappa \leq \kappa_0 = \kappa_0(n,p)$. Then there exist constants $r_0 = r_0(n,p,b) > p$ and $C = C(n,p,b) > 0$ such that $g \in L^r_{\mathrm{loc}}$ for all $1 < r < r_0$ and

$$\fint_{B_\varrho} g^r\, dx \leq C\left(\fint_{B_{4\varrho}} g^p\, dx\right)^{r/p}.$$

Lemma 2.3. *Let Ω be an open set in $\mathbb{R}^n$, $1 \le p < \infty$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. If for some $\alpha \in (0,1)$ there holds*

$$\int_{B(x,\varrho)} |f - (f)_{B(x,\varrho)}|^p \, dx \le C\varrho^{n+p\alpha}$$

for all $x \in \Omega$ and all $\varrho < \min\{R_0, \operatorname{dist}(x, \partial\Omega)\}$ then $f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

At a certain point we will need a result of Muckenhoupt and Wheeden [12] which provides estimates for Riesz potentials in terms of the fractional maximal function

$$M_\alpha f(x) = \sup_{r>0} r^{\alpha-n} \int_{B(x,r)} |f(y)| \, dy \quad \text{for } 0 \le \alpha \le n.$$

Lemma 2.4. *Let $0 < q < \infty$ and I_α denote the Riesz potential operator, i.e.,*

$$I_\alpha * f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

There exists a constant $C > 0$, independent of f, such that

$$\|I_\alpha * f\|_q \le C \|M_\alpha f\|_q.$$

Remark. The above result is valid for *all* exponents $q > 0$, cf. [12, Theorem 1]; however, we shall use it only for $q > 1$.

We shall use the Green function $G_{2,n}$ for the bi-Laplacian with zero boundary data on the unit ball of $\mathbb{R}^n$, which is defined as follows: if $u : \mathbb{R}^n \supset B \equiv B(0,1) \to \mathbb{R}$ satisfies

$$\Delta^2 u = f \quad \text{in } B, \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B,$$

then

$$u(x) = \int_B G_{2,n}(x,y) f(y) \, dy \, .$$

Such Green functions for biharmonic and polyharmonic operators were first explored by Boggio. Later Grunau and Sweers [6] provided precise estimates for them; in particular for $n > 4$ and the bi-Laplacian operator we have

$$G_{2,n}(x,y) \sim |x-y|^{4-n} \min\left\{1, \frac{(1-|x|)^2(1-|y|)^2}{|x-y|^4}\right\}. \tag{2.1}$$

The following estimate for biharmonic functions in the unit ball of $\mathbb{R}^n$ is standard. A proof can be found e.g. in [19, Claim 2.4].

Lemma 2.5. *If $h : B_1 \equiv B(0,1) \to \mathbb{R}$ is biharmonic, then for every $\theta \in (0, \frac{1}{4})$ we have*

$$\int_{B_\theta} |\nabla h|^2 \, dx \le C \, \theta^n \int_{B_1} |\nabla h|^2 \, dx, \tag{2.2}$$

where the constant C depends only on n.

To estimate the term $|\nabla u|^4$ coming out from the growth condition (1.2) for Q we will employ the Gagliardo–Nirenberg inequality in the sharp form, proved originally by Meyer and Rivière in [10]. We will use the local version from Rivière and Strzelecki [16].

Lemma 2.6. *Assume $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{BMO}$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \le \eta \le 1$. There exists a constant $C = C(n)$ such that*

$$\int \eta^4 |\nabla u|^4 \, dx \le C \|u\|^2_{\mathrm{BMO}} \left\{ \int \eta^4 |D^2 u|^2 \, dx + \|\nabla \eta\|^2_\infty \int |\nabla u|^2 \, dx \right\}. \tag{2.3}$$

Here BMO stands for the space of functions of bounded mean oscillation with the seminorm

$$\|u\|_{\mathrm{BMO}} := \sup_{\varrho > 0} \left(⨍_{B_\varrho} |u(x) - (u)_{B_\varrho}| \, dx \right) < \infty.$$

Note that (2.3) with $\|u\|_{L^\infty}$ instead of $\|u\|_{\mathrm{BMO}}$ can be easily proved using integration by parts. Inequalities similar to (2.3) hold not only for $p = 2$, but also for other exponents $p > 1$, see [13, 18].

3 Reverse Hölder inequalities

We begin with a simple extension lemma which is included for the sake of completeness (we were not able to pinpoint a precise reference to the existing literature).

Without loss of generality assume that $0 \in \Omega$ and fix two concentric balls

$$B_r \subset B_{4r} \Subset \Omega, \qquad \text{where} \quad r < \frac{1}{16} \mathrm{dist}\,(0, \partial\Omega).$$

Lemma 3.1. *Assume that $u \in W^{1,4}(\Omega)$ satisfies*

$$K^4 := \sup_{x \in \Omega, \varrho > 0} \varrho^{4-n} \int_{\Omega \cap B(x,\varrho)} |\nabla u(\xi)|^4 \, d\xi < \infty.$$

Then there exists a constant $C = C(n)$ such that the extension of u defined as

$$\tilde{u} := u_{B_r} + \zeta(u - (u)_{B_r}),$$

where $\zeta \in C_0^\infty(B_{8r})$ is a cutoff function such that $\zeta \equiv 1$ on some neighbourhood of B_{4r} and $|\nabla \zeta| \lesssim 1/r$, satisfies

$$\sup_{x \in \mathbb{R}^n, \varrho > 0} \varrho^{4-n} \int_{B(x,\varrho)} |\nabla \tilde{u}(\xi)|^4 \, d\xi \le C(n) K^4. \tag{3.1}$$

Proof. We have $\nabla\tilde{u} = \zeta\nabla u + (u - u_{B_r})\nabla\zeta$. Thus, $\nabla\tilde{u} \equiv 0$ outside B_{8r}. In fact, changing ζ if necessary we may assume that $\operatorname{supp}\nabla\tilde{u}$ is a compact subset of B_{7r}. Then, it is easy to see that it is enough to check the inequality in (3.1) only for balls $B(x,\varrho)$ with $x \in B_{8r}$ and $\varrho < r/2$. From now on, $B = B(x,\varrho)$ denotes such a ball.

We fix $k \in \mathbb{N}$ such that

$$\log_2 \frac{r}{\varrho} + 1 \geq k > \log_2 \frac{r}{\varrho}$$

and write

$$\nabla\tilde{u} = \zeta\nabla u + \nabla\zeta\big(u-(u)_B+(u)_B-(u)_{2B}+\cdots+(u)_{2^{k-1}B}-(u)_{2^kB}+(u)_{2^kB}-(u)_{B_r}\big).$$

Applying Minkowski inequality to the whole long sum above, we obtain[2]

$$\begin{aligned} \varrho\left(\fint_{B(x,\varrho)} |\nabla\tilde{u}|^4\,dy\right)^{1/4} \lesssim\ & \varrho\left(\fint_{B(x,\varrho)} |\nabla u|^4\,dy\right)^{1/4} \\ & + \frac{\varrho}{r}\sum_{l=0}^{k}\left(\fint_{2^lB} |u-(u)_{2^lB}|^4\,dy\right)^{1/4} \\ & + \frac{\varrho}{r}\left(\fint_{B_r} |u-(u)_{2^kB}|^4\,dy\right)^{1/4}. \end{aligned} \tag{3.2}$$

Since $B = B(x,\varrho) \subset 2^kB \subset B(x,2r) \subset B_{10r} \subset \Omega$, we have

$$\varrho\left(\fint_{B(x,\varrho)} |\nabla u|^4\,dy\right)^{1/4} \leq K\,.$$

Moreover, by Poincaré inequality, each term in the sum appearing in the right hand side of (3.2) satisfies

$$\frac{\varrho}{r}\left(\fint_{2^lB} |u-(u)_{2^lB}|^4\,dy\right)^{1/4} \leq C(n)\frac{\varrho}{r}K\,.$$

Finally, since $B_r \subset 9\cdot 2^k B(x,\varrho)$, the last term on the right hand side of (3.2) does not exceed a constant multiple of $\varrho K/r$ (this follows directly from Poincaré inequality). Putting all these estimates together, we obtain

$$\begin{aligned} \varrho^{1-\frac{n}{4}}\left(\int_{B(x,\varrho)} |\nabla\tilde{u}|^4\,dy\right)^{1/4} &\leq CK\left(1+k\frac{\varrho}{r}\right) \\ &\leq C\,K\left(1+\frac{\varrho}{r}\log_2\frac{r}{\varrho}\right) \leq CK, \end{aligned}$$

since $x\log_2\frac{1}{x}$ is bounded on $[0,2]$. $\square$

[2] Such a telescopic splitting of $u - u_{B_r}$ is necessary since we cannot compare u to u_{B_r} in L^4 norm on a much smaller ball B_ϱ – note that the ratio ϱ/r is not a priori bounded away from zero.

Lemma 3.2. *Assume $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega)$ to be a weak solution of* (1.1). *Denote $\|u\|_\infty = M$. There exist constants $\varepsilon_0 = \varepsilon_0(n, N, \Lambda, M)$ and $C, s > 0$, depending also on n, N, Λ, M such that if u satisfies the smallness condition* (1.4), *then*

$$\left(\fint_{B_r} \big(|D^2u| + |\nabla u|^2\big)^{2(1+s)}\, dx\right)^{1/(1+s)} \le C \fint_{B_{4r}} \big(|D^2u|^2 + |\nabla u|^4\big)\, dx \tag{3.3}$$

for every ball $B_r \subset B_{4r} \Subset \Omega$.

Proof. To derive reverse Hölder inequalities for weak solutions of (1.1), we fix two balls $B_r \Subset B_{2r} \Subset \Omega$, assuming w.l.o.g. that they are centered at $0 \in \mathbb{R}^n$ and that $r < \frac{1}{4} \operatorname{dist}(0, \partial\Omega)$.

Fix a standard cutoff function $\zeta \in C_0^\infty(B_{2r})$ such that $\zeta \ge 0$, $\zeta \equiv 1$ on B_r, $|\nabla\zeta| \lesssim 1/r$ and $|D^2\zeta| \lesssim 1/r^2$. Set

$$\varphi := \zeta^2\big(u - T^1_{B_{2r}} u\big), \qquad \text{where} \quad T^1_{B_{2r}} u(x) := u_{B_{2r}} + \big(\nabla u\big)_{B_{2r}} x.$$

By a density argument, φ is admissible as a test map in (1.3) and we have

$$\int \Delta u \Delta\varphi\, dx = \int \varphi Q(x, u, \nabla u)\, dx. \tag{3.4}$$

The left hand side of (3.4) equals

$$\int \zeta^2 |\Delta u|^2 + I_1 + I_2,$$

where

$$\begin{aligned} |I_1| &\lesssim \frac{1}{r} \int |\Delta u|\, \big|\nabla u - \big(\nabla u\big)_{B_{2r}}\big|\, dx, \\ |I_2| &\lesssim \frac{1}{r^2} \int |\Delta u|\, \big|u - T^1_{B_{2r}} u\big|\, dx. \end{aligned}$$

Setting $p = 2n/(n+1)$ (so that $p'_* := np'/(n+p') = p$) and applying Hölder, Poincaré and Sobolev inequalities in a standard way, we estimate these integrals as follows:

$$\begin{aligned} |I_1| &\lesssim r^{n-1} \left(\fint_{B_{2r}} |\Delta u|^p\, dx\right)^{1/p} \left(\fint_{B_{2r}} \big|\nabla u - \big(\nabla u\big)_{B_{2r}}\big|^{p'}\, dx\right)^{1/p'} \\ &\lesssim r^n \left(\fint_{B_{2r}} |\Delta u|^p\, dx\right)^{1/p} \left(\fint_{B_{2r}} |D^2 u|^{p'_*}\, dx\right)^{1/p'_*} \\ &\lesssim r^n \left(\fint_{B_{2r}} |D^2 u|^p\, dx\right)^{2/p}; \end{aligned}$$

$$
\begin{aligned}
|I_2| &\lesssim r^{n-2}\Big(\fint_{B_{2r}} |\Delta u|^p\,dx\Big)^{1/p}\Big(\fint_{B_{2r}} \big|u - T^1_{B_{2r}}u\big|^{p'}\,dx\Big)^{1/p'}\\
&\lesssim r^{n-1}\Big(\fint_{B_{2r}} |\Delta u|^p\,dx\Big)^{1/p}\Big(\fint_{B_{2r}} \big|\nabla u - \big(\nabla u\big)_{B_{2r}}\big|^{p'}\,dx\Big)^{1/p'}\\
&\lesssim r^{n}\Big(\fint_{B_{2r}} |D^2 u|^p\,dx\Big)^{2/p}.
\end{aligned}
$$

Next, setting $w = \zeta(u - T^1_{B_{2r}}u)$, we have

$$
\Delta w = \zeta\Delta u + 2\nabla\zeta\cdot\big(\nabla u - \big(\nabla u\big)_{B_{2r}}\big) + \Delta\zeta(u - T^1_{B_{2r}}u).
$$

Therefore,

$$
\zeta^2|\Delta u|^2 = |\Delta w|^2 + \text{lower order terms},
$$

where – as in the estimates of I_1 and I_2 – one can check that

$$
\int_{B_{2r}} \big|\,\text{lower order terms}\,\big|\,dx \lesssim r^n\Big(\fint_{B_{2r}} |D^2u|^p\,dx\Big)^{2/p}.
$$

Thus, an easy Fourier transform argument gives

$$
\begin{aligned}
\int_{B_r} |D^2u|^2\,dx &= \int_{B_r} |D^2 w|^2\,dx \le \int_{\mathbb{R}^n} |D^2 w|^2\,dx\\
&\le \int_{\mathbb{R}^n} |\Delta w|^2\,dx \qquad (3.5)\\
&\le \int_{B_{2r}} \zeta^2|\Delta u|^2\,dx + C\,r^n\Big(\fint_{B_{2r}} |D^2u|^p\,dx\Big)^{2/p}.
\end{aligned}
$$

Combining this inequality with estimates of I_1, I_2, we finally obtain an estimate of the left hand side of (3.4),

$$
\int \Delta\varphi\Delta u\,dx \ge \int_{B_r} |D^2u|^2\,dx - C\,r^n\Big(\fint_{B_{2r}} |D^2u|^{\frac{2n}{n+1}}\,dx\Big)^{(n+1)/n}. \qquad (3.6)
$$

Now, to estimate the right hand side of (3.4), we employ the Gagliardo–Nirenberg inequalities in their sharp form. Due to Lemma 3.1 we may assume that u satisfies the equation in B_{2r} and ∇u has compact support in B_{8r} and satisfies the smallness condition (1.4) in the whole space $\mathbb{R}^n$, i.e.

$$
\sup_{\varrho>0, z\in\mathbb{R}^n} \varrho^{4-n}\int_{B(z,\varrho)} |\nabla u|^4\,dx < C(n)\varepsilon_0^4. \qquad (3.7)
$$

Set $v = u - T^1_{B_{2r}} u$; then, using the above inequality, we have

$$D^2 v = D^2 u, \qquad \nabla v = \nabla u - \big(\nabla u\big)_{B_{2r}}, \tag{3.8}$$

$$\|v\|_{\mathrm{BMO}} \lesssim \|u\|_{\mathrm{BMO}} + r\big|\big(\nabla u\big)_{B_{2r}}\big| \le C(n)\varepsilon_0\,, \tag{3.9}$$

where the BMO norm of v is taken over the whole space.

Noting that $\|\varphi\|_\infty \le \|v\|_\infty \le 2(M+\varepsilon_0)$, where $M = \|u\|_\infty$, we have

$$\begin{aligned} \int_{B_{2r}} \varphi\, Q(x,u,\nabla u)\, dx &\lesssim (M+\varepsilon_0)\int_{B_{2r}} |\nabla u|^4\, dx \\ &\lesssim (M+\varepsilon_0)\left\{\int_{B_{2r}} |\nabla v|^4\, dx + r^n\big|\big(\nabla u\big)_{B_{2r}}\big|^4\right\}. \end{aligned} \tag{3.10}$$

Pick a standard cutoff function $\eta \in C_0^\infty(B_{4r})$ with $\eta \equiv 1$ on B_{3r}. By the Gagliardo–Nirenberg inequality (2.3) and Poincaré inequality,

$$\begin{aligned} \int_{B_{2r}} |\nabla v|^4\, dx &\le \int \eta^4 |\nabla v|^4\, dx \\ &\lesssim \|v\|^2_{\mathrm{BMO}}\left\{\int_{B_{4r}} |D^2 v|^2\, dx + \frac{1}{r^2}\int_{B_{4r}}\big|\nabla u - \big(\nabla u\big)_{B_{2r}}\big|^2\, dx\right\} \\ &\lesssim \varepsilon_0^2 \int_{B_{4r}} |D^2 u|^2\, dx\,. \end{aligned} \tag{3.11}$$

Inserting this estimate into the previous one and assuming w.l.o.g. $M \ge 1$, we obtain

$$\begin{aligned} \left|\int \varphi Q(x,u,\nabla u)\, dx\right| &\lesssim M\varepsilon_0^2 \int_{B_{4r}} |D^2 u|^2\, dx + M r^n \big|\big(\nabla u\big)_{B_{2r}}\big|^4 \\ &\lesssim M\varepsilon_0^2 \int_{B_{4r}} |D^2 u|^2\, dx \\ &\quad + M r^n \left(\fint_{B_{2r}} |\nabla u|^{4n/(n+1)}\, dx\right)^{(n+1)/n}. \end{aligned} \tag{3.12}$$

Finally, combining inequalities (3.6) and (3.12), and assuming that $M\varepsilon_0^2$ is small enough (smaller that κ_0 in Lemma 2.2), we obtain a reverse Hölder inequality for

$$g := |D^2 u| + |\nabla u|^2$$

which proves that g is not just in $L^2_{\mathrm{loc}}(\Omega)$, but also in $L^{2(1+s)}_{\mathrm{loc}}(\Omega)$ for some $s > 0$. □

4 Morrey estimates for derivatives of u

In dimension $n = 4$ higher integrability of ∇u yields immediately the desired result. Indeed, if $\nabla u \in L^{4(1+\delta)}_{\mathrm{loc}}$, then the right hand side of (1.1) is in $L^{1+\delta}_{\mathrm{loc}}$, and by Calderón–Zygmund theory $D^4 u \in L^{1+\delta}_{\mathrm{loc}}$. Applying Sobolev imbedding theorem in dimension 4, we obtain $\nabla u \in L^{4p}_{\mathrm{loc}}$, where $p = (1+\delta)/(1-3\delta) > 1+2\delta$. Thus, we have $Q(x,u,\nabla u) \in L^{1+2\delta}_{\mathrm{loc}}$, and therefore $D^4 u \in L^{1+2\delta}_{\mathrm{loc}}$. After finitely many steps, we obtain $\nabla u \in C^\alpha$ for some $\alpha > 0$ and then Theorem 1.2 follows by classical bootstrap using Schauder estimates.

For $n > 4$ more work is necessary.

Lemma 4.1. *Assume $u \in W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, where $\Omega \subset \mathbb{R}^n$, $n > 4$, to be a weak solution of* (1.1) *satisfying $\|u\|_{L^\infty(\Omega)} \le M$ and the smallness condition* (1.4). *If $M\varepsilon_0^2$ is sufficiently small, then there exists a number $\delta = \delta(n) \in (0, \frac{1}{16})$ such that*

$$\varrho^{4-n} \int_{B(a,\varrho)} \left(|D^2u|^2 + |\nabla u|^4\right) dx \le \delta^3 \cdot R^{4-n} \int_{B(a,R)} \left(|D^2u|^2 + |\nabla u|^4\right) dx \quad (4.1)$$

whenever $B(a,\varrho) \Subset B(a,R) \Subset \Omega$ and $\varrho/R = \delta$.

Proof. The crucial difficulty is to obtain appropriate decay estimates for integrals of $|D^2u|^2$. Suppose w.l.o.g. that $a = 0$ and let

$$\varrho = \theta r, \qquad R = 4r < \operatorname{dist}(a, \partial\Omega), \qquad \theta = \frac{\varrho}{r} < \frac{1}{4}, \qquad \delta = \frac{\varrho}{R} = \frac{\theta}{4}.$$

We write $u = u_0 + u_1$, where u_0 is a minimizer of the energy functional

$$E(w) = \int_{B_r} |D^2 w|^2 \, dx$$

in the class of admissible mappings

$$\mathcal{A} := \{w \in W^{2,2}(B_r, \mathbb{R}^N) : w - u \in W^{2,2}_0(B_r, \mathbb{R}^N)\}.$$

Then u_0 is smooth and $\Delta^2 u_0 = 0$ in B_r; $u_1 \in W^{2,2}_0(B_r, \mathbb{R}^N)$ and $\Delta^2 u_1 = \Delta^2 u$ in B_r.

Step 1 (decay estimates for D^2u_0). We apply Lemma 2.5 noting that a similar estimate holds also for D^2h (the derivatives of a biharmonic function are again biharmonic). Setting $h(x) = u_0(rx)$ for $x \in B_1$ and scaling from B_1 to B_r, we obtain

$$\begin{aligned}
\varrho^{4-n} \int_{B_\varrho} |D^2u_0|^2 \, dx &\le C\theta^4 r^{4-n} \int_{B_r} |D^2u_0|^2 \, dx \\
&\le C\theta^4 r^{4-n} \int_{B_r} |D^2u|^2 \, dx \\
&\le 4^n C \cdot \delta^4 \cdot R^{4-n} \int_{B_R} |D^2u|^2 \, dx . \qquad (4.2)
\end{aligned}$$

Step 2. Decay estimates for $|D^2 u_1|$. We have

$$u_1 = G_{2,n} * Q(x, u, \nabla u) \qquad \text{in } B_r$$

where $G_{2,n}$ denotes the Green function of Δ^2 with zero boundary conditions in B_r. Rescaling to B_1, i.e., setting $v_1(x) = u_1(rx)$, and using the Green function estimates obtained by Grunau and Sweers (2.1), we estimate u_1 as follows

$$\begin{aligned} |u_1(rx)| &= |v_1(x)| \\ &\leq \int_{B(0,1)} |G_{2,n}(x,y)|\, |\Delta^2 v_1(y)|\, dy \\ &\lesssim \int_{B(0,1)} |G_{2,n}(x,y)|\, \underbrace{r^4 |\nabla u(ry)|^4}_{=:f(y)}\, dy \\ &\lesssim \int_{B(0,1)} |x-y|^{4-n} f(y)\, dy. \end{aligned} \tag{4.3}$$

Now we use the fact that f can be extended from B_1 onto $\mathbb{R}^n$ so that the smallness condition (3.7) is satisfied – see Lemma 3.1. (Note that $R = 4r$ is fixed and the extended function stays unchanged in $B_R = B_{4r}$.) Thus, $|u_1(rx)| \lesssim I_4 * f(x)$ and $f \geq 0$ satisfies

$$\begin{aligned} M_4 f(x) &= \sup_{\varrho > 0} \varrho^{4-n} \int_{B(x,\varrho)} f(y)\, dy \\ &\leq \sup_{x \in \mathbb{R}^n, \varrho > 0} (r\varrho)^{4-n} \int_{B(rx, r\varrho)} |\nabla u(\xi)|^4\, d\xi \;\leq\; \varepsilon_0^4. \end{aligned} \tag{4.4}$$

Therefore, by Lemma 2.4,

$$\begin{aligned} \left(\int_{B(0,1)} |u_1(rx)|^q\, dx \right)^{1/q} &\lesssim \|I_4 * f\|_{L^q(B(0,1))} \\ &\lesssim \|M_4 f\|_{L^q(B(0,1))} \lesssim \varepsilon_0^4 \end{aligned}$$

by (4.4). Scaling back to B_r, we obtain

$$\left(\int_{B_r} |u_1(x)|^q\, dx \right)^{1/q} \lesssim r^{n/q} \varepsilon_0^4. \tag{4.5}$$

(By Lemma 2.4, such an estimate holds for any $q > 1$, with a constant depending on q.)

Choosing q as the Hölder conjugate of $1+s$ and applying the results of the previous section, we now write

$$
\begin{aligned}
\int_{B_r} |D^2 u_1|^2\,dx &\le \int_{B_r} |\Delta u_1|^2\,dx \\
&= \left| \int_{B_r} u_1 \Delta^2 u\,dx \right| \\
&\lesssim \left(\int_{B_r} |u_1|^q\,dx \right)^{1/q} \left(\int_{B_r} |\nabla u|^{4(1+s)}\,dx \right)^{1/(1+s)} \\
&\lesssim r^{n/q} \varepsilon_0^4 \left(\int_{B_r} |\nabla u|^{4(1+s)}\,dx \right)^{1/(1+s)} \\
&\lesssim \varepsilon_0^4 \int_{B_R} \big(|D^2 u|^2 + |\nabla u|^4 \big)\,dx\,.
\end{aligned}
\tag{4.6}
$$

Putting together inequalities (4.2) and (4.6), we obtain

$$
\begin{aligned}
\varrho^{4-n} \int_{B_\varrho} |D^2 u|^2\,dx &\le C\delta^4 R^{4-n} \int_{B_R} |D^2 u|^2\,dx \\
&\quad + C\varepsilon_0^4 \delta^{4-n} R^{4-n} \int_{B_R} \big(|D^2 u|^2 + |\nabla u|^4 \big)\,dx\,.
\end{aligned}
\tag{4.7}
$$

Step 3. Decay estimates for the gradient. To finish the proof, we now need to combine (4.7) with a decay estimate for $|\nabla u|^4$. Reasoning as in (3.10) and (3.11), we write

$$
\begin{aligned}
\int_{B_\varrho} |\nabla u|^4\,dx &\lesssim \int_{B_\varrho} |\nabla u - (\nabla u)_{B_\varrho}|^4\,dx + \varrho^n |(\nabla u)_{B_\varrho}|^4 \\
&\lesssim \varepsilon_0^2 \int_{B_{4\varrho}} |D^2 u|^2\,dx \; + \; \varrho^n \left(\fint_{B_\varrho} |\nabla u|^2\,dx \right)^2 .
\end{aligned}
\tag{4.8}
$$

To cope with the last term in the right hand side, we use a different splitting of u. Namely, set $u = u_2 + u_3$, where u_2 is harmonic in B_r and $u_3 \in W_0^{1,2}(B_r, \mathbb{R}^N)$. Since $|\nabla u_2|^2$ is subharmonic, the mean value property and Dirichlet principle yield

$$
\begin{aligned}
\varrho^{2-n} \int_{B_\varrho} |\nabla u_2|^2\,dx &\le \theta^2 r^{2-n} \int_{B_r} |\nabla u_2|^2\,dx \\
&\le \theta^2 r^{2-n} \int_{B_r} |\nabla u|^2\,dx \\
&\le 4^n \delta^2 R^{2-n} \int_{B_R} |\nabla u|^2\,dx\,.
\end{aligned}
\tag{4.9}
$$

Next, since $\Delta u_3 = \Delta u$, we have

$$\int_{B_r} |\nabla u_3|^2\,dx = \left|\int_{B_r} u_3 \Delta u\,dx\right| \lesssim \left(\int_{B_r} |u_3|^2\,dx\right)^{1/2} \left(\int_{B_r} |D^2 u|^2\,dx\right)^{1/2}. \tag{4.10}$$

As $u_3 \in W_0^{1,2}(B_r, \mathbb{R}^N)$ (= the closure of C_0^∞ in the norm of $W^{1,2}$), we can invoke the standard Green's representation formula and, after one integration by parts, conclude that

$$|u_3(x)| \lesssim \int |\nabla \Gamma(x-y)| \cdot |\nabla u(y)|\,dy,$$

where Γ stands for the fundamental solution of the Laplace operator in $\mathbb{R}^n$. Now, a reasoning mimicking the proof of (4.3)–(4.5) (employing the result of Muckenhoupt and Wheeden on Riesz potentials, cf. Lemma 2.4) leads to

$$\left(\fint_{B_r} |u_3|^2\,dx\right)^{1/2} \lesssim \varepsilon_0\,; \tag{4.11}$$

thus, coming back to (4.10), we obtain

$$\begin{aligned} \varrho^4 \left(\fint_{B_\varrho} |\nabla u_3|^2\,dx\right)^2 &\le C\theta^{4-2n} \varepsilon_0^2 r^{4-n} \int_{B_r} |D^2 u|^2\,dx \\ &\le 4^n C \delta^{4-2n} \varepsilon_0^2 R^{4-n} \int_{B_R} |D^2 u|^2\,dx. \end{aligned} \tag{4.12}$$

Estimates (4.9) and (4.12) combined with (4.8) give

$$\begin{aligned} \varrho^{4-n} \int_{B_\varrho} |\nabla u|^4\,dx \lesssim{}& \varepsilon_0^2 (\delta^{4-n} + \delta^{4-2n}) R^{4-n} \int_{B_R} |D^2 u|^2\,dx \\ &+ \delta^4 R^{4-n} \int_{B_R} |\nabla u|^4\,dx\,. \end{aligned}$$

Adding this inequality to (4.7), we finally obtain

$$\begin{aligned} \varrho^{4-n} \int_{B(a,\varrho)} \left(|D^2 u|^2 + |\nabla u|^4\right) dx \le{}& C_1 \Big(\delta^4 + \delta^{4-n}(\varepsilon_0^4 + \varepsilon_0^2)\Big) R^{4-n} \int_{B_R} |D^2 u|^2\,dx \\ &+ C_2 \Big(\delta^4 + \delta^{4-n}\varepsilon_0^4 + \delta^{4-2n}\varepsilon_0^2\Big) R^{4-n} \int_{B_R} |\nabla u|^4\,dx\,. \end{aligned} \tag{4.13}$$

Fixing first a small number $\theta \in (0, \frac{1}{4})$ so that $C_i \delta = C_i \theta/4 < \frac{1}{3}$ for $i = 1, 2$, and then choosing $\varepsilon_0 < 1$ so that

$$\varepsilon_0 \delta^{-n} < \min\Big(\frac{1}{3C_i}, \frac{1}{4}\Big) \qquad \text{for} \quad i = 1, 2,$$

we complete the proof of (4.1). □

Lemma 4.2. *If $u \in W^{2,2}(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.1) *satisfying*

$$\|u\|_{L^\infty(\Omega)} \le M, \qquad \sup_{\varrho>0,\ a\in\mathbb{R}^n} \varrho^{4-n} \int_{B(a,\varrho)} |\nabla u|^4\, dx \le \varepsilon_0^4,$$

and $M\varepsilon_0^2$ is sufficiently small, then there exists a constant $C = C(n)$ such that

$$⨍_{B_\varrho} |\nabla u - (\nabla u)_{B_\varrho}|^2\, dx \le C\varrho R ⨍_{B_R} \big(|D^2u|^2 + |\nabla u|^4\big)\, dx \tag{4.14}$$

for each pair of concentric balls $B_\varrho \Subset B_R \Subset \Omega$.

Proof. Iterating inequality (4.1), we obtain

$$\varrho^{4-n} \int_{B(a,\varrho)} \big(|D^2u|^2 + |\nabla u|^4\big)\, dx \lesssim \left(\frac{\varrho}{R}\right)^3 R^{4-n} \int_{B(a,R)} \big(|D^2u|^2 + |\nabla u|^4\big)\, dx$$

whenever $\varrho < R < \operatorname{dist}(a, \partial\Omega)$. By Poincaré inequality,

$$⨍_{B_\varrho} |\nabla u - (\nabla u)_{B_\varrho}|^2\, dx \lesssim \varrho^2 ⨍_{B_\varrho} |D^2u|^2\, dx\,.$$

Putting these two estimates together, we obtain (4.14). □

Proof of the main theorem. The inequality (4.14) together with Theorem 2.3 gives $\nabla u \in C^{0,1/2}_{\mathrm{loc}}$. Since we assume that $Q \in C^\infty$ it follows that the right hand side of (1.1) is Hölder continuous, i.e. $Q(x, u, \nabla u) \in C^{0,1/2}_{\mathrm{loc}}$. Then Schauder theory yields $u \in C^{4,1/2}_{\mathrm{loc}}$. The standard bootstrap technique gives $u \in C^\infty_{\mathrm{loc}}(\Omega)$. □

Acknowledgments. Part of this work was done while both authors were visiting the Department of Mathematics of Helsinki University of Technology. The authors thank Juha Kinnunen for his hospitality and for excellent working conditions.

The first author is also grateful to Tristan Rivière for numerous stimulating discussions concerning Gagliardo–Nirenberg inequalities and their applications.

References

[1] F. Bethuel, On the singular set of stationary harmonic maps, *Manuscripta Math.* **78** (1993), 417–443.

[2] S.-Y. A. Chang, L. Wang and P. Yang, A regularity theory of biharmonic maps, *Comm. Pure Appl. Math.* **52** (1999), 1113–1137.

[3] L. C. Evans, Partial regularity for stationary harmonic maps into spheres, *Arch. Rat. Mech. Anal.* **116** (1991), 101–113.

[4] J. Frehse, A discontinuous solution of a mildly nonlinear elliptic system, *Math. Z.* **134** (1973), 229–230.

[5] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, Princeton, New Jersey, 1983.

[6] H-Ch. Grunau and G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, *Math. Ann.* **307** (1997), 589–626.

[7] F. Hélein, *Harmonic Maps, Conservation Laws and Moving Frames*, Cambridge Tracts in Mathematics 150, Cambridge University Press, Cambridge, 2002.

[8] F. Hélein, Regularité des applications faiblement harmoniques entre une surface et une variété riemannienne, *C. R. Acad. Sci. Paris* **312** (1991), 591–596.

[9] T. Lamm and T. Rivière, Conservation laws for fourth order systems in four dimensions, to appear in *Comm. PDE.*

[10] Y. Meyer and T. Rivière, A partial regularity result for a class of stationary Yang–Mills fields in high dimension, *Rev. Mat. Iberoamericana* **19** (2003), no. 1, 195–219.

[11] R. Moser, Remarks on the regularity of biharmonic maps in four dimensions, *Comm. Pure Appl. Math.* **26** (2006), no. 3, 317–329.

[12] B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, *Proc. Amer. Math. Soc.* **192** (1974), 261–274.

[13] D. Pumberger, Regularity results for stationary harmonic and J-holomorphic maps, preprint, ETH Zuerich, 2004.

[14] T. Rivière, Conservation laws for conformally invariant variational problems, *Inventiones Math.* **168** (2007), 1–22.

[15] T. Rivière and M. Struwe, Partial regularity for harmonic maps, and related problems, preprint, 2006, see `http://www.math.ethz.ch/~riviere/pub.html`.

[16] T. Rivière and P. Strzelecki, A sharp nonlinear Gagliardo–Nirenberg-type estimate and applications to the regularity of elliptic systems, *Communications in Partial Differential Equations* **30** (2005), 589–604.

[17] P. Strzelecki, On biharmonic maps and their generalizations, *Calc. Var. Partial Differential Equations* **18** (2003), 401–432.

[18] P. Strzelecki, Gagliardo–Nirenberg inequalities with a BMO term, *Bull. London Math. Soc.* **38** (2006), 294–300.

[19] Ch. Wang, Remarks on biharmonic maps into spheres, *Calc. Var. Partial Differential Equations* **21** (2004), 221–242.

[20] Ch. Wang, Biharmonic maps from $\mathbf{R}^4$ into a Riemannian manifold, *Math. Z.* **247** (2004), no. 1, 65–87.

Received June 26, 2007; revised December 11, 2007.

Author information

Paweł Strzelecki, Institute of Mathematics,
Warsaw University,
ul. Banacha 2,
02–097 Warszawa, Poland.
E-mail: pawelst@mimuw.edu.pl

Anna Zatorska-Goldstein, Institute of Applied Mathematics and Mechanics,
Warsaw University,
ul. Banacha 2,
02–097 Warszawa, Poland.
E-mail: azator@mimuw.edu.pl