A FUNCTION NOT CONSTANT ON A CONNECTED SET OF CRITICAL POINTS

By Hassler Whitney

1. **Introduction.** Let $f(x_1, \cdots, x_n)$ be a function of class C^m (i.e., with continuous partial derivatives through the mth order) in a region R. Any point at which all its first partial derivatives vanish is called a *critical point* of f. Suppose every point of a connected set A of points in R is a critical point. It is natural to suspect then that f is a constant on A. But this need not be so. We construct below an example with $n = 2$, $m = 1$, $A =$ an arc. The example may be extended to the case $n = n$, $m = n - 1$, $A =$ an arc. The arc and the function on the arc are easily defined. The extension of the function through the rest of the plane or space is given by a theorem of the author.[1]

The question settled in this paper was raised implicitly in a paper of W. M. Whyburn.[2] It is brought up by his definition of critical sets as the maximal connected subsets of the set of critical points on which the function takes a single critical value. Theorem 2 of Whyburn's paper shows that an example of the type given in the present paper can be constructed only by using critical sets which have points that cannot be joined in these sets by rectifiable arcs. It would be interesting to discover how far from rectifiable a closed set must be to be a set of critical points of some function but not a critical set of the function. It may be remarked that any closed set may be a critical set.[3]

For fixed n and m large enough, $m \geqq [(n - 3)^2/16 + n]$, where $[n]$ is the integral part of n, f must be constant on any connected critical set, as shown by M. Morse and A. Sard in an unpublished paper.

The example shows that it is in general impossible to express the values of a function $f(x_1, \cdots, x_n)$ along a curve which is not rectifiable by means of an integral of a function of partial derivatives of f of order $\leqq n - 1$ along the curve.[4]

2. **The arc.** Let Q be a square of side 1 in the plane. Let Q_0, Q_1, Q_2, Q_3 be squares of side 1/3 lying interior to Q in cyclical order, each a distance 1/12

Received May 16, 1935; presented to the American Mathematical Society, October 26, 1935. The example given here was discovered in 1932 while the author was a National Research Fellow.

[1] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 63–89, Lemma 2. We refer to this paper as AE.

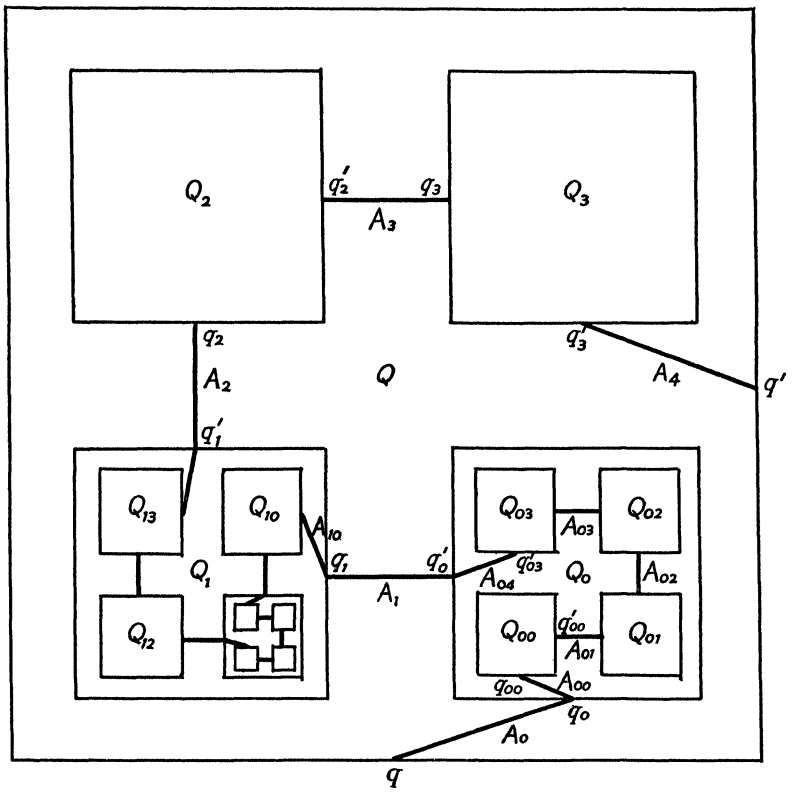
[2] W. M. Whyburn, Bull. Amer. Math. Soc., vol. 35 (1929), pp. 701–708.

[3] See A. Ostrowski, Bull. des Sciences Math., Feb. (1934), pp. 64–72.

[4] For such an expression (using partial derivatives of any desired order) along a rectifiable curve, see H. Whitney, *Functions differentiable on the boundaries of regions*, Annals of Mathematics, vol. 35 (1934), pp. 482–485, (1) and (3).

from the boundary of Q. Let q and q' be the centers of the sides of Q along Q_0, Q_1, and along Q_3, Q_0. Let q_i and q_i' be centers of adjacent sides of Q_i ($i = 0, 1, 2, 3$) so that q_{i-1}' and q_i face each other ($i = 1, 2, 3$), and q_0 is near q, q_3' is near q'. Let A_0 be a line joining q and q_0, let A_i join q_{i-1}' and q_i ($i = 1, 2, 3$), and let A_4 join q_3' and q'.

Suppose we have constructed squares $Q_{i_1 \cdots i_t}$, points $q_{i_1 \cdots i_t}$, $q_{i_1 \cdots i_t}'$, and lines $A_{j_1 \cdots j_t}$ (each $i_k = 0, 1, 2, 3$; each $j_k = 0, 1, 2, 3, 4$) for $t < s$. By taking a square $Q_{i_1 \cdots i_{s-2}}$, shrinking it to a third its size, and turning it around and upside down if necessary, we may place it in $Q_{i_1 \cdots i_{s-1}}$ so that $q_{i_1 \cdots i_{s-2}}$ and $q_{i_1 \cdots i_{s-2}}'$ go into $q_{i_1 \cdots i_{s-1}}$



and $q_{i_1 \cdots i_{s-1}}'$, and thus construct squares $Q_{i_1 \cdots i_s}$, etc. We continue this process indefinitely. Let $Q_{i_1 i_2 \cdots}$ be the point common to Q, Q_{i_1}, $Q_{i_1 i_2}$, $\cdots$ for each $(i_1, i_2, \cdots)$.

The line segments $A_{i_1 \cdots i_s}$ together with the points $Q_{i_1 i_2 \cdots}$ form an arc A. It may be represented as the topological image of the segment $(0, 1)$ by letting $A_{i_1 \cdots i_s}$ correspond to the segment

$$\left(\frac{2i_1 + 1}{9} + \cdots + \frac{2i_{s-1} + 1}{9^{s-1}} + \frac{2i_s}{9^s}, \frac{2i_1 + 1}{9} + \cdots + \frac{2i_{s-1} + 1}{9^{s-1}} + \frac{2i_s + 1}{9^s}\right),$$

and letting $Q_{i_1 i_2 \ldots}$ correspond to the number

$$\frac{2i_1+1}{9} + \frac{2i_2+1}{9^2} + \cdots .$$

3. **The function** $f(x, y)$. We first define $f(x, y)$ along the arc A as follows:

$$\text{on } A_{i_1 \ldots i_s}, \qquad f = \frac{i_1}{4} + \cdots + \frac{i_s}{4^s};$$

$$\text{at } Q_{i_1 i_2 \ldots}, \qquad f = \frac{i_1}{4} + \frac{i_2}{4^2} + \cdots .$$

f increases from 0 to 1 as we run along A from q to q'. Set $f_{00}(x,y) = f(x,y)$, $f_{10}(x,y) = f_{01}(x,y) = 0$ on A. We shall show that f_{00} *is of class* C^1 *on* A *in terms of* (f_{00}, f_{10}, f_{01}) (see AE). It will follow from Lemma 2 of AE that the definition of $f(x,y)$ may be extended over the plane (in particular, over Q) so that f is of class C^1; also $\partial f/\partial x = f_{10} = 0$, $\partial f/\partial y = f_{01} = 0$ on A, and hence each point of A is a critical point of f.

As f_{10} and f_{01} are continuous in A, we need merely prove that for each $\epsilon > 0$ there is a $\delta > 0$ such that if (x,y) and (x',y') are points of A whose distance apart is $r < \delta$, then

$$|f(x',y') - f(x,y)| < r\epsilon, \tag{1}$$

(see AE (3.1) and (3.2)).

The proof rests on the following two facts.

(a) If (x,y) and (x',y') are points of A in $Q_{i_1 \ldots i_s}$, then

$$|f(x',y') - f(x,y)| \leqq 1/4^s. \tag{2}$$

(b) If (x,y) and (x',y') are points of A separated by some point $Q_{i_1 i_2 \ldots}$, and if $Q_{i_1 \ldots i_s}$ is the smallest square containing them both, then

$$r > \frac{1}{12}\frac{1}{3^{s+1}}. \tag{3}$$

Assume (a) and (b) are true. Given $\epsilon > 0$, choose s_0 and δ so that

$$36\left(\frac{3}{4}\right)^{s_0} < \epsilon, \qquad \delta < \frac{1}{12}\frac{1}{3^{s_0+1}}. \tag{4}$$

Now let (x,y) and (x',y') be any two points of A distant $r < \delta$. If no point $Q_{i_1 i_2 \ldots}$ separates them, then $f(x',y') = f(x,y)$, and (1) holds. Otherwise, let $Q_{i_1 \ldots i_s}$ be the smallest square containing them both. (b) gives

$$\frac{1}{12}\frac{1}{3^{s+1}} < r < \delta < \frac{1}{12}\frac{1}{3^{s_0+1}}, \quad \therefore\ s > s_0 .$$

Hence

$$\frac{|f(x',y') - f(x,y)|}{r} \leqq \frac{12 \cdot 3^{s+1}}{4^s} = 36\left(\frac{3}{4}\right)^s < \epsilon .$$

It remains to prove (a) and (b). (a) is obvious from the definition of f. To prove (b), we consider three cases: neither point is in any square $Q_{i_1 \dots i_s i_{s+1}}$; each point is in such a square; one point is in such a square, and the other is not. In the first two cases we see that $r > 1/(2 \cdot 3^{s+1})$; in the third case, (3) holds.

4. **Generalization to higher dimensions.** We shall indicate how the corresponding example is constructed for $n = 3$, $m = 2$; the generalization to higher dimensions is obvious.

Let Q be a cube of side 1. Let $Q_0, \cdots, Q_7$ be cubes of side 2/5 arranged in Q so that Q_i is adjacent to Q_{i-1}. Let q and q' be the centers of faces of Q which are adjacent and adjacent respectively to Q_0 and Q_7. Define q_i and q_i' $(i = 0, \cdots, 7)$ as before, and similarly for $A_0, \cdots, A_8$. The process is continued indefinitely, as before. A is again an arc. We set

$$\text{on } A_{i_1 \dots i_s}, \quad f = \frac{i_1}{8} + \cdots + \frac{i_s}{8^s},$$

$$\text{at } Q_{i_1 i_2 \dots}, \quad f = \frac{i_1}{8} + \frac{i_2}{8^2} + \cdots.$$

Set $f_{000} = f$, $f_{\alpha\beta\gamma} = 0$ $(\alpha + \beta + \gamma = 1 \text{ or } 2)$ on A. To prove that $f = f_{000}$ is of class C^2 in terms of these functions we need merely show that

$$\Delta = | f(x', y', z') - f(x, y, z) | < r^2 \epsilon \text{ on } A \qquad (r < \delta). \tag{5}$$

Note that f varies by at most $1/8^s$ in any $Q_{i_1 \dots i_s}$. Now if the smallest cube containing (x,y,z) and (x',y',z') is $Q_{i_1 \dots i_s}$, r is of the order $(2/5)^s$, while $\Delta \leqq 1/8^s$; hence Δ/r^2 is of the order $(25/32)^s$, which is $< \epsilon$ for r small enough and hence s large enough.

Note that *all partial derivatives of* f (of order $\leqq n - 1 = 2$) *vanish on* A.

HARVARD UNIVERSITY.