Null Lagrangians
Definitions, Examples and Applications

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Learning mathematics by only reading about it is like making love by e-mail

(A paraphrase of Luciano Pavarotti on music)
Nonlinear Hyperelasticity

One enquires into deformations \( h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y} \) of smallest stored energy

\[
\mathcal{E}[h] = \int_{\mathbb{X}} E(x, h, Dh) \, dx, \quad E : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{m \times n} \to \mathbb{R}
\]

where the accustomed hypothesis is Morrey’s Quasiconvexity, which yields lower semicontinuity of the energy functional

\[
\int \mathbb{X} E(x, h, Dh) \, dx \leq \lim \inf \int \mathbb{X} E(x, h_k, Dh_k) \, dx
\]

whenever \( h_k \rightharpoonup h \), weakly in \( \mathcal{W}^{1,p}(\mathbb{X}) \). Special (more pragmatic) cases are **polyconvex functionals**. These are convex functions of subdeterminants of the differential matrix.
The term *null Lagrangian* pertains to a nonlinear differential expression whose integral mean over any open region is determined by the boundary values, like integral of an exact differential form. Because of fine technics, developed with great imagination and mathematical taste, the present topics could reasonably be subtitled:

**THE ART OF INTEGRATING BY PARTS.**

An important special case is furnished by the *Jacobian determinant* of a vector field
Here is how integration by parts (Stokes’ formula) reduces the integral means of $J(x, f)$ over any subregion $U \subset \Omega$ to its boundary:

$$
\int_U J(x, f) \, dx = \int_U df^1 \wedge df^2 \wedge ... \wedge df^n = \int_{\partial U} f^1 df^2 \wedge ... \wedge df^n
$$
More refine calculation then reveals that

\[ \int_{\mathbb{U}} \mathbf{J}(x, f) \, dx = \int_{\mathbb{U}} \mathbf{J}(x, g) \, dx, \quad \text{whenever } f = g \text{ on } \partial \mathbb{U} \]

This identity lies fairly deep in the concept of the topological degree. Many more differential expressions enjoy the identity such as this. For reasons to be understood later, we call such expressions null Lagrangians. Jacobian determinants and other null Lagrangians owe much of their importance to prominent advances in the calculus of variations (polyconvex functions), nonlinear PDEs (compensated compactness), geometric function theory (quasiconformal mappings) and several fields of applied mathematics: continuum mechanics, nonlinear elasticity, composites (rank-one connections), micro-structure of crystals, and much more.
In nonlinear elasticity we speak of \( f : \Omega \to \mathbb{R}^n \) as a deformation of an elastic body \( \Omega \subset \mathbb{R}^n \), where the present knowledge of null Lagrangians tells us something about topological behavior of such deformations. One rather surprising discovery, that brought null Lagrangians to the attention of many researchers is the phenomenon of selfimprovement of the degree of integrability. In this category of important results we ought to emphasize the \( \mathcal{L}^1 \)-integrability of \( J(x, f) \) under minimal regularity hypothesis on the mapping \( f \).

Any (the slightest possible) self-improvement of integrability of null Lagrangians turns out to be a useful bonus for the existence of energy-minimal deformations.
Yet, within various different contexts we ought to mention the fundamental role of null Lagrangians in understanding

**Very weak solutions of nonlinear elliptic PDEs**

The present lectures will certainly fresh light on these connections.

New and useful connections with partial differential equations, calculus of variations, material science, yet to be discovered.

**What better way to enhance current trends in nonlinear analysis?**
Null Lagrangians evolve from a subtle play of differentiation and integration. Its simplest case and intrinsic beauty begins with the Fundamental Theorem of Calculus

*Every absolutely continuous function* \( f : [a, b] \to \mathbb{R} \) *is differentiable almost everywhere and its derivative belongs to* \( \mathcal{L}^1[a, b] \). *Moreover,*

\[
\int_a^b f'(x) \, dx = f(b) - f(a)
\]
The class of absolutely continuous functions, denoted by $A C[a, b]$, is none other than the Sobolev space $W^{1,1}(a, b)$. The Fundamental Theorem of Calculus yields the following identity

$$
\int_a^b f'(x) \, dx = \int_a^b g'(x) \, dx
$$

whenever two functions $f, g \in A C[a, b]$ coincide at the endpoints of the interval. One might ask which integral expressions enjoy the identity such as this? Let us clear up this question with somewhat less obvious example. Suppose $F = F(x, y)$ is a function of class $C^1([a, b] \times \mathbb{R})$. The Fundamental Theorem of Calculus yields, for every $f \in A C[a, b]$

$$
\int_a^b \left[ F_x(x, f) + F_y(x, f) f' \right] \, dx = \int_a^b \frac{d}{dx} F(x, f) \, dx = F(b, f(b)) - F(a, f(a))
$$
With the notation

$$E(x, y, z) = F_x(x, y) + zF_y(x, y), \quad F \in C^1([a, b] \times \mathbb{R}) \quad (*)$$

we again find ourselves in a situation when the values of $f$ inside the domain $(a, b)$ have no impact on the values of the energy integral

$$\mathcal{E}[f] = \int_a^b E(x, f, f') \, dx$$

In these words, two functions $f, g \in AC[a, b]$ that are equal at the endpoints store the same energy,

$$\mathcal{E}[f] = \int_a^b E(x, f, f') \, dx = \int_a^b E(x, g, g') \, dx = \mathcal{E}[g] \quad (**)$$
It is of interest to inquire as to whether these are the only integrands $E : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that possess such a distinctive property. Clearly, $E$ should be sufficiently regular in order to define $E[f]$. Let us examine the case when $E = E(x, y, z)$ and its partial derivatives $E_y$ and $E_z$ are continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$. Under these conditions it becomes legitimate to speak of the energy of $f, g \in \mathcal{C}^1[a, b]$, and we have:

**Proposition**
The identity (**) holds for every $f, g \in \mathcal{C}^1[a, b]$ if and only if $E$ takes the form (†).
Proof. Our line of reasoning is quite reminiscent of the variational approach by Euler (1742) and Lagrange (1755); the technique we shall exploit greatly in a variety of settings. Fix a test function $\eta \in C^1(a, b)$. For every real parameter $t$ we consider a function $f + t\eta$, a variation of $f$ with the same values at the endpoints. Accordingly, the energy $E[f + t\eta] = \int_a^b E(x, f + t\eta, f' + t\eta') \, dx$ remains unchanged, so its derivative at $t = 0$ vanishes. By the Chain Rule applied under the integral sign we obtain

$$\int_a^b \left[ \eta(x) E_y(x, f, f') + \eta'(x) E_z(x, f, f') \right] \, dx = 0$$

This integral identity is often referred to as the weak form of the Lagrange-Euler equation. It is worth noticing that the term $E_y(x, f, f')$ is a continuous
function on \([a, b]\). Next, we integrate by parts

\[
\int_a^b \left[ E_z(x, f, f') - \int_a^x E_y(s, f, f') \, ds \right] \eta'(x) \, dx = 0
\]

where we have suppressed the dependence of \(f\) and \(f'\) on \(x\). As this equation holds for all \(\eta \in C^1(a, b)\) the entire expression in the brackets must be constant; an old tradition calls this fact the \textit{Lemma of du Bois-Reymond}. Hence,

\[
E_z(x, f, f') = \int_a^x E_y(s, f, f') \, ds + E_z(a, f(a), f'(a)) \tag{1}
\]

It then follows that \(E_z(x, f, f')\) is a continuously differentiable function in \(x\). This gives us the \textit{point-wise form} of the Lagrange equation:
\[
\frac{d}{dx} E_z(x, f, f') = E_y(x, f, f'), \quad \text{whenever } f \in C^1[a, b] \quad (2)
\]

Just because this Lagrange equation is identically satisfied we shall give the name \textit{null Lagrangian} to the function \( E : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \).

(Let us skip the "only if" part; the proof being presented below in detail for the interested students)

To fully explore the null Lagrangian identity (2) we now test it with functions \( f_\varepsilon \in C^1[a, b] \) of the form

\[
f_\varepsilon(x) = \begin{cases} 
\alpha(x - x_\circ) + y_\circ & \text{if } a \leq x \leq x_\circ - \varepsilon \\
\beta(x - x_\circ) + y_\circ & \text{if } x_\circ + \varepsilon \leq x \leq b
\end{cases}
\]
where $a < x_\circ < b$, $y_\circ \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ are arbitrary though fixed, whereas $\varepsilon > 0$ will go to zero. A gap $x_\circ - \varepsilon \leq x \leq x_\circ + \varepsilon$ in our definition can easily be filled up with a quadratic polynomial. In this interval we set

$$f_\varepsilon(x) = \frac{\beta - \alpha}{4\varepsilon}(x - x_\circ)^2 + \frac{\beta + \alpha}{2}(x - x_\circ) + \frac{\beta - \alpha}{4}\varepsilon + y_\circ$$

Letting $\varepsilon$ go to zero the limit of $f_\varepsilon$ becomes a piecewise linear function, Lipschitz continuous but not $C^1$-regular. It is to be noted that both $f_\varepsilon(x)$ and its derivative $f'_\varepsilon(x)$ stay bounded as $\varepsilon \to 0$. With the aid of formula (1) we evaluate $E_z(x, f, f')$ at $x = x_\circ \pm \varepsilon$, to obtain

$$E_z(x_\circ, y_\circ + \varepsilon \beta, \beta) - E_z(x_\circ, y_\circ - \varepsilon \alpha, \alpha) = \int_{x_\circ - \varepsilon}^{x_\circ + \varepsilon} E_y(s, f_\varepsilon, f'_\varepsilon) \, ds \to 0$$
Hence \( E_z(x_\circ, y_\circ, \beta) = E_z(x_\circ, y_\circ, \alpha) \), for all \( \alpha, \beta \in \mathbb{R} \). In other words, \( E_z \) does not depend on the \( z \)-variable, so \( E \) is affine with respect to \( z \). Let us write it as \( E(x, y, z) = A(x, y) + zB(x, y) \). Proceeding further in this computation we observe that both partials \( A_y(x, y) \) and \( B_x(x, y) \) exist and are continuous on \([a, b] \times \mathbb{R}\), just test (2) with \( f(x) \equiv \text{const.} \). Moreover, identity (2) gives now the relation \( A_y(x, y) = B_x(x, y) \). In the theory of differential equations this relation simply means that the differential form \( A(x, y) \, dx + B(x, y) \, dy \) is exact. Precisely it says that \( A(x, y) \, dx + B(x, y) \, dy = F_x(x, y) \, dx + F_y(x, y) \, dy = dF(x, y) \), where the potential function \( F \in C^1([a, b] \times \mathbb{R}) \) can be explicitly defined by elementary integration. \( F(x, y) = \int_a^x A(s, y) \, ds + \int_0^y B(a, t) \, dt \).

Hence \( E(x, y, z) = F_x(x, y) + z F_y(x, y) \)

completing the proof of the "only if" part.
The Jacobian Determinant

In higher dimensions there are many examples of nonlinear differential expressions whose integral depends only on the boundary datum. The fundamental one, and most familiar, is the Jacobian determinant. Consider a smooth mapping $f = (f^1, f^2, ..., f^n) : \Omega \rightarrow \mathbb{R}^n$ defined on an open region $\Omega \subset \mathbb{R}^n$, briefly $f \in C^\infty(\Omega, \mathbb{R}^n)$. Its differential, sometimes called the gradient matrix, consists of the first order partial derivatives of the coordinate functions
\[ Df(x) = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n} \end{bmatrix} = \left[ \frac{\partial f^i}{\partial x_j} \right] \] (3)

The determinant of this matrix is called the Jacobian of \( f \). We recall several different symbols to denote the Jacobian, most common are:

\[ J(x, f) = \det[Df(x)] = \frac{\partial(f^1, f^2, \ldots, f^n)}{\partial(x_1, x_2, \ldots, x_n)} = J_f(x) = J(f^1, f^2, \ldots, f^n) \]

From a geometric point of view, we see that the integral of the Jacobian of an orientation preserving diffeomorphism \( f : \Omega \to \mathbb{R}^n \) represents the volume
of \( f(\Omega) \).

\[
\int_{\Omega} J(x, f) \, dx = |f(\Omega)|
\]

Therefore, if two diffeomorphisms \( f, g : \Omega \to \mathbb{R}^n \) coincide on \( \partial \Omega \), then \( f(\Omega) = g(\Omega) \). In particular, they have the same volume integrals.

\[
\int_{\Omega} J(x, f) \, dx = \int_{\Omega} J(x, g) \, dx \tag{4}
\]

This identity is further brought to light by the fact that it holds without any topological assumption (once the mappings coincide on the boundary). Also the \( \mathcal{C}^\infty \)-regularity of \( f \) and \( g \) can be relaxed. Observe that the Jacobian determinant is a polynomial of degree \( n \) with respect to the partial derivatives \( \frac{\partial f^i}{\partial x^j} \). Therefore, the Sobolev class \( \mathcal{W}^{1,n}(\Omega, \mathbb{R}^n) \) of weakly differentiable mappings whose differential is integrable with power \( n \) should
be regarded as the natural domain of the definition of the energy functional

$$E[f] = \int_{\Omega} J(x, f) \, dx$$  \hspace{1cm} (5)$$

The integration of the Jacobian in the Sobolev space $W^{1, n}(\Omega, \mathbb{R}^n)$ is pretty much a commencement to the geometric function theory \cite{?} in $\mathbb{R}^n$, and manifolds as well. In topology \cite{?}, on the other hand, we have the notion of the degree, which leads to clear and distinct interpretation of the integral at (5). Nothing more than Stokes’ formula gives the identity at (4) for non-diffeomorphic mappings. Of course, the language of the exterior forms is best suited for such computation. We write the $n$-form $J(x, f) \, dx$ as wedge product of the differentials of the coordinate functions. Then, for
each \( i = 1, 2 \ldots, n \), we express it in the following exact form,

\[
\mathbf{J}(x, f) \, dx = df^1 \wedge df^2 \wedge \ldots \wedge df^n = \\
= (-1)^{i-1} \, d \left( f^i \, df^1 \wedge df^2 \wedge \ldots \wedge df^{i-1} \wedge df^{i+1} \wedge \ldots \wedge df^n \right),
\]

(6)

Next, for two Jacobians, we may decompose their difference in a telescoping fashion

\[
\left[ \mathcal{J}(x, f) - \mathcal{J}(x, g) \right] \, dx = \\
\sum_{i=1}^{n} \, dg^1 \wedge \ldots \wedge dg^{i-1} \wedge (f^i \, df^1 \wedge df^2 \wedge \ldots \wedge df^n)
\]

(7)

By Stokes’ theorem the integral of each term vanishes. Precisely, given
smooth mappings $f, g : \Omega \to \mathbb{R}^n$ such that $f - g \in \mathcal{C}_\infty(\Omega, \mathbb{R}^n)$, we obtain

$$\int_{\Omega} [ J(x, f) - J(x, g) ] \, dx = 0$$

Finally, by an approximation argument this identity extends to mappings in the Sobolev space $\mathcal{W}^{1,n}(\Omega, \mathbb{R}^n)$, provided $f - g \in \mathcal{W}^{1,n}_0(\Omega, \mathbb{R}^n)$ -the completion of $\mathcal{C}_\infty(\Omega, \mathbb{R}^n)$ in $\mathcal{W}^{1,n}(\Omega, \mathbb{R}^n)$.

Let us reflect for a moment on the null Lagrangian identity for the Jacobian determinant.
Fix a mapping \( f = (f^1, f^2, \ldots, f^n) \in C^\infty(\Omega, \mathbb{R}^n) \). For every test mapping \( \eta = (\eta^1, \eta^2, \ldots, \eta^n) \in C^\infty(\Omega, \mathbb{R}^n) \) and numbers \( t_1, t_2, \ldots, t_n \in \mathbb{R} \), we have

\[
\int_\Omega d(f^1 + t_1 \eta^1) \wedge \ldots \wedge d(f^n + t_n \eta^n) = \int_\Omega J(x, f) \, dx
\]

As this integral is a constant function in \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \) its gradient \( \nabla_t = (\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_n}) \) vanishes at \( t_1 = t_2 = \ldots = t_n = 0 \). This gives a weak form of the Lagrange equations. Let us illustrate explicit computation with the derivative in the \( t_1 \)-variable. Since the determinant depends linearly on the entries in the first row, we obtain the identity
\begin{align*}
\int_{\Omega} \left| \begin{array}{cccc}
\frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} & \cdots & \frac{\partial \theta}{\partial x_n} \\
\frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n}
\end{array} \right| \, dx = 0 , \quad \text{where} \quad \theta = \eta^1 \in C_\infty(\Omega) \quad (9)
\end{align*}

One may appeal to the Laplace expansion of determinants to express this identity in an elegant way by using cross product notation of the row vectors \( \nabla f^2, \ldots, \nabla f^n \)

\begin{align*}
\int_{\Omega} \left\langle \nabla \theta \mid \nabla f^2 \times \nabla f^3 \times \cdots \times \nabla f^n \right\rangle \, dx = 0 , \quad \text{for every} \quad \theta \in C_\infty(\Omega) \quad (10)
\end{align*}
Integration by parts yields the desired point-wise null Lagrangian equation, which we phrase as:

**The cross product** \( F = \nabla f^2 \times \nabla f^3 \times \ldots \times \nabla f^n \)

*of gradient fields is divergence free.*

Yet in other notation, for ever mapping \((f^2, f^3, \ldots, f^n) \in C^\infty(\Omega, \mathbb{R}^{n-1})\), we write

\[
\text{div} \left( \nabla f^2 \times \nabla f^3 \times \ldots \times \nabla f^n \right) = \left\langle \nabla \mid F \right\rangle = 0
\] (11)

Here and in the sequel we shall consistently mimic a notation from differential calculus of vector fields. For instance, the dot product notation \(\left\langle \nabla \mid F \right\rangle = \sum_{i=1}^{n} \frac{\partial F^i}{\partial x_i}\) stands for the divergence of a vector field \(F = (F^1, \ldots, F^n)\).
While the above derivation of the null Lagrangian identity may look mysterious, the use of differential forms will give us something much more tangible.

Indeed, equation (11) is none other than the statement that the wedge product

\[ df^2 \wedge df^3 \wedge ... \wedge df^n \]

is a closed form of degree \( n - 1 \).
Some further generalizations are evident. Consider a mapping $f = (f^1, ..., f^n) : \Omega \to \mathbb{R}^m$ of an open region $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Using the standard bases for the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ we identify the linear differential map $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ with an $m \times n$ matrix

$$Df(x) = \begin{bmatrix} \frac{\partial f^i}{\partial x^j} \end{bmatrix} \in \mathbb{R}^{m \times n}, \text{ where } i = 1, ..., m \text{ and } j = 1, ..., n. \quad (12)$$

Observe that the notation $\mathbb{R}^{m \times n}$ is being used for the space of $m \times n$ matrices. To every pair $(I, J)$ of ordered $\ell$-tuples $I : 1 \leq i_1 < i_2 < ... < i_\ell \leq m$ and $J : 1 \leq j_1 < j_2 < ... < j_\ell \leq n$, with $1 \leq \ell \leq \min\{m, n\}$, there corresponds an $\ell \times \ell$ -minor of $Df(x)$, denoted by
\[
\frac{\partial f^I}{\partial x_J} = \frac{\partial (f_{i_1}, \ldots, f_{i_\ell})}{\partial (x_{j_1}, \ldots, x_{j_\ell})}
\]  

(13)

These minors, also called \textit{subdeterminants}, are the coefficients of the wedge product:

\[
df_{i_1} \wedge \ldots \wedge df_{i_\ell} = \sum_{1 \leq j_1 < \ldots < j_\ell \leq n} \frac{\partial (f_{i_1}, \ldots, f_{i_\ell})}{\partial (x_{j_1}, \ldots, x_{j_\ell})} \, dx_{j_1} \wedge \ldots \wedge dx_{j_\ell} = \sum_J \frac{\partial f^I}{\partial x_J} \, dx_J
\]

where the \( \ell \)-tuple \( I : 1 \leq i_1 < \ldots < i_\ell \leq m \) is fixed.

Essentially the same arguments combined with Fubini’s theorem show that for \( f, g \in \mathcal{W}^{1, \ell}(\Omega, \mathbb{R}^m) \) it holds
\[
\int_{\Omega} \frac{\partial f^I}{\partial x_j} \, dx = \int_{\Omega} \frac{\partial g^I}{\partial x_j} \, dx, \quad \text{provided } f - g \in \mathcal{W}^{1,\ell}(\Omega, \mathbb{R}^m) \quad (14)
\]

A little reflection on the above thinking leads us to affine combinations of the subdeterminants as examples of null Lagrangians:

\[
N(x, Df) = \sum_{\ell = 0}^{\min\{m,n\}} \sum_{1 \leq i_1 < \ldots < i_\ell \leq m \atop 1 \leq j_1 < \ldots < j_\ell \leq n} \lambda_{i_1 \ldots i_\ell} \frac{\partial (f^{i_1}, \ldots, f^{i_\ell})}{\partial (x_{j_1}, \ldots, x_{j_\ell})} \quad (15)
\]

where we adhere to the convention that the term with \( \ell = 0 \) is a constant.
Now, given any matrix \( X = [X^i_j] \in \mathbb{R}^{m \times n} \), we denote by \( X \boxplus \) the list of all subdeterminants of \( X \). This includes the number 1 as \( 0 \times 0 \)-minor and the entries of \( X \) as \( 1 \times 1 \)-minors. The highest order subdeterminants considered are the \( \ell \times \ell \)-minors, where \( \ell = \min \{m, n\} \). In what follows \( X \boxplus \) will be viewed as a point in the Euclidean space \( \mathbb{R}^{\binom{m+n}{n}} \). By elementary combinatorial consideration the reader may wish to figure out that the number of all subdeterminants is:

\[
\dim X \boxplus = \sum_{\ell \geq 0} \binom{m}{\ell} \binom{n}{\ell} = \binom{m+n}{n} = \frac{(m+n)!}{m! n!}.
\] (16)

We accordingly associate with every pair of ordered \( \ell \)-tuples \( I : 1 \leq i_1 <, \ldots, < i_\ell \leq m \) and \( J : 1 \leq j_1 <, \ldots, < j_\ell \leq n \), where
$0 \leq \ell \leq \min\{m, n\}$, the $(I, J)$-minor function

$M_{IJ}^I : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$, defined as $M_{IJ}^I(X) = \det X_{IJ}^I$

with the convention that $M_{IJ}^I \equiv 1$, if $\ell = 0$. The following theorem provides us with a complete algebraic description of the first order null Lagrangians, of which we shall make repeated use throughout these lectures.

**THEOREM Null Lagrangians** $N : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ form a linear space of dimension $(m+n)$. Its basis is furnished by the minor functions $M_{IJ}^I$. 31
This result goes back to Landers [], Ericksen [], Edelen [] and Rund [].

Should it be desired to construct more general null Lagrangians of the type 
\[ E(x, f, Df) \], one might consider the subdeterminants of the differential of a mapping \( F = F(x, f) \). Precise characterization of all first order null Lagrangians of the form \( E(x, f, Df) \) can be found in de Franchis [], [],[]. Needless to say, higher order null Lagrangians are much harder to characterize algebraically. Those of the form \( E(\nabla^k f) \), depending only on derivatives of order precisely equal to \( k \), were successfully treated in [Ball, Currie, Olver].
Wedge products

Besides the subdeterminants of a differential matrix, the nonlinear differential expressions such as wedge products of exact differential forms will also motivate our approach to general null Lagrangians.

Briefly, let $\Lambda^\ell = \wedge^\ell \mathbb{R}^n$ denote the space of all $\ell$-covectors in $\mathbb{R}^n$, $\ell = 1, ..., n$. View them as $\ell$-linear alternating forms on the vector space $\mathbb{R}^n$; thus $\wedge^1 \mathbb{R}^n$ is the dual space to $\mathbb{R}^n$. By convention, $\Lambda^0 = \wedge^0 \mathbb{R}^n = \mathbb{R}$ - the space of scalars. A differential $\ell$-form on $\Omega \subset \mathbb{R}^n$ is simply a function (later it will be a Schwartz distribution) with values in $\Lambda^\ell = \wedge^\ell \mathbb{R}^n$. A detailed review of differential forms is given in Chapter ?. For now, we will have to presuppose some familiarity with the language of differential forms.

As a substitute of differential matrices let us take on stage the $m$-tuples of exact differential forms $d\Phi = (d\varphi^1, ..., d\varphi^m)$ on $\Omega \subset \mathbb{R}^n$. Here each $d\varphi^i$
has degree \( \ell_i \geq 1 \) and \( \ell_1 + \ldots + \ell_m = \ell \leq n \). Since the components \( \varphi^i \) are considered as functions on \( \Omega \) with values in the space \( \wedge^{\ell_i-1} = \wedge^{\ell_i-1} \mathbb{R}^n \), it will be natural to denote: \( \varphi^i \in \mathcal{C}^\infty(\Omega, \wedge^{\ell_i-1}) \), \( \Phi \in \mathcal{C}^\infty(\Omega, \wedge^{\ell_1-1} \times \ldots \times \wedge^{\ell_m-1}) \) and \( d\Phi \in \mathcal{C}^\infty(\Omega, \wedge^{\ell_1} \times \ldots \times \wedge^{\ell_m}) \).

Fix an arbitrary closed form \( \Upsilon \in \mathcal{C}^\infty(\Omega, \wedge^{n-\ell}) \), so that \( d\varphi^1 \wedge \ldots \wedge d\varphi^m \wedge \Upsilon = d(\varphi^1 \wedge d\varphi^2 \wedge \ldots \wedge d\varphi^m \wedge \Upsilon) \) becomes an exact \( n \)-form on \( \Omega \). We then define the null Lagrangian \( \mathbf{N} = \mathbf{N}_{\Upsilon} : \Omega \times \wedge^{\ell_1} \times \ldots \times \wedge^{\ell_m} \to \mathbb{R} \) by the rule

\[
\mathbf{N}(x, d\Phi) \ dx = d\varphi^1 \wedge \ldots \wedge d\varphi^m \wedge \Upsilon(x)
\]

The following notation emphasizes a resemblance between these wedge products and the Jacobian determinants

\[
\mathcal{J}(x, \Phi) \ dx = d\varphi^1 \wedge d\varphi^2 \wedge \ldots \wedge d\varphi^m , \quad \text{where} \quad \ell_1 + \ell_2 + \cdots + \ell_m = \dim \Omega
\]
for $\Phi = (\phi^1, \ldots, \phi^m)$. While it is beyond the scope of this survey to provide details, we can at least state the null Lagrangian identity

$$\int_{\Omega} N(x, d\Phi) \, dx = \int_{\Omega} N(x, d\Psi) \, dx$$  \hspace{1cm} (17)$$

provided $\Phi$ and $\Psi$ share the same tangential component on $\partial\Omega$. In particular, when $\Phi - \Psi \in C^\infty(\Omega, \wedge^{\ell_1-1} \times \ldots \times \wedge^{\ell_m-1})$. 

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Coffee is proof that God loves mathematicians and wants us to be happy.

*Benjamin Franklin* (paraphrase)
The div-curl product

The wedge product \( du \wedge dv \) of exact differential forms \( du \in C^\infty(\Omega, \wedge^1) \) and \( dv \in C^\infty(\Omega, \wedge^{n-1}) \) takes special place in the theory of null Lagrangians. It is often preferable to work with vector fields instead of 1-forms and \((n-1)\)-forms. First, the exterior derivative of \( u \):

\[
du = u_{x_1} dx_1 + u_{x_2} dx_2 + \cdots + u_{x_n} dx_n
\]

\[
= E_1 dx_1 + E_2 dx_2 + \cdots + E_n dx_n
\]

may be viewed as a gradient field \( E = (E_1, E_2, ..., E_n) = \nabla u \in C^\infty(\Omega, \mathbb{R}^n) \).

Also, the \((n-1)\) -form \( dv \) can be written as:

\[
B^1 dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n - B^2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n - (-1)^n B^n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}
\]
and then identified with the vector field \( \mathbf{B} = (B^1, B^2, ..., B^n) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n) \).

Now the wedge product translates into the dot product of \( \mathbf{B} \) and \( \mathbf{E} \),

\[
du \wedge dv = (B^1 E_1 + B^2 E_2 + ... + B^n E_n) \ dx_1 \wedge dx_2 \wedge ... \wedge dx_n = \langle \mathbf{B} | \mathbf{E} \rangle \ dx
\]

Since the exterior derivative applied twice gives zero, \( du \) and \( dv \) are closed differential forms. Let us rephrase this fact in words of the field theory. Accordingly,

\[
\text{curl } \mathbf{E} = \left[ \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right]_{i,j=1,...,n} = \mathbf{E} \otimes \nabla - \nabla \otimes \mathbf{E} = 0 \quad (18)
\]
and

\[
\text{div } \mathbf{B} = \frac{\partial B^1}{\partial x_1} + \frac{\partial B^2}{\partial x_2} + \cdots + \frac{\partial B^n}{\partial x_n} = \left\langle \nabla \mid \mathbf{B} \right\rangle = 0 \quad (19)
\]

We say that \( \mathbf{B} \) is divergence-free and \( \mathbf{E} \) is curl-free. The curl and div operators acting on vector fields \( \mathbf{E}, \mathbf{B} \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n) \) owe much of their importance to Maxwell’s equations. The commonly used letters, \( \mathbf{E} \) for the electric field and \( \mathbf{B} \) for the magnetic field, will be adopted to reflect on these connections. We call the pair \( (\mathbf{B}, \mathbf{E}) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n \times \mathbb{R}^n) \) a div-curl couple. Of course, the true significance of the div-curl couples goes beyond this mere connection with electromagnetism. For instance, the \textit{compensated compactness} theory has originated and developed primary from the Div-Curl Lemma by F. Murat and L. Tartar in 1972-1980.
An alternative view on the Jacobian determinant now emerges by looking again at the Laplace expansion:

\[
\mathbf{J}(x, f) = \begin{vmatrix}
\frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\
\frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n}
\end{vmatrix} = \langle \mathbf{B} | \mathbf{E} \rangle \quad (20)
\]

where \( \mathbf{B} = \nabla f^1 \) is curl-free while the cross product of the remaining gradient fields \( \mathbf{B} = \nabla f^2 \times \nabla f^3 \times \ldots \times \nabla f^n \) is divergence-free. This view puts the Jacobian determinants in larger perspectives, sometimes very helpful.
Elliptic PDEs

In the theory of partial differential equations (PDEs) the div-curl products and other null Lagrangians occur naturally. Perhaps most elegant and clear example is provided by the nonlinear second order elliptic equation:

\[ \text{div} \ A(x, \nabla u) = 0, \quad \text{where} \quad A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n \]  

(21)

In the linear case \( A(x, \xi) = A(x) \xi \), where \( A = A(x) \) is a symmetric positive definite measurable matrix function defined on \( \Omega \). Every solution generates a div-curl couple \( (B, E) \); with \( E = \nabla u(x) \) as curl-free and \( B = A(x, \nabla u) \) as divergence-free. The natural domain of the definition of the equation (21) consists of weakly differentiable functions having finite
energy,

$$\mathcal{E}[u] = \int_\Omega \langle \nabla u \mid A(x, \nabla u) \rangle \, dx = \int_\Omega \langle E \mid B \rangle = \int_\Omega du \wedge dv$$

The point to make here is that, under sufficient regularity, we have:

**The energy of any solution is determined by its boundary values.**

This principle has been evident to researchers from the very beginning, and used effectively in establishing a priori estimates. That the div-curl product $\langle B\mid E \rangle$ assumes only nonnegative values on the solution $u$ is critical for establishing its regularity properties (higher integrability of the gradient).
Another point to make here is that the equation (21) reduces to a first order differential system for the vector fields $B$ and $E$.

$$B = \mathcal{A}(x, E), \quad \text{where} \quad \text{curl } E = 0, \quad \text{div } B = 0 \quad (22)$$

To illustrate, the Laplace equation yields $B = E = \nabla u$, and we are reduced to the familiar Cauchy-Riemann system for the gradient of a harmonic function

$$\text{div } E = 0, \quad \text{curl } E = 0$$

What really matters is not the precise relation $B = \mathcal{A}(x, E)$ between the vector fields $B$ and $E$ but the ellipticity condition that couples these fields in the so-called distortion inequality. For the linear equation
\text{div } A(x) \nabla u = 0 \text{ the distortion inequality reads as:}

\|B\|^2 + \|E\|^2 \leq 2K \langle B \mid E \rangle \quad \text{with } 1 \leq K = K(x) < \infty \quad (23)

The recent remarkable qualitative analysis of elliptic PDEs [CLMS] [IS], especially in the degenerate case when \( K = K(x) \) is unbounded, rely fundamentally on integral estimates of the div-curl products.
Geometric Function Theory

One might observe similar important role of null Lagrangians in the geometric function theory (GFT), largely concerned with \textit{quasiregular mappings} in $\mathbb{R}^n$.

\textbf{DEFINITION}

A mapping $f : \Omega \rightarrow \mathbb{R}^n$ of Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$ is said to be \textit{K-quasiregular}, $1 \leq K < \infty$, if it satisfies the distortion inequality

$$|Df(x)|^n \leq K J(x, f) \quad \text{for almost every } x \in \Omega \quad (24)$$

The operator norm of the differential matrix $Df(x) \in \mathbb{R}^{n \times n}$ is being used here, $|Df(x)\xi| = \max\{|Df(x)\xi|; |\xi| = 1\}$. 
Geometrically, the distortion inequality says that the linear differential map $Df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ deforms spheres into ellipsoids of uniformly controlled eccentricity. There is no distortion when $K = 1$, which results in conformal deformations. We indulge ourselves by putting Jacobian subdeterminants of quasiregular mapping into play here.

First we take into consideration a Sobolev mapping $f \in \mathcal{W}^{1,\ell}(\Omega, \mathbb{R}^n)$ in even dimension $n = 2\ell$. If $f$ is conformal then at almost every point $x \in \Omega$ its differential $Df(x)$ is a scalar multiple of an orthogonal transformation. By elementary algebra we split the differential matrix into four submatrices and infer the following relations:

$$Df(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix},$$

\[
\begin{cases} 
\det A = \det D \\
\det B = (-1)^{\ell} \det C 
\end{cases}
\]
where $A, B, C$ and $D$ are the $\ell \times \ell$-submatrices. Notice the resemblance to the Cauchy-Riemann system in $\mathbb{R}^2$, in which case

$$Df(x) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

When the rows and columns are permuted in $Df(x)$, more linear relations between the $\ell \times \ell$-minors are obtained. If there is some distortion ($K > 1$) these linear relations have variable bounded coefficients. Such equations, when lifted to the exterior $\ell$-forms, lead to many interesting first order systems of PDEs, linear and nonlinear. They are far more geometric than their second order cousins. But it would take us a bit away from these lectures to expand into such discussion.
Higher order null Lagrangians

Now that we know what null Lagrangians of the first order look like, the next step is to examine the possible null Lagrangians with higher order derivatives involved. One general method of making new null Lagrangians from the known ones is to replace the functions with derivatives of new functions. It seems at first that this method leads to only limited class of higher order null Lagrangians. The inescapable reality is that these are basically the only null Lagrangians of higher order [Ball Curie and Olver].

We take into consideration an arbitrary linear differential operator of order $k = 1, 2, ...$,

$$\mathcal{D} : C^\infty(\Omega, \mathbb{R}^m) \longrightarrow C^\infty(\Omega, \mathbb{R}^n), \quad \Omega \subset \mathbb{R}^n$$
It assigns to each $u \in \mathcal{C}^{\infty}(\Omega, \mathbb{R}^m)$ a vector field $f = \mathfrak{D}u : \Omega \rightarrow \mathbb{R}^n$, by the rule

$$\mathfrak{D}u = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha}, \quad \text{where} \quad \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{(25)}$$

The coefficients $A_\alpha = A_\alpha(x)$ are $n \times m$-matrix functions of class $\mathcal{C}^{\infty}(\Omega, \mathbb{R}^{n \times m})$. On substituting $f = \mathfrak{D}u$ into the Jacobian determinant $J(x, f)$ we obtain a null Lagrangian of order $k + 1$, in symbols:

$$N(x, \mathfrak{D}u) = \frac{\partial (\ldots \mathfrak{D}u \ldots)}{\partial (x_1, \ldots, x_n)}, \quad \text{for} \; u \in \mathcal{C}^{\infty}(\Omega, \mathbb{R}^m)$$

Precisely, $N(x, \mathfrak{D}u)$ is a polynomial of degree $n$ with respect to the partial
derivatives $\partial^{|\beta|} u / \partial x^\beta$, with $|\beta| \leq k + 1$. We have the desired identity

$$\int_{\Omega} \left[ N(x, \nabla u) - N(x, \nabla v) \right] \, dx = 0,$$

whenever

$$u, \ v \in C_\infty(\Omega, \mathbb{R}^m)$$

and

$$u - v \in C_\infty^\circ(\Omega, \mathbb{R}^m).$$
**Hessians and Gaussian curvature**

Let us say a few words about the case when $\mathcal{D}$ is the gradient operator, $\mathcal{D} = \nabla : C^\infty(\Omega) \longrightarrow C^\infty(\Omega, \mathbb{R}^n)$. It gives us the second order Hessian operator

$$\nabla \otimes \nabla = \left[ \frac{\partial^2}{\partial x_i \partial x_j} \right] : C^\infty(\Omega) \longrightarrow C^\infty(\Omega, \mathbb{R}^{n \times n})$$

**Hessians**

With every $u \in C^\infty(\Omega)$ there is associated a mapping $\nabla u : \Omega \longrightarrow \mathbb{R}^n$. Its Jacobian matrix is called Hessian matrix of $u$. The determinant of the Hessian matrix is a null Lagrangian of order 2, which we shall denote by
Using wedge product notation we write it as

\[
\mathbf{H}(x, u) \, dx = du_{x_1} \wedge du_{x_2} \wedge ... \wedge du_{x_n}
\]

Hessian determinant arises in a large variety of geometric problems (Gaussian curvature) and applied PDEs. For example, the *Monge-Amperè* equation

\[
\mathbf{H}(x, u) = J(x, \nabla u) = \begin{vmatrix}
\frac{\partial^2 u}{\partial x_1 \partial x_1} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\
\frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 u}{\partial x_n \partial x_1} & \frac{\partial^2 u}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_n \partial x_n}
\end{vmatrix}
\]
\[
\det \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} \right] = h(x, u, \nabla u)
\]

arises in optics \[\] and transport optimization problems [Caffarelli]. Yet in a domain \( \Omega \) of the complex space \( \mathbb{C}^n = \{ z = (z_1, z_2, \ldots, z_n); \quad z_k = x_k + iy_k, \quad k = 1, \ldots, n \} \) we have an exact 2-form associated with a real valued function \( u \in C^\infty(\Omega) \)

\[
d\varphi = d d^c u = 2i \sum_{j, k = 1}^{n} \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} d z_j \wedge d \bar{z}_k = 2i \sum_{j, k = 1}^{n} u_{z_j \bar{z}_k}(z) d z_j \wedge d \bar{z}_k
\]
where the 1-form $\varphi$ is given by

$$\varphi = d^c u = i \sum_{\alpha=1}^{n} \left( \frac{\partial u}{\partial \overline{z}_{\alpha}} d\overline{z}_{\alpha} - \frac{\partial u}{\partial z_{\alpha}} dz_{\alpha} \right) = \sum_{\alpha=1}^{n} \left( \frac{\partial u}{\partial x_{\alpha}} dy_{\alpha} - \frac{\partial u}{\partial y_{\alpha}} dx_{\alpha} \right)$$

Recall, for comparison, that the exterior derivative applied to $u$ would give us the 1-form:

$$du = \sum_{\beta=1}^{n} \left( \frac{\partial u}{\partial z_{\beta}} dz_{\beta} + \frac{\partial u}{\partial \overline{z}_{\beta}} d\overline{z}_{\beta} \right) = \sum_{\beta=1}^{n} \left( \frac{\partial u}{\partial x_{\beta}} dx_{\beta} + \frac{\partial u}{\partial y_{\beta}} dy_{\beta} \right)$$
The complex Hessian determinant of a real function $u \in C^\infty(\Omega)$ is again a real function on $\Omega$ given by:

$$H_C(z, u) = \begin{vmatrix} \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_n} \\ \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial z_n \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_n} \end{vmatrix}$$

(27)

Notice that the real Hessian matrix has size $2n \times 2n$, so its determinant is a
polynomial of degree $2n$ with respect to the second order partial derivatives

\[ \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \frac{\partial^2 u}{\partial x_\alpha \partial y_\beta}, \frac{\partial^2 u}{\partial y_\alpha \partial x_\beta} \] \text{ and } \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \\

On the other hand, the complex Hessian determinant is a real polynomial of degree $n$ with respect to the second order derivatives, which is immediate from the following identity

\[ 4^n n! \, H_C(z, u) \, dx_1 \wedge dy_1 \ldots \wedge dx_n \wedge dy_n = \underbrace{d\phi \wedge d\phi \wedge \ldots \wedge d\phi}_{n \text{ times}} \]

We are again in the situation of the wedge product of exact differential
forms. Thus the complex Hessian determinant

\[ \det \left[ \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \right] : \mathcal{W}^2, n(\Omega) \rightarrow \mathcal{L}^1_{\text{loc}}(\Omega) \]

is a null Lagrangian of order two. It is also curious to learn that \( H_C(z,u) \) can be expressed as a sum of certain \( n \times n \) minors of the real Hessian matrix.

The equation \( H_C(z,u) = 0 \), developed primarily by E. Bedford and B.A. Taylor, plays fundamental role in complex analysis as it describes the maximal plurisubharmonic functions. A smooth real valued function \( u \) is said to be plurisubharmonic if \( dd^c u \geq 0 \), meaning that the following
Hermitian form is positive semi-definite:

\[ \sum_{\alpha, \beta = 1}^{n} \left( \frac{\partial^2 u(z)}{\partial z_\alpha \partial \overline{z}_\beta} \right) \xi_\alpha \overline{\xi}_\beta \geq 0 , \quad \text{for every } \xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n \]

As a matter of fact every $C^\infty$-smooth closed form $\sum \Gamma_{\alpha \beta}(z) \, dz_\alpha \wedge d\overline{z}_\beta$ which is positive semi-definite in a domain like ball in $\mathbb{C}^n$ always comes that way [Lelong, Gruman].
In certain natural situations the second order null Lagrangians also contain the first order derivatives of the variable function $u \in C^\infty(\Omega)$. This is most often the case in differential geometry. The curvature of a surface is the first example we encounter. Recall that the **Gaussian curvature** of a surface $z = u(x, y)$ in $\mathbb{R}^3$ is equal to

$$K = \frac{u_{xx} u_{yy} - u_{xy} u_{xy}}{(1 + u_x^2 + u_y^2)^2} = \frac{H(z,u)}{(1 + |\nabla u|^2)^2}, \quad z = x + iy$$

(28)

In terms of the gradient map $f = (u_x, u_y)$, this formula takes the form
\[
K = \frac{\det Df}{(1 + |f|^2)^2} = J(x, \hat{f}), \quad \text{where} \quad \hat{f} = \frac{f}{\sqrt{1 + |f|^2}}
\]

gaussian curvature provides us with a good example of the geometric significance of the sign of a null Lagrangian. If \( K > 0 \) at \((x_0, y_0) \in \Omega\) then the surface lies on one side of the tangent plane near the given point \((x_0, y_0, z_0) \in \mathbb{R}^3\). One says that the surface is strictly convex if there is a positive constant \( c \) such that \( K(x, y) \geq c \), everywhere. If \( K(x_0, y_0) < 0 \) the surface has a saddle at the point \((x_0, y_0, z_0)\). For instance, this actually happens at every point of the graph of a harmonic function.

The mean curvature of the surface \( z = u(x, y) \) is also a null Lagrangian
\[ H = \text{div} \left( \frac{(u_x, u_y)}{\left(1 + u_{x}^2 + u_{y}^2\right)^{\frac{1}{2}}} \right) = \text{div} \left( \frac{f}{\sqrt{1 + |f|^2}} \right) = \text{div} \hat{f} \]

Different metrics can be imposed on the same vector space, leading to new curvature formulas. The curvature for a semi-Riemannian surface is a null Lagrangian as well. Recall that the Minkowski 3-space is the linear space with coordinates \((x, y, z)\) in terms of which the standard basis of tangent vectors \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\) gives rise to an indefinite scalar product (non-degenerate symmetric bilinear form):

\[
\left\langle a \mid b \right\rangle = a_3 \cdot b_3 - a_2 \cdot b_2 - a_1 \cdot b_1,
\text{ where } \ \left\{ \begin{array}{l}
a = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z} \\
b = b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + b_3 \frac{\partial}{\partial z}
\end{array} \right.
\]

A surface is space-like if every nonzero tangent vector \(v\) satisfies \(\left\langle v \mid v \right\rangle > 0\).
0. If a surface is given in the form \( z = u(x, y) \), then \( u_x^2 + u_y^2 < 1 \). The Gaussian curvature of a semi-Riemannian space-like surface is defined by

\[
K = \frac{u_{xy} u_{xy} - u_{xx} u_{yy}}{(1 - u_x^2 - u_y^2)^2} = \frac{-\mathbf{H}(z, u)}{(1 - |\nabla u|^2)^2}
\]  

(29)

For example, the hyperboloid \( z^2 = 1 + x^2 + y^2 \) has constant negative curvature \( K \equiv -1 \). As before, an elementary computation reveals that

\[
-K = \frac{\det Df}{(1 - |f|^2)^2} = \mathbf{J}(z, \tilde{f}) , \quad \text{where} \quad \tilde{f} = \frac{f}{\sqrt{1 - |f|^2}}
\]

Thus \( K \) is a null Lagrangian of the form \( \mathcal{N}(\nabla u, \nabla^2 u) \).
We end this list of classical examples with a very notable although analytically evident fact that the null Lagrangians are of local type, in the sense that when restricted to any subregion of $\Omega$ they continue to be null Lagrangians therein. This local property is very characteristic to differential operators. That is why we shall also view null Lagrangians as nonlinear differential operators, like the Jacobian and Hessian determinants:

$$J : W^{1,n}(\Omega, \mathbb{R}^n) \rightarrow L^1(\Omega) \quad \text{and} \quad H : W^{2,n}(\Omega) \rightarrow L^1(\Omega)$$

These natural domains of definition of $J$ and $H$ will soon be extended and investigated more effectively in various spaces of weakly differentiable functions.
The general concept of null Lagrangians makes use of linear differential operators as substitutes for the gradient $\nabla$. Of course, a proper setting requires that the differential operators act on sections of vector bundles over a given manifold. But such an attempt would obscure some interesting details in specific cases. Nevertheless, the definitions we offer here will prepare us for a study of null Lagrangians on manifolds.

Let $V$ and $W$ be finite dimensional real vector spaces furnished with the inner products $\langle \cdot | \cdot \rangle_V$ and $\langle \cdot | \cdot \rangle_W$. We shall often suppress the subscripts $V$ and $W$ in the notation of the inner products if no confusion can arise.
**Linear differential operators**

Given a nonempty open set \( \Omega \subset \mathbb{R}^n \), we consider two spaces \( \mathcal{C}^\infty(\Omega, V) \) and \( \mathcal{C}^\infty(\Omega, W) \) that consist of smooth functions defined on \( \Omega \) and valued in \( V \) and \( W \), respectively. These spaces will be confirmed or replaced by suitable Sobolev classes as the discussion goes on. A linear partial differential operator of order \( k = 0, 1, 2, \ldots \), denoted by

\[
\mathcal{D} : \mathcal{C}^\infty(\Omega, V) \longrightarrow \mathcal{C}^\infty(\Omega, W)
\]

assigns to every \( v \in \mathcal{C}^\infty(\Omega, V) \) a function \( w \in \mathcal{C}^\infty(\Omega, W) \) by the rule

\[
w = \mathcal{D}v = \sum_{|\alpha| \leq k} A_\alpha(x) \frac{\partial^{|\alpha|} v}{\partial x^\alpha}, \text{ where } A_\alpha(x) \in \text{Hom} (V, W)
\]
The coefficients \( A_\alpha : \Omega \to \text{Hom}(\mathbf{V}, \mathbf{W}) \) are smooth functions valued in the space of linear transformations from \( \mathbf{V} \) into \( \mathbf{W} \). Most differential operators that we shall meet in this text have constant coefficients. The \textit{Lagrange adjoint} of \( \mathcal{D} \) is a differential operator

\[
\mathcal{D}^* : C^\infty(\Omega, \mathbf{W}) \longrightarrow C^\infty(\Omega, \mathbf{V})
\]

\[
\mathcal{D}^* w = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \frac{\partial^{|\alpha|} [ A^T_\alpha(x) w ]}{\partial x^\alpha}, \quad A^T_\alpha(x) \in \text{Hom}(\mathbf{W}, \mathbf{V})
\]

(30)

Here the transpose coefficients \( A^T_\alpha(x) : \mathbf{W} \to \mathbf{V} \) are determined through the scalar products in \( \mathbf{V} \) and \( \mathbf{W} \). One may use Leibnitz’ formula to write \( \mathcal{D}^* w \) explicitly as a linear combination of the partial derivatives \( \frac{\partial^{|\alpha|} w}{\partial x^\alpha} \).
However, this would give us nothing deeper than involvement of the derivatives of the coefficients $A_\alpha$. Obviously, one could not speak of the Lagrange adjoint if the coefficients $A_\alpha$ were not $k$ times differentiable. Nevertheless, applying Lagrange adjoint twice brings us back to the original operator, in symbols $(\mathcal{D}^*)^* = \mathcal{D}$. Derivatives of $A_\alpha(x)$ cancel out. The easiest way to see this phenomenon is through *integration by parts*,

\[ \int_{\Omega} \left\langle \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Phi \mid \Psi \right\rangle \, dx = (-1)^{|\alpha|} \int_{\Omega} \left\langle \Phi \mid \frac{\partial^{|\alpha|}}{\partial x^\alpha} \Psi \right\rangle \, dx \quad (31) \]

which holds if one of the functions $\Phi, \Psi \in C^{\infty}(\Omega, V)$ has compact support.
Optional

For a given Hölder conjugate pair of exponents $1 \leq p, q \leq \infty$, $p+q = p\cdot q$, we define the scalar product of $\Phi \in L^p(\Omega, V)$ and $\Psi \in L^q(\Omega, V)$,

$$\langle \langle \Phi \mid \Psi \rangle \rangle_V = \langle \langle \Phi \mid \Psi \rangle \rangle_V = \int_\Omega \langle \Phi(x) \mid \Psi(x) \rangle_V \, dx \quad (32)$$

by analogy to the notation $\| \|_{L^p}$ for the $L^p$-norm. In PDEs the name formal adjoint is being used for $\mathcal{D}^*$, which is natural since $\mathcal{D}^*$ is completely characterized by the identity

$$\langle \langle \mathcal{D}^* w \mid v \rangle \rangle_V = \langle \langle w \mid \mathcal{D} v \rangle \rangle_W, \quad \text{for} \quad \begin{cases} w \in C_\infty^\infty(\Omega, W) \\ v \in C_\infty^\infty(\Omega, V) \end{cases} \quad (33)$$

Integration by parts just tells us that the formal adjoint of the partial
differentiation \( \partial^{\nu}\alpha / \partial x^{\alpha} : \mathcal{C}^{\infty}(\Omega, \mathbf{V}) \to \mathcal{C}^{\infty}(\Omega, \mathbf{V}) \) is the operator \((-1)^{\nu}\alpha \partial^{\nu}\alpha / \partial x^{\alpha}\).

Our detailed account concerns the case in which the underlying operator \( \mathcal{D} \) is the first order gradient

\[\nabla : \mathcal{C}^{\infty}(\Omega, \mathbf{V}) \rightarrow \mathcal{C}^{\infty}(\Omega, \mathbf{V} \oplus \mathbf{V} \oplus \cdots \oplus \mathbf{V}), \quad \nabla v = \left( \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n} \right)\]

In particular, taking \( \mathbf{V} = \mathbb{R}^m \) and identifying the direct sum \( \mathbb{R}^m \oplus \mathbb{R}^m \oplus \cdots \oplus \mathbb{R}^m \) with the space of \( m \times n \) -matrices, we recover the differential operator \( D : \mathcal{C}^{\infty}(\Omega, \mathbb{R}^m) \to \mathcal{C}^{\infty}(\Omega, \mathbb{R}^{m \times n}) \). In this notation, the gradient \( \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) \) consists of none other than the column vectors of the Jacobian matrix. Similarly, the \( k \)-th order gradient makes a list of all \( k \)-th partial derivatives of a function \( v \in \mathcal{C}^{\infty}(\Omega, \mathbf{V}) \).
\[ \nabla^k : \mathcal{C}^\infty(\Omega, \mathbf{V}) \rightarrow \mathcal{C}^\infty(\Omega, \bigoplus_{|\alpha|=k} \mathbf{V}) \], \quad \nabla^k \mathbf{v} = \left\{ \frac{\partial|\alpha|\mathbf{v}}{\partial x^\alpha} \right\}_{|\alpha|=k}

\[ |\nabla^k \mathbf{v}|^2 = \sum_{|\alpha|=k} \left\langle \frac{\partial|\alpha|\mathbf{v}}{\partial x^\alpha} \left| \frac{\partial|\alpha|\mathbf{v}}{\partial x^\alpha} \right\rangle \right. \] (34)
More General Null Lagrangians

Our most general formulation of null Lagrangians demands a few precautionary assumptions concerning integrands. We say that $\mathcal{N} : \Omega \times \mathcal{W} \to \mathbb{R}$ satisfies the Carathéodory regularity hypotheses if:

i) The function $x \mapsto \mathcal{N}(x, \xi)$ is measurable in $x \in \Omega$, for every $\xi \in \mathcal{W}$.

ii) The function $\xi \mapsto \mathcal{N}(x, \xi)$ is continuous on $\mathcal{W}$, for almost every $x \in \Omega$.

iii) The function $(x, \xi) \mapsto \mathcal{N}(x, \xi)$ is bounded on compact subsets of $\Omega \times \mathcal{W}$.
We point out that the hypotheses \( i \) and \( ii \) are imposed in order to ensure measurability of the composite function \( x \mapsto N(x, w(x)) \), whenever \( w : \Omega \mapsto W \) is measurable. This fact is well known as Scorza-Dragoni Theorem \([\cdot]\). We now introduce a nonlinear differential operator \( N(x, D) \) and refer to \( N(x, \xi) \) as its symbol. In what follows the conditions \( i \), \( ii \), \( iii \) will be in force without explicit mention. As the text progresses we shall impose further conditions on the symbol \( N(x, \xi) \) as to be able to place \( N(x, D) \) in suitable Sobolev classes.

\[ \text{With all these preliminaries, we now come to the following general definition.} \]
Definition I

The nonlinear differential operator \( N(x, \mathcal{D}) : \mathcal{C}^\infty(\Omega, \mathcal{V}) \rightarrow \mathcal{L}^1_{\text{loc}}(\Omega) \) is called null Lagrangian if:

\[
\int_{\Omega} \left[ N(x, \mathcal{D}u) - N(x, \mathcal{D}v) \right] \, dx = 0 , \quad \text{whenever} \quad u, v \in \mathcal{C}^\infty(\Omega, \mathcal{V}) \quad u-v \in \mathcal{C}^\infty_0(\Omega, \mathcal{V})
\]

The observant reader may have noticed that we need only integrate over the support of \( u - v \); thus the integral converges thanks to the condition \( \text{iii} \). For the first order null Lagrangians, we can take \( u \) and \( v \) from \( \mathcal{W}^{1,\infty}_{\text{loc}}(\Omega, \mathcal{V}) \), provided \( u-v \) vanishes outside a compact subset of \( \Omega \).
OPTIONAL

One may allow $N(x, \xi)$ to take values in a finite dimensional vector space $\mathcal{R}$ rather than real numbers, $N : \Omega \times \mathcal{W} \to \mathcal{R}$. Definition I extends in the same guise to nonlinear differential operators

$$N(x, \mathcal{D}) : \mathcal{C}^\infty(\Omega, \mathcal{V}) \to L^1_{\text{loc}}(\Omega, \mathcal{R})$$

We can already bring on stage the null Lagrangians which are valued in the space $\wedge^\ell \mathbb{R}^n$ of $\ell$-covectors. Consider the $m$-tuples $\Phi = (\phi^1, \phi^2, ..., \phi^m) \in \mathcal{C}^\infty(\Omega, \mathcal{V})$, where $\mathcal{V} = \wedge^{\ell_1-1} \times \wedge^{\ell_2-1} \times \cdots \times \wedge^{\ell_m-1}$ and $\ell = \ell_1 + \ell_2 + \cdots + \ell_m \leq n$. Let the underlying linear differential operator $\mathcal{D} : \mathcal{C}^\infty(\Omega, \mathcal{V}) \to \mathcal{C}^\infty(\Omega, \mathcal{W})$ be the exterior derivative $d\Phi = (d\phi^1, d\phi^2, ..., d\phi^m) \in \mathcal{C}^\infty(\Omega, \mathcal{W})$, where $\mathcal{W} = \wedge^{\ell_1} \wedge^{\ell_2} \wedge \cdots \wedge^{\ell_m}$. 74
The wedge product \(d\phi^1 \wedge d\phi^2 \wedge ... \wedge d\phi^m\) assumes values in \(\wedge^\ell \mathbb{R}^n = \mathcal{R}\). In this way we arrive at the null Lagrangian whose symbol is the wedge product

\[
N(x, \xi) = \xi^1 \wedge \xi^2 \wedge ... \wedge \xi^m \in \wedge^\ell \mathbb{R}^n , \quad \text{for } \xi = (\xi^1, \xi^2, ..., \xi^m) \in \mathcal{W}.
\]

The wedge null Lagrangian will be denoted by

\[
\Lambda(x, d) : C^\infty(\Omega, V) \rightarrow L^1_{\text{loc}}(\Omega, \wedge^\ell \mathbb{R}^n) , \quad V = \wedge^{\ell_1 - 1} \times \wedge^{\ell_2 - 1} \times \cdots \times \wedge^{\ell_m - 1}
\]

\[
\ell = \ell_1 + \ell_2 + ... + \ell_m \leq n
\]

\[
\Lambda(x, d\Phi) = d\phi^1 \wedge d\phi^2 \wedge ... \wedge d\phi^m , \quad \text{for } \Phi = (\phi^1, \phi^2, ..., \phi^m) \in C^\infty(\Omega, V) \quad (35)
\]
The Lagrange identity

Variational equations are used for the study of the critical points of the integral functionals such as this:

$$E[u] = \int_\Omega N(x, \mathcal{D}u) \, dx$$

But this requires differentiability of the integrand $N(x, \xi)$ with respect to the variable $\xi \in W$. For this purpose we assume that

\begin{itemize}
  \item[ii)\] The function $\xi \mapsto N(x, \xi)$ is continuously differentiable for a.e. $x \in \Omega$.
\end{itemize}

The inner product in $W$ allows us to think of the gradient $\nabla_\xi N(x, \xi)$ as a vector in $W$ rather than an element of the dual space. For notational
simplicity we denote $\nabla_\xi N(x, \xi) = N'(x, \xi)$. Recall from elementary calculus that

$$\langle N'(x, \xi) | h \rangle_W = \frac{d N(x, \xi + th)}{dt} \bigg|_{t=0}$$

for almost every $x \in \Omega$ and all $\xi, h \in W$. This vector is determined from the Taylor expansion formula

$$N(x, \xi + h) = N(x, \xi) + \langle N'(x, \xi) | h \rangle_W + o(h), \quad \text{for } h \in W \ (36)$$

It holds

$$\int_\Omega \left[ N(x, Dv + t D\eta) - N(x, Dv) \right] dx = 0, \quad \text{whenever } \begin{cases} v \in C^\infty(\Omega, V) \\ \eta \in C^\infty(\Omega, V) \end{cases}$$
We want to apply $\frac{d}{dt}\bigg|_{t=0}$ under the integral sign. This is certainly legitimate if we further assume that:

iii) The function $(x, \xi) \mapsto \mathbf{N}'(x, \xi)$ is bounded on compact subsets of $\Omega \times \mathbf{W}$.

Now the integral form of the Lagrange-Euler equation reads as:

$$\int_{\Omega} \left\langle \mathbf{N}'(x, \mathcal{D}v) \big| \mathcal{D}\eta \right\rangle \, dx = 0, \quad \text{for all } \eta \in \mathcal{C}_\infty(\Omega, \mathbf{V}) \quad (37)$$

Our discussion of the formal adjoint operator $\mathcal{D}^*$ suggests investigating
the point-wise equation of order $2k$;

$$\mathcal{D}^* [ N'(x, \mathcal{D}v) ] = 0, \text{ for all } v \in C^\infty(\Omega, \mathcal{V})$$

However, without further differentiability hypotheses on the symbol $N(x, \xi)$, we cannot speak of this equation in the point-wise sense. Schwartz distributions come in handy. For the time being, let us think of this equation as convenient notation of (37).

Now, assuming the regularity hypotheses at i) ii) and iii), our Definition I amounts to the following statement
Definition II

The nonlinear differential operator $N(x, \mathcal{D}) : \mathcal{C}^\infty(\Omega, \mathcal{V}) \rightarrow \mathcal{L}^1_{\text{loc}}(\Omega)$

whose Lagrange-Euler equation

$$\mathcal{D}^* [N'(x, \mathcal{D}v)] = 0$$

(38)

holds for all $v \in \mathcal{C}^\infty(\Omega, \mathcal{V})$ is called null Lagrangian.

We call (38) the Lagrange identity. Historically, calculus of variations grew
out of the classical task to minimize a given energy functional

\[ \mathcal{E}[v] = \int_{\Omega} E(x, \mathcal{D}v) \, dx \]

for \( v \in \mathcal{C}^\infty(\Omega) \) subject to prescribed boundary conditions. Mathematical principles of continuum mechanics, elasticity theory, microstructure of materials, and so forth are formulated upon appropriate choice of the governing energy integrals. When studying such integrals, the following characterization of null Lagrangians is frequently quoted; we view it as the third definition.

Adding a null Lagrangian \( N(x, \mathcal{D}v) \)

to any energy integrand \( E(x, \mathcal{D}v) \)

will affect the energy but not its minimizers.