Higher-Order Nonemptiness Step by Step

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What is it about?

Higher-Order = we consider higher-order recursion schemes

Nonemptiness = we solve the acceptance problem for alternating reachability automata (= language nonemptiness)

Step by Step = we give a new method, working in multiple simple steps

Higher-order recursion schemes – what is this?

Definition

Higher-order recursion schemes = a generalization of context-free grammars, where nonterminals can take arguments. We use them to generate trees.

Equivalent definition: simply-typed lambda-calculus + recursion

In other words:

- programs with recursion
- higher-order functions (i.e., functions taking other functions as parameters)
- every function/parameter has a fixed type
- no data values, only functions

Higher-order recursion schemes – example

Ranked alphabet: (rank = number of children)

a of rank 2, b of rank 1, c of rank 0

Nonterminals:

S (starting), A, D

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$$S \to A\ b$$
$$A\ f \to a\ (A\ (D\ f))\ (f\ c)$$
$$D\ f\ x \to f\ (f\ x)$$

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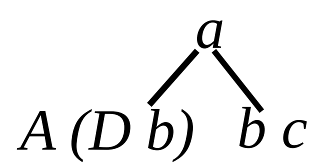

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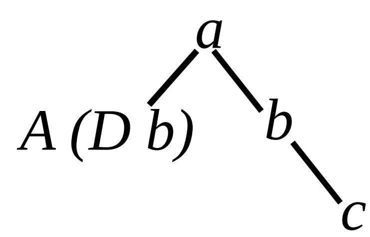

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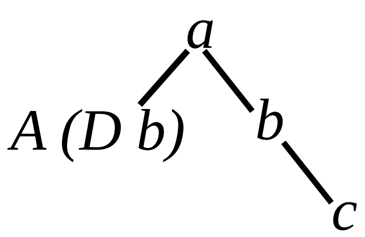

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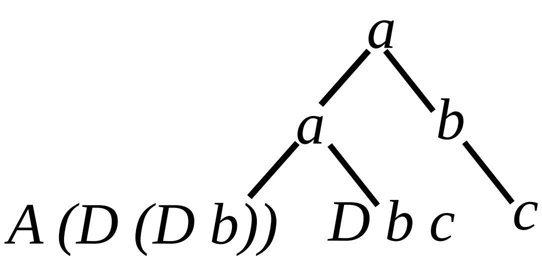

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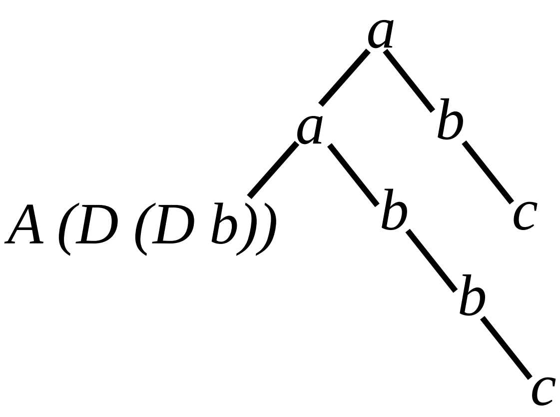

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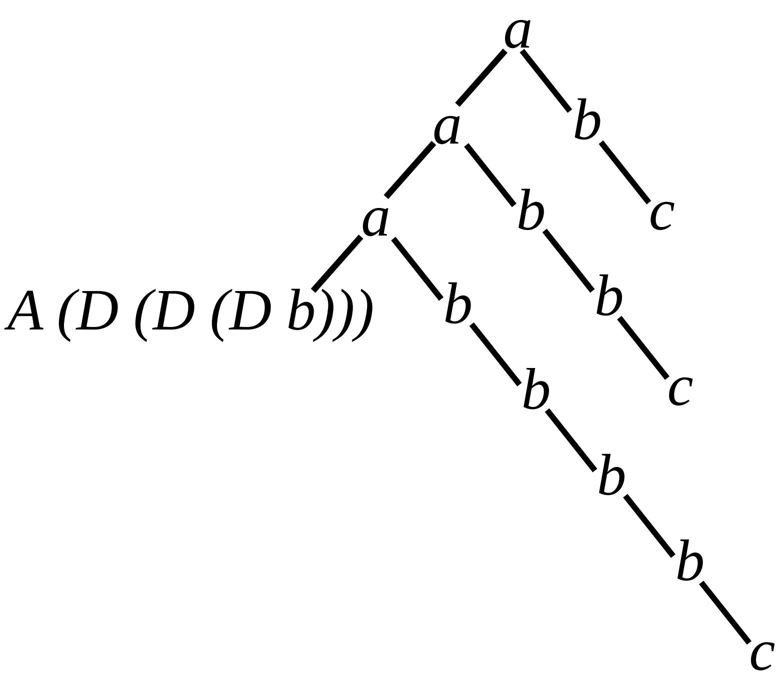

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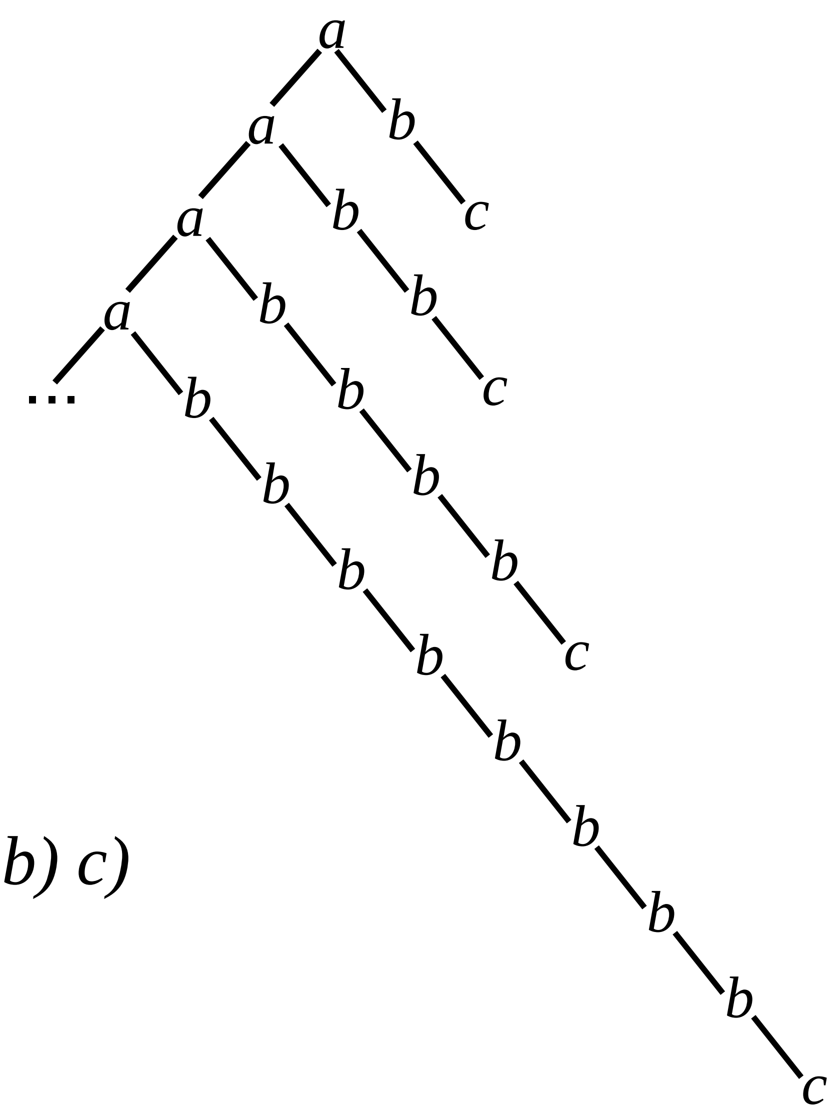

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Types

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Every nonterminal (every argument) has assigned some type, for example:

- o – a tree
- $o \rightarrow o$ – a function that takes a tree, and produces a tree
- $o \rightarrow (o \rightarrow o) \rightarrow o$ – a function that takes a tree and a function of type $o \rightarrow o$, and produces a tree

Order of a type

$$\mathrm{ord}(o) = 0$$
$$\mathrm{ord}(\alpha_1 \to \ldots \to \alpha_k \to o) = 1+\max(\mathrm{ord}(\alpha_1), \ldots, \mathrm{ord}(\alpha_k))$$

For example:

- $\mathrm{ord}(o) = 0$,
- $\mathrm{ord}(o \to o) = \mathrm{ord}(o \to o \to o) = 1$,
- $\mathrm{ord}(o \to (o \to o) \to o) = 2$

Order of a recursion scheme
= maximal order of (a type of) its nonterminal

Model-checking for recursion schemes

General goal: verifying properties of trees generated by schemes

Why? Recursion schemes are decidable models (abstractions) of programs using higher-order recursion

Model-checking for recursion schemes

Input: alternating tree automaton (ATA) $\mathcal{A}$, recursion scheme $\mathcal{G}$
Qestion: does $\mathcal{A}$ accept the tree generated by $\mathcal{G}$?

Theorem [Ong 2006]
This problem is decidable for parity ATA (i.e., for MSO).

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Several proofs, using:
- game semantics
- collapsible pushdown automata
- intersection types
- Krivine machines

and several extensions.
Some proofs only for reachability ATA.

We show another, very simple algorithm for reachability ATA.

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Complexity:
- n-EXPTIME-complete for recursion schemes of order n,
- FTP: linear in the size of $\mathcal{G}$, when size of $\mathcal{A}$ and maximal arity of types in $\mathcal{G}$ are fixed,
- the same for parity ATA and for reachability ATA
- (algorithms based on intersection types perform relatively well in practice)

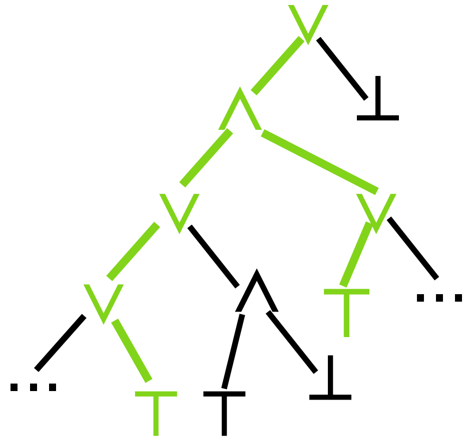
Our algorithm achieves the same complexity.

Preprocessing

We consider an (appropriately defined) product of $\mathcal{G}$ and $\mathcal{A}$.

It is a recursion scheme generating a tree labeled by:
$\wedge$ (AND),
$\vee$ (OR),
with $\top$ (empty AND), $\bot$ (empty OR) as special cases

We ask about alternating reachability.



General idea

We replace the recursion scheme $\mathcal{G}_n$ of order n by an equivalent recursion scheme $\mathcal{G}_{n-1}$ of order n-1. Size grows exponentially.

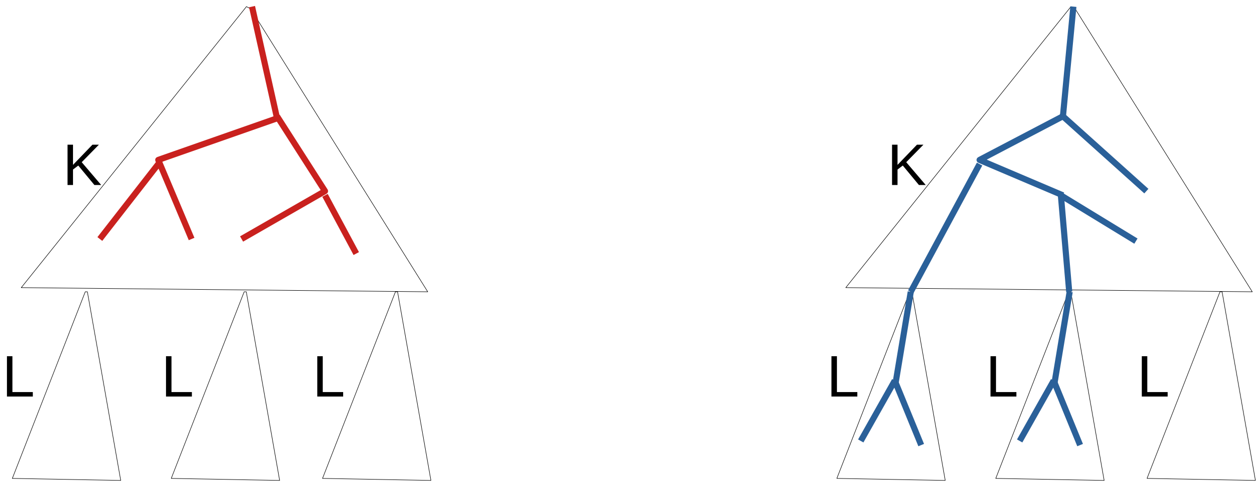
$$\mathcal{G}_n \to \mathcal{G}_{n-1} \to \mathcal{G}_{n-2} \to \cdots \to \mathcal{G}_1 \to \mathcal{G}_0$$

For recursion schemes of order 0 the problem becomes trivial.

Transformation

Consider an application KL, where L is of order 0 (generates a tree).
When is the tree generated by KL accepting?

- When K⊥ is accepting (i.e., K is accepting without using the argument)
- When both KT and L are accepting



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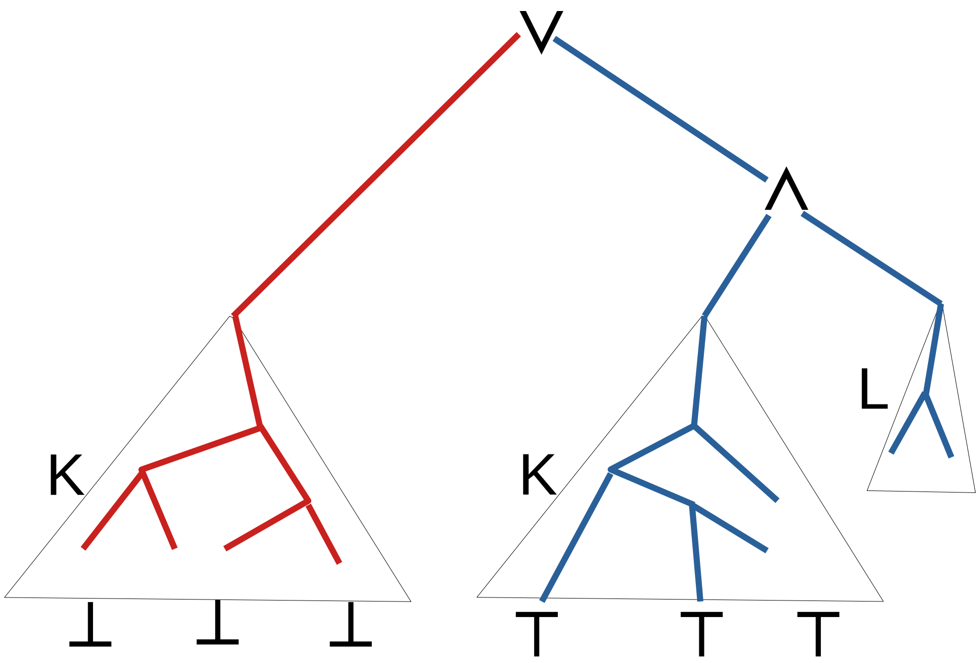
We change KL into ∨ (K⊥) (∧ (KT) L)



Complete example (order 1)

order-0 argument

$$X \rightarrow Y\ Z$$
$$Y\ x \rightarrow \vee\ T\ x$$
$$Z \rightarrow T$$

$$\Rightarrow$$

$$X \rightarrow \vee\ Y_0\ (\wedge\ Y_1\ Z)$$
$$Y_0 \rightarrow \vee\ T\ \bot \qquad Y_1 \rightarrow \vee\ T\ T$$
$$Z \rightarrow T$$

(k order-0 arguments $\Rightarrow$ 2^k variants of the nonterminal)

Complete example (order 2)

$X \rightarrow Z\ Y$

$Y\ x \rightarrow \vee\ T\ x$

$Z\ y \rightarrow y\ (y\ T)$

order-0 arguments

$X \rightarrow Z\ Y_0\ Y_1$

$Y_0 \rightarrow \vee\ T\ \bot \qquad Y_1 \rightarrow \vee\ T\ T$

$Z\ y_0\ y_1 \rightarrow \vee\ y_0\ (\wedge\ y_1\ (\vee\ y_0\ (\wedge\ y_1\ T)))$

Complete example (order 2)

$$X \to Z\,Y$$
$$Y\,x \to \vee\,\top\,x$$
$$Z\,y \to y\,(y\,\top)$$

order-0 arguments

$$X \to Z\,Y_0\,Y_1$$
$$Y_0 \to \vee\,\top\,\bot \qquad Y_1 \to \vee\,\top\,\top$$
$$Z\,y_0\,y_1 \to \vee\,y_0\,(\wedge\,y_1\,(\vee\,y_0\,(\wedge\,y_1\,\top)))$$

- easy to generalize
- easy (syntactical) correctness proof
- verified in Coq

Conclusion

- We consider the model-checking problem for recursion schemes + reachability ATA
- We propose a new, simpler method algorithm solving this problem: we repeatedly reduce the order of a recursion scheme by one, increasing its size exponentially
- We obtain optimal complexity
- Future work: extend this method to parity ATA / to the diagonal problem (SUP)

Thank you!