Extensions of the Caucal Hierarchy?

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Caucal hierarchy – a hierarchy of graphs

We consider directed, edge-labeled graphs without isolated vertices

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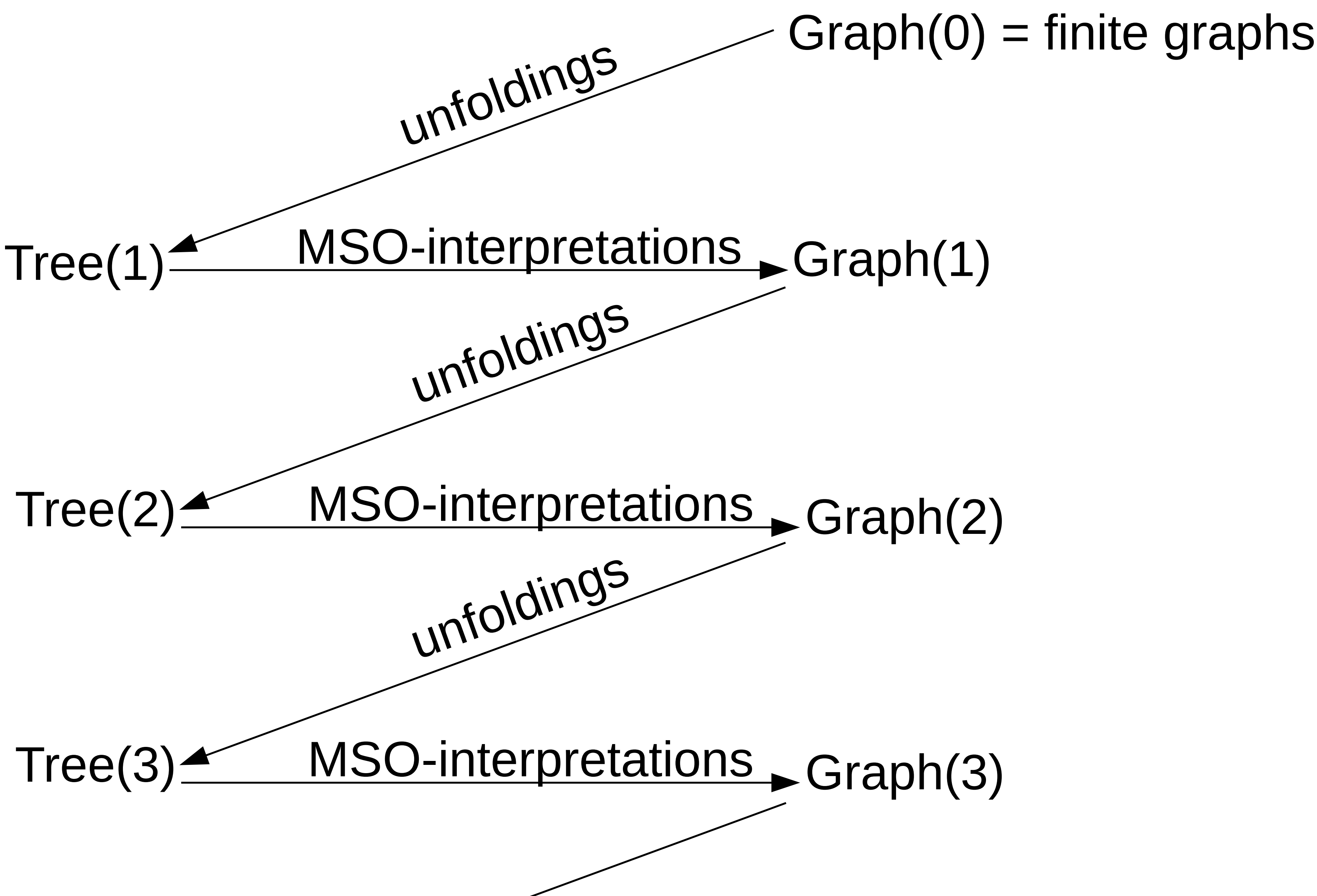

Caucal hierarchy – a hierarchy of graphs

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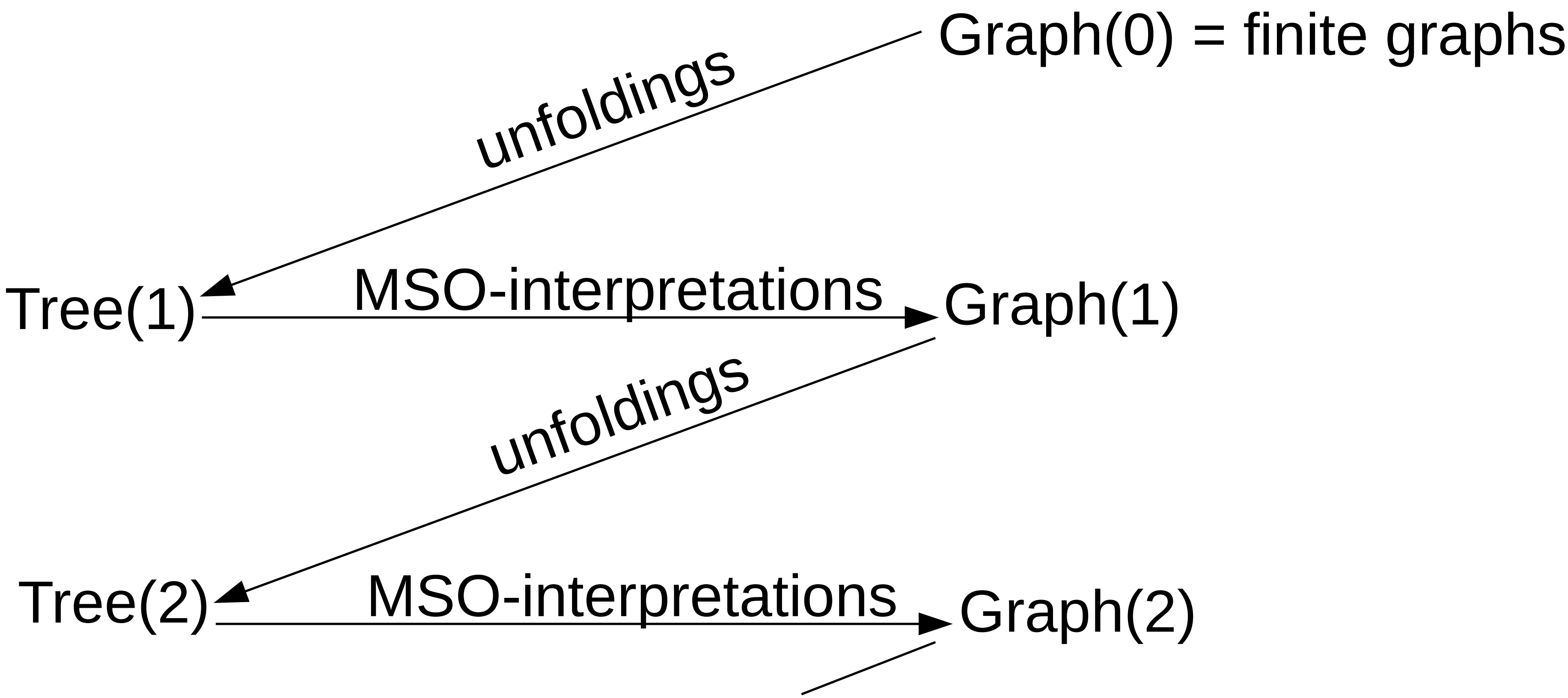


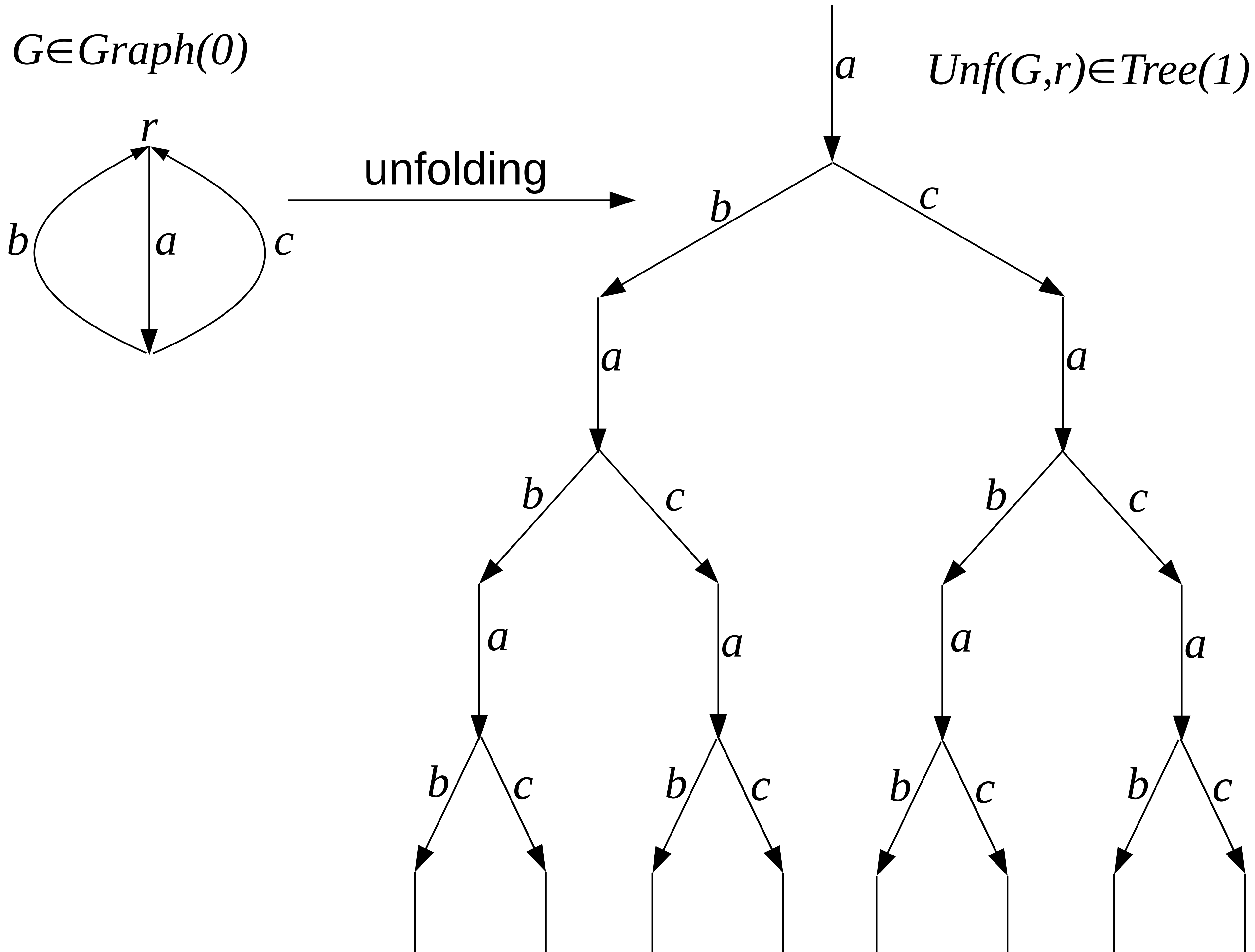
Defined by Caucal (2002)
Studied intensively by Carayol & Wöhrle (2003)

Unfoldings

G – graph
r – a selected node in G

Unf(G, r) – unfolding of G from r (a new graph)
nodes: paths in G starting from r
edges: for every edge $u \xrightarrow{a} v$ in G,
and for every path p ending in u
if p‘ is the extension of p by the edge $u \xrightarrow{a} v$
we create an a-labeled edge from p to p‘

Unfoldings



G∈Graph(0)
r
b
a
c
unfolding
Unf(G,r)∈Tree(1)
a
b
c
a
a
b
c
b
c
a
a
a
a
b
c
b
c
b
c
b
c

MSO-interpretations

MSO logic – a logic, where you can quantify over nodes
and over sets of nodes, and reason about edges between nodes

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and over sets of nodes, and reason about edges between nodes

MSO interpretation:

- a graph G
- a tuple of MSO formulas $\phi_a(x,y)$, for every letter $a \in \Sigma$

This defines a new graph:

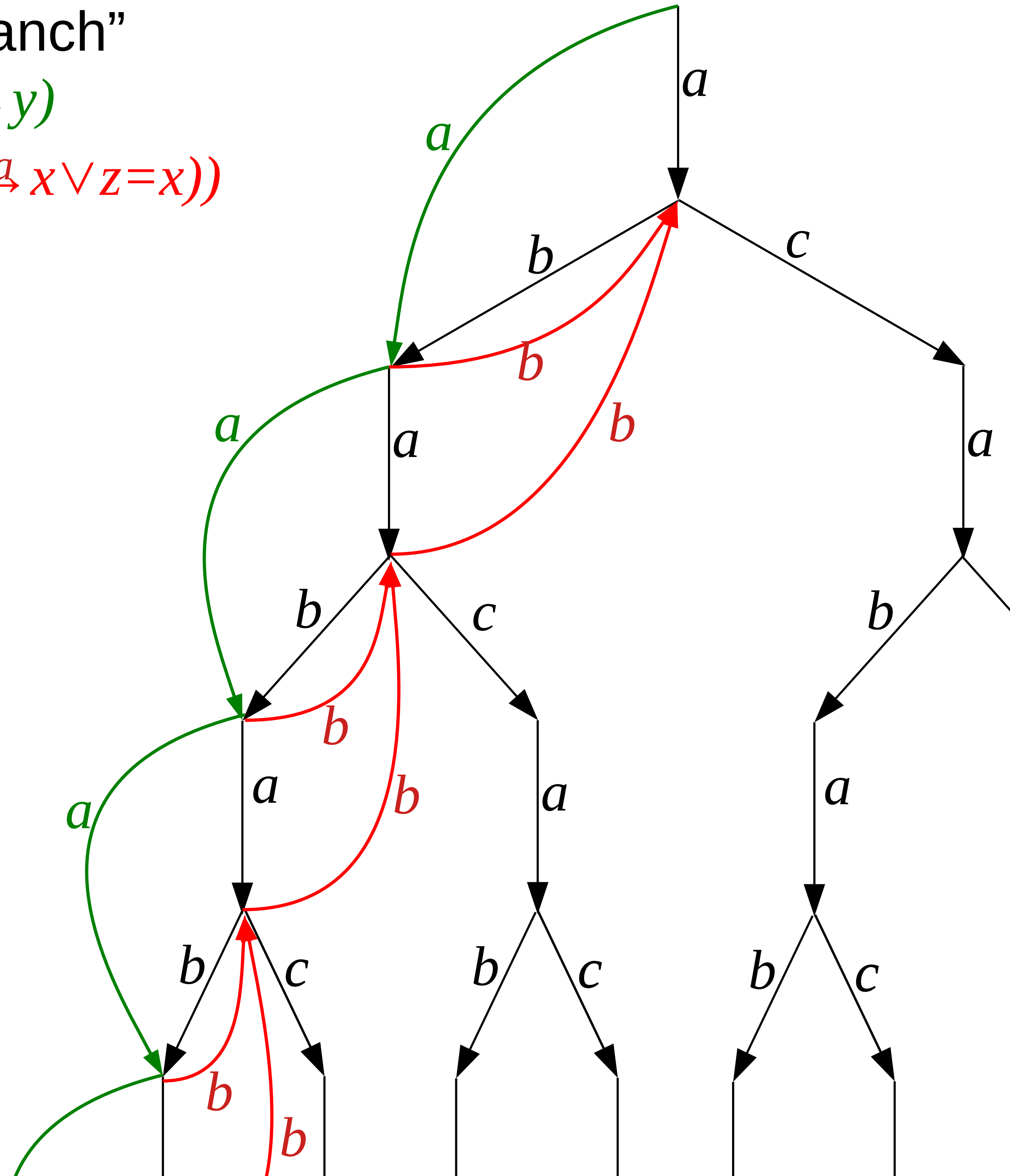
- there is an a-labeled edge between x and y if $\phi_a(x,y)$ holds
- nodes = nodes of G incident with at least one edge

MSO-interpretations

$\psi(x)$ = “x is on the $(ab)^*$ branch”

$\phi_a(x,y) = \psi(x) \wedge \exists z\, (x \xrightarrow{a} z \wedge z \xrightarrow{b} y)$

$\phi_b(x,y) = \psi(x) \wedge \exists z\, (y \xrightarrow{b} z \wedge (z \xrightarrow{a} x \vee z = x))$



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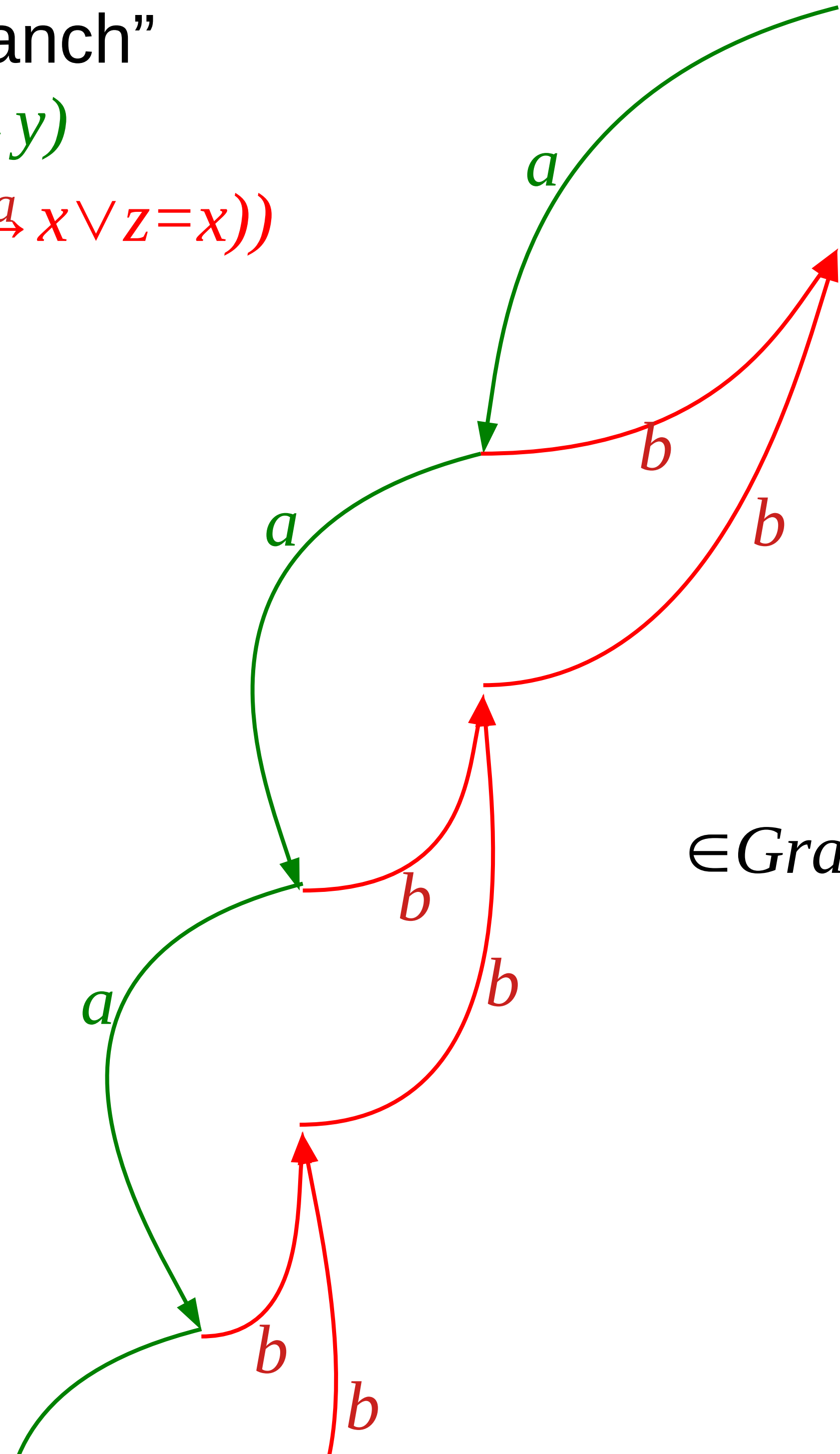
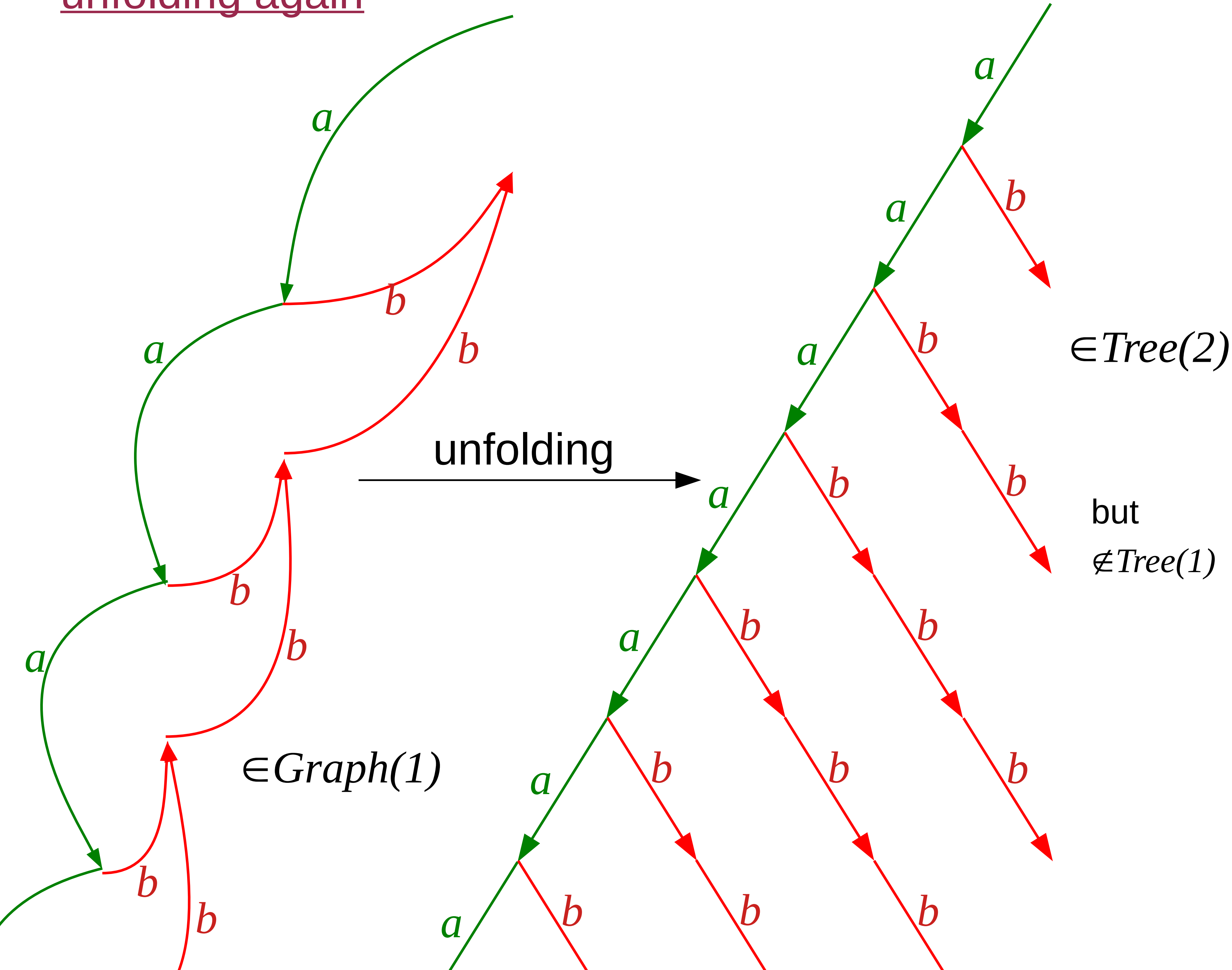


$\in Graph(1)$

unfolding again



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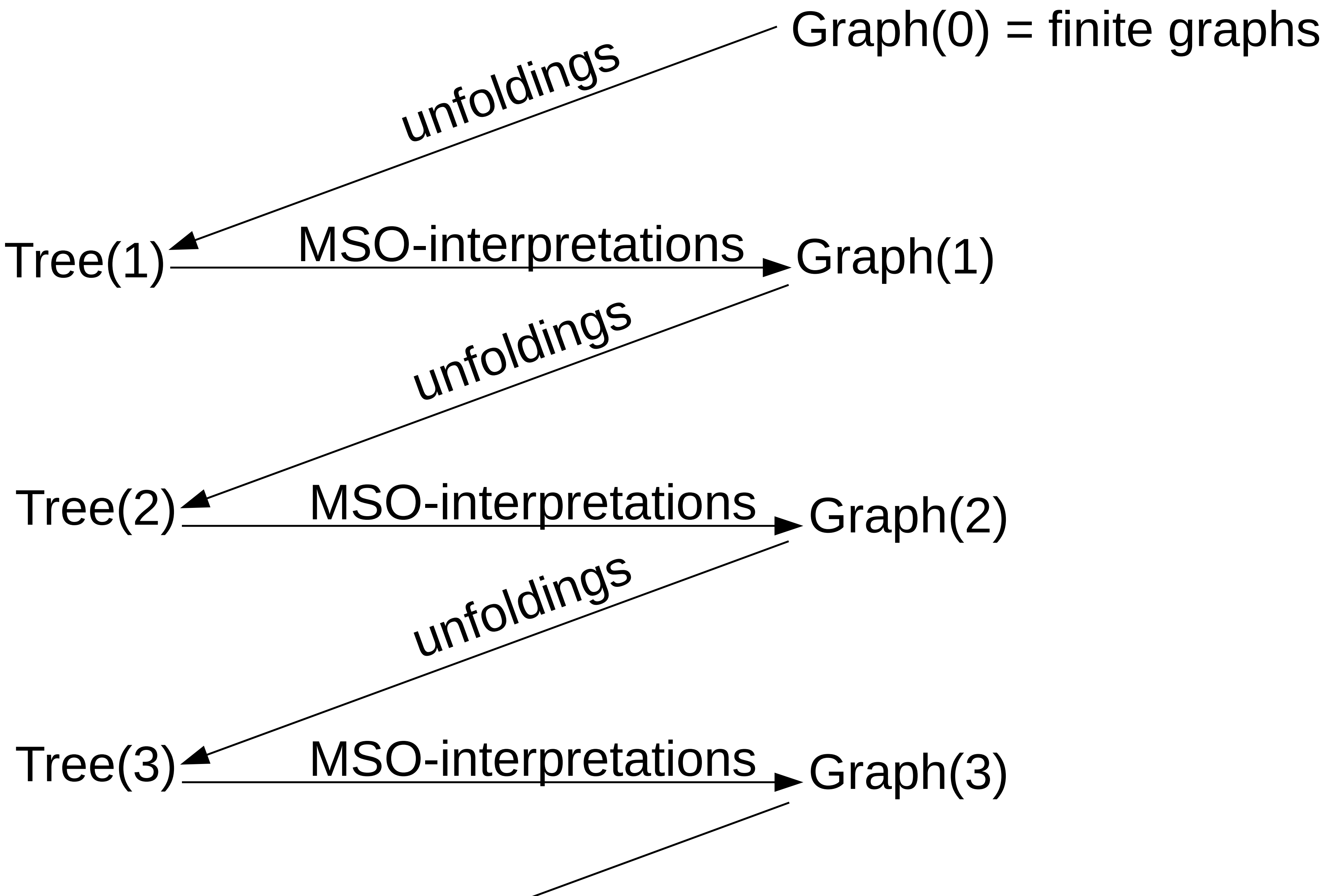

What is interesting about the Caucal hierarchy?

Graphs in the Caucal hierarchy have decidable MSO theory
i.e. for every graph G in the hierarchy there is a procedure that given an MSO sentence ϕ says whether ϕ holds in G

Reason:
- unfoldings preserve decidability of MSO [Courcelle & Walukiewicz 1998]
- MSO-interpretations preserve decidability of MSO

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Remark:
For many graphs the MSO theory is undecidable,
e.g. for the infinite grid (thus the infinite grid is not in the Caucal hierarchy)

What is interesting about the Caucal hierarchy?

There are other, equivalent definitions of the hierarchy:

- instead of MSO-interpretations we can use:
  - MSO-transductions (=create multiple copies + MSO-interpretation) [Courcelle 1994]
  - inverse rational mappings (a “special form” of MSO-interpretations: we can only analyze some path between the nodes) [Caucal 1996]
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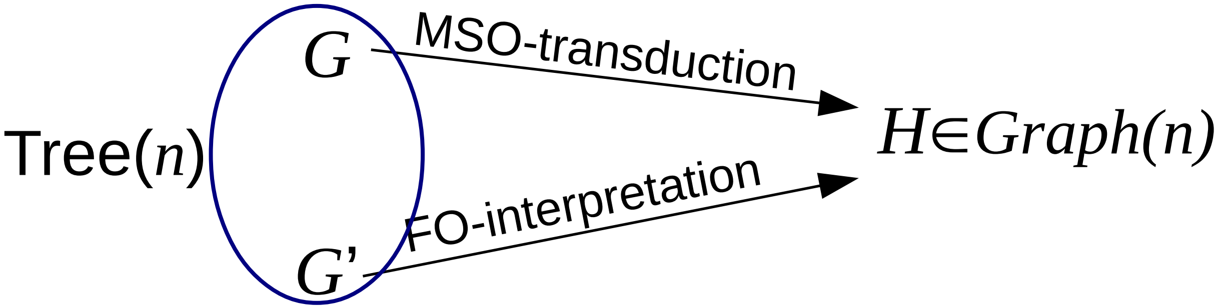
Remark: when we have a fixed graph G, then by using MSO-transductions from G we can obtain more graphs than by using FO-interpretations from G

But: there is another graph G' on the same level of the hierarchy such that the MSO-transduction in G can be replaced by an FO-interpretation in G'



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- Graph(n) = ε-closures of configuration graphs of order-n pushdown automata (a generalization of pushdown automata: they use a stack of order n – a stack of stacks of … of stacks) [Maslov 1976]

Our contribution:

1) Using WMSO+U-interpretations, we obtain the same hierarchy
(i.e. every level of the hierarchy is closed under WMSO+U-interpretations)

2) Using MSO+U-interpretations, we can obtain graphs with undecidable MSO theory
(i.e. we obtain more graphs, but without nice properties)

Logic MSO+U

MSO+U extends MSO by a new quantifier „U” [Bojańczyk, 2004]

$$\mathsf{U}X.\phi(X)$$

$\phi(X)$ holds for finite sets of arbitrarily large size

$\forall n \in \mathbb{N}\ \exists X\ (\ n < |X| < \infty\ \wedge\ \phi(X)\ )$

WMSO+U = “weak” MSO+U – we can quantify only over finite sets
($\exists X$ / $\forall X$ means: exists a finite set X / for all finite sets X)

Decision problems for MSO+U

Satisfiability (the problem usually considered for MSO+U):
input: sentence ϕ, question: is ϕ true in some tree?

- undecidable for MSO+U, even for words [Bojańczyk, P., Toruńczyk 2016]
  some fragments of MSO+U decidable for words [Bojańczyk, Colcombet 2006]
- decidable for WMSO+U [Bojańczyk, Toruńczyk 2012]
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HORS model-checking is decidable [P. 2018]
input: sentence $\phi \in$MSO+U, higher-order recursion scheme G,
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We use this result here!!

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Moreover, we have the reflection property:
input: formula $\phi(x) \in$MSO+U, higher-order recursion scheme G,
output: a scheme G_+ generating a tree of the same shape, but with additional label in every node saying whether ϕ holds for this node

We use this result here!!

Higher-order recursion schemes

A generalization of context-free grammars:

- nonterminals can take arguments
- these arguments may be used on the right side of productions
- arguments may take further arguments
- deterministic (one rule for every nonterminal)
- we want to generate an infinite tree:
  on the right side of productions we may use constructors of nodes

[Damm 1986, Knapik, Niwiński, Urzyczyn 2002]

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Thm. Trees generated by deterministic order-n pushdown automata
= trees generated by safe order-n recursion schemes
where "safe" is some syntactic restriction on the schemes

How do we prove our theorems?

Thm 1. Every level of the Caucal hierarchy is closed under WMSO+U-interpretations

Step 1: establish relation between the Caucal hierarchy and trees generated by recursion schemes

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We know that:

Fact 1. Every $G \in$Graph(n) is an ε-closure of the configuration graph of some order-n pushdown automaton

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Problems here:

1) deterministic vs nondeterministic automata
2) recursion schemes & deterministic automata can generate only finitely branching trees, while in Tree(n) we also have infinitely branching trees
3) (superficial) node-labeled / edge-labeled trees

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Step 1: establish relation between the Caucal hierarchy and trees generated by recursion schemes

It is possible to prove that:

Lemma 1. A graph is in Graph(n) iff it can be obtained by applying an MSO-interpretation to a tree generated by some safe order-(n-1) recursion scheme.

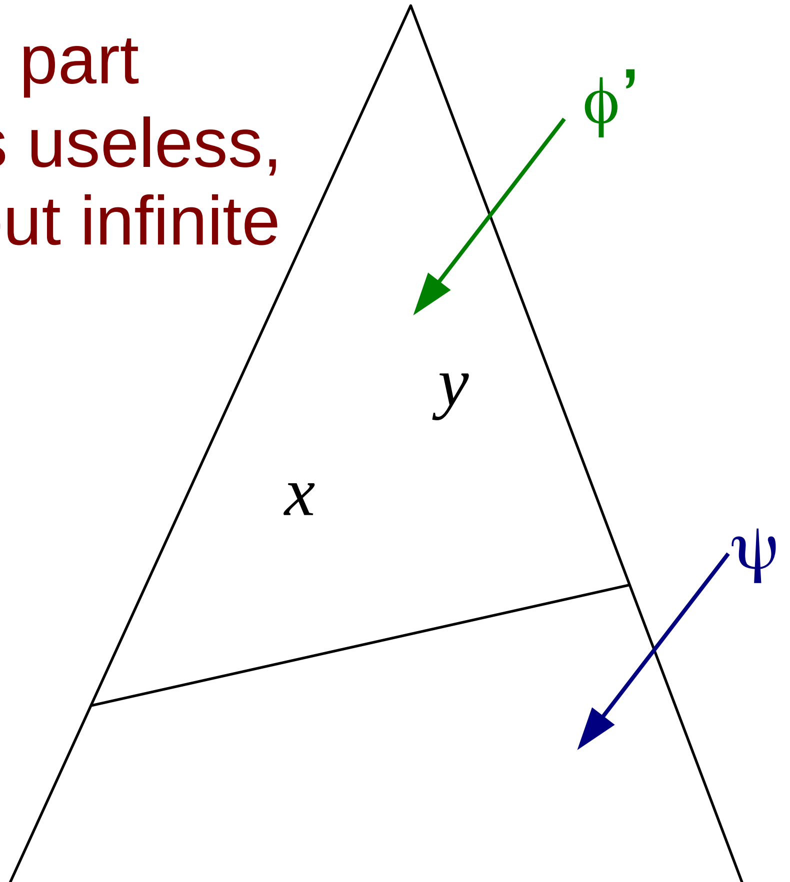
How do we prove our theorems?

Thm 1. Every level of the Caucal hierarchy is closed under WMSO+U-interpretations

Step 2 – we prove that:

Lemma 2. Every WMSO+U formula $\phi(x,y)$ can be rewritten as an MSO formula $\phi'(x,y)$ having WMSO+U subformulas $\psi(z)$.

Idea: Using ϕ' we describe the top (finite) part of the tree, containing x and y, where U is useless, and we use subformulas ψ to reason about infinite subtrees.



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Using the WMSO+U-reflection property for recursion schemes [P. 2008], we obtain:

Lemma 3. Every WMSO+U interpretation in a tree T generated by a safe order-n recursion scheme G can be rewritten as an MSO-interpretation in a tree T_+ generated by a safe order-n recursion scheme G_+.

How do we prove our theorems?

Thm 2. By applying some MSO+U-interpretation to some tree $T \in \text{Tree}(2)$ we can obtain a graph with undecidable MSO theory

This is obtained by inspecting / modifying the proof that satisfiability of MSO+U is undecidable.

Conclusions

Thm 1. Every level of the Caucal hierarchy is closed under WMSO+U-interpretations.

Thm 2. By applying some MSO+U-interpretation to some tree $T \in$ Tree(2) we can obtain a graph with undecidable MSO theory (hence outside of the Caucal hierarchy).

Open problem:

- Find a larger class of graphs with decidable MSO theory.
- In particular, trees generated by all recursion schemes have decidable MSO theory. But only trees generated by safe recursion schemes are in the Caucal hierarchy.
  Is there a class with a nice logical characterization that contains all these trees?

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