Recursion Schemes and the WMSO+U Logic

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Higher-order recursion schemes – what is this?

Definition

Recursion schemes = simply-typed lambda-calculus + recursion

In other words:

- programs with recursion
- higher-order functions (i.e., functions taking other functions as parameters)
- every function/parameter has a fixed type
- no data values, only functions

Higher-order recursion schemes – example

```
fun f(x) {
    a(x);
    if * then f(x);
    b(x);
}
f(x)
```

branching (we are not sure what will be chosen)

recursion

uniterpreted constants (unknown functions)

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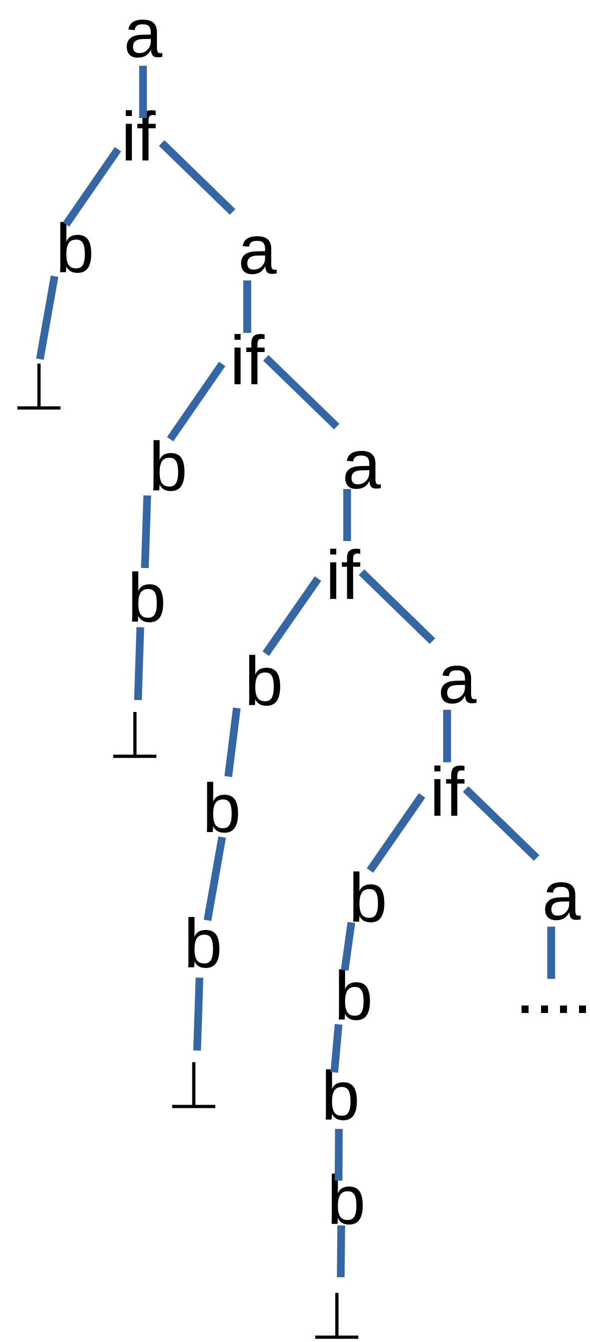
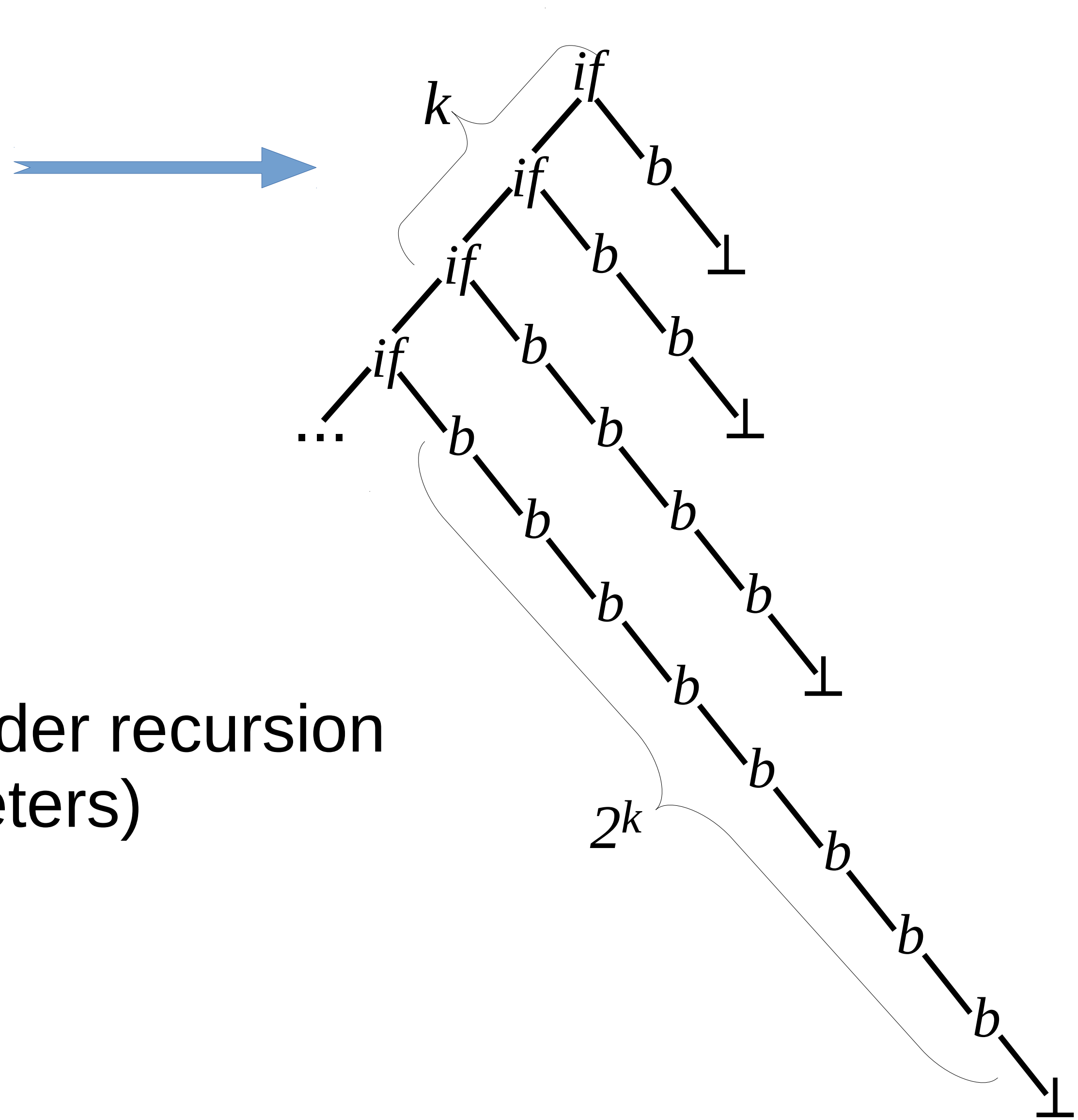


We are interested in trees representing the control flow of such programs.

Observation: these trees need not to be regular

Higher-order recursion schemes – example

```
fun A(f,x) {
   if * then A(D(f),x) else f(x);
}
fun D(f)(x) {
   f(x); f(x);
}
fun P(x) {
   b(x);
}
A(P,x)
```



This program uses higher-order recursion (passes functions as parameters)

Model-checking

Theorem [Ong 2006]
MSO model-checking on trees generated by recursion schemes is decidable.

Input: MSO formula ϕ, recursion scheme $\mathcal{G}$

Question: is ϕ true in the (infinite) tree generated by $\mathcal{G}$?

Model-checking

- a program in a functional programming language (e.g. OCAML)
- a property ψ

does the program satisfy ψ?

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Approximation

ignore some details,

simulate some details using functions

- a recursion scheme $\mathcal{G}$
- a formula ϕ

does the program satisfy ψ?

is ϕ true in the tree generated by $\mathcal{G}$?

decidable

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Approximation

ignore some details,

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- a recursion scheme $\mathcal{G}$
- a formula ϕ

does the program satisfy ψ?

- yes
- ?

is ϕ true in the tree generated by $\mathcal{G}$?

- yes
- no

decidable

There exist tools that take (short) programs in Ocaml and can verify some useful properties.

This work – can we go beyond MSO?

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We consider the WMSO+U logic.

“+U” = we add a new quantifier „U” [Bojańczyk, 2004]

$$UX.\phi(X)$$

$\phi(X)$ holds for finite sets of arbitrarily large size

$$\forall n \in \mathbb{N}\ \exists X\ (\ n<|X|<\infty\ \land\ \phi(X)\ )$$

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$$\forall n \in \mathbb{N}\ \exists X\ (\ n<|X|<\infty\ \wedge\ \phi(X)\ )$$

“W” = weak – we can quantify only over finite sets
($\exists X$ / $\forall X$ means: exists a finite set X / for all finite sets X)

Decision problems for MSO+U

Satisfiability (the problem usually considered for MSO+U):

input: formula ϕ, question: is ϕ true in some tree?

- undecidable for MSO+U, even for words [Bojańczyk, P., Toruńczyk 2016]
  some fragments of MSO+U decidable for words [Bojańczyk, Colcombet 2006]
- decidable for WMSO+U [Bojańczyk, Toruńczyk 2012]
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HORS model-checking

input: formula ϕ, HORS $\mathcal{G}$,

question: is ϕ true in the tree generated by $\mathcal{G}$

- decidable for $\phi \in$ MSO [Ong 2006]
- undecidable for $\phi \in$ MSO+U (generalizes satifiability)
- **Contribution: decidable for $\phi \in$ WMSO+U**

About the proof

Theorem – the following problem is decidable:
input: formula ϕ, HORS $\mathcal{G}$,
question: is ϕ true in the tree generated by $\mathcal{G}$?

Key ingredients:

- decidability of the "diagonal problem" for HORSes:
  input: HORS $\mathcal{G}$, letter a
  question: are there paths with arbitrarily many letters a in the tree generated by $\mathcal{G}$?
  [Hague, Kochems, Ong 2016, Clemente, P., Salvati, Walukiewicz 2016]
  Remark 1: this property is not regular
  Remark 2: this is a „universal" property that can be expressed by a single „U" quantifier

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- „reflection" for the diagonal problem: [P. 2016]

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- „reflection" for (W)MSO: [Broadbent, Carayol, Ong, Serre 2010]
  input: HORS $\mathcal{G}$, formula $\psi(x) \in$ WMSO
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- induction on the structure of ϕ – we add labels with information about subformulas (here it is useful that the logic is „weak")

Future work

The diagonal problem for HORS is decidable in a more general version:

input: HORS $\mathcal{G}$, letters $a_1,...,a_k$

question: are there paths with arbitrarily many appearances of every letter $a_1,...,a_k$ in the tree generated by $\mathcal{G}$?

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Question: Design a more general logic, capable to express the multi-letter diagonal problem (and prove its decidability for trees generated by HORSes, via a reduction to this version of the diagonal problem)

Thank you!