# The MSO+U Theory of (N,<) Is Undecidable

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STACS 2016

# MSO+U logic

MSO+U extends MSO by the following „U" quantifier:

$$UX.\phi(X)$$

$\phi(X)$ holds for sets of arbitrarily large size

$$\forall n \in \mathbb{N}\ \exists X\ (\ n<|X|<\infty\ \wedge\ \phi(X)\ )$$

This construction may be nested inside other quantifiers,
and $\phi$ may have free variables other than X.

(MSO+U was introduced by Bojańczyk in 2004)

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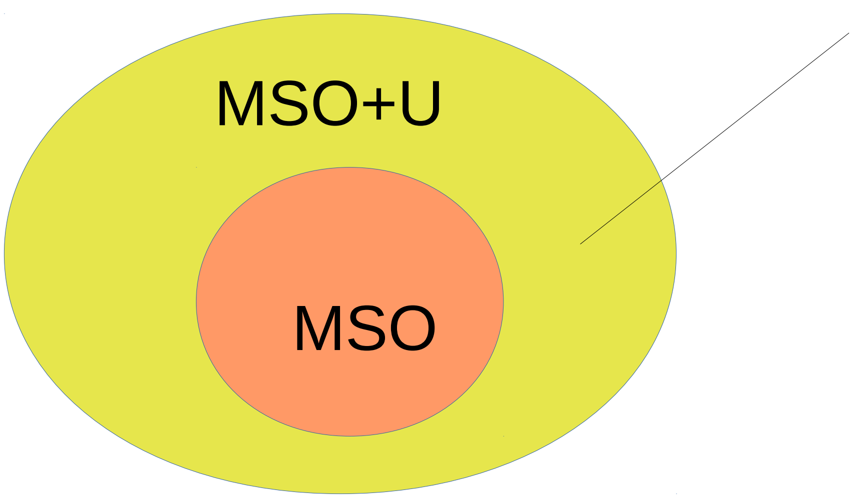


words in $(b^*a)^\omega$ where $b^*$ blocks have unbounded size

$$\{\ b^{n_1}\ a\ b^{n_2}\ a\ \ldots\ |\ \limsup n_i=\infty\}$$

(contains no ultimately periodic word)

## MSO+U logic

Consider the following “Myhill-Nerode” relation:

$v \sim v'$ when for all words $u \in A^*$, $w \in A^\omega$ $uvw \models \phi \Leftrightarrow uv'w \models \phi$

This relation has finitely many equivalence classes.

**Slogan:** The non-regularity of MSO+U is seen only in the asymptotic behavior.

<u>Considered problem</u>: satisfiability for $\omega$-words

Input: formula $\phi \in$ MSO+U

Question: $\exists\, w \in A^\omega \,.\, w \models \phi$ ?

Equivalently:

Input: formula $\phi \in$ MSO+U

Question: $a^\omega \models \phi$ ?

**<u>Our result:</u> This problem is undecidable!!!**

# MSO+U logic

Plan of the talk:

1) Some fragments of MSO+U are decidable – *earlier work*
   a) BS-formulas
   b) WMSO+U

2) MSO+U is not decidable over $\omega$-words – ***this paper***

# Decidable fragments of MSO+U

negation allowed

BS-formulas: boolean combinations of formulas
in which U appears positively
(+ existential quantification outside)

**Theorem** (Bojańczyk & Colcombet, 2006):
Satisfiability of BS-formulas is decidable over ω-words.

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**Theorem** (Bojańczyk & Colcombet, 2006):
Satisfiability of BS-formulas is decidable over $\omega$-words.

Solution: $\omega$BS-automata
Nondeterministic automata with counters that
can be incremented and reset to 0, but cannot be read.
Accepting condition: counter is bounded/unbounded.

(Colcombet & others) Automata with counters were developed
into a theory of „regular cost functions" of the form:

$$f : A^* \to \mathbb{N}$$

# Decidable fragments of MSO+U

Weak logics: ∃/∀ quantifier range only over finite sets.

Satisfiability is decidable for:

WMSO+U on infinite words (Bojańczyk, 2009)

WMSO+R on infinite words (Bojańczyk & Toruńczyk, 2009)

R = exists infinitely many sets of bounded size

WMSO+U on infinite trees (Bojańczyk & Toruńczyk, 2012)

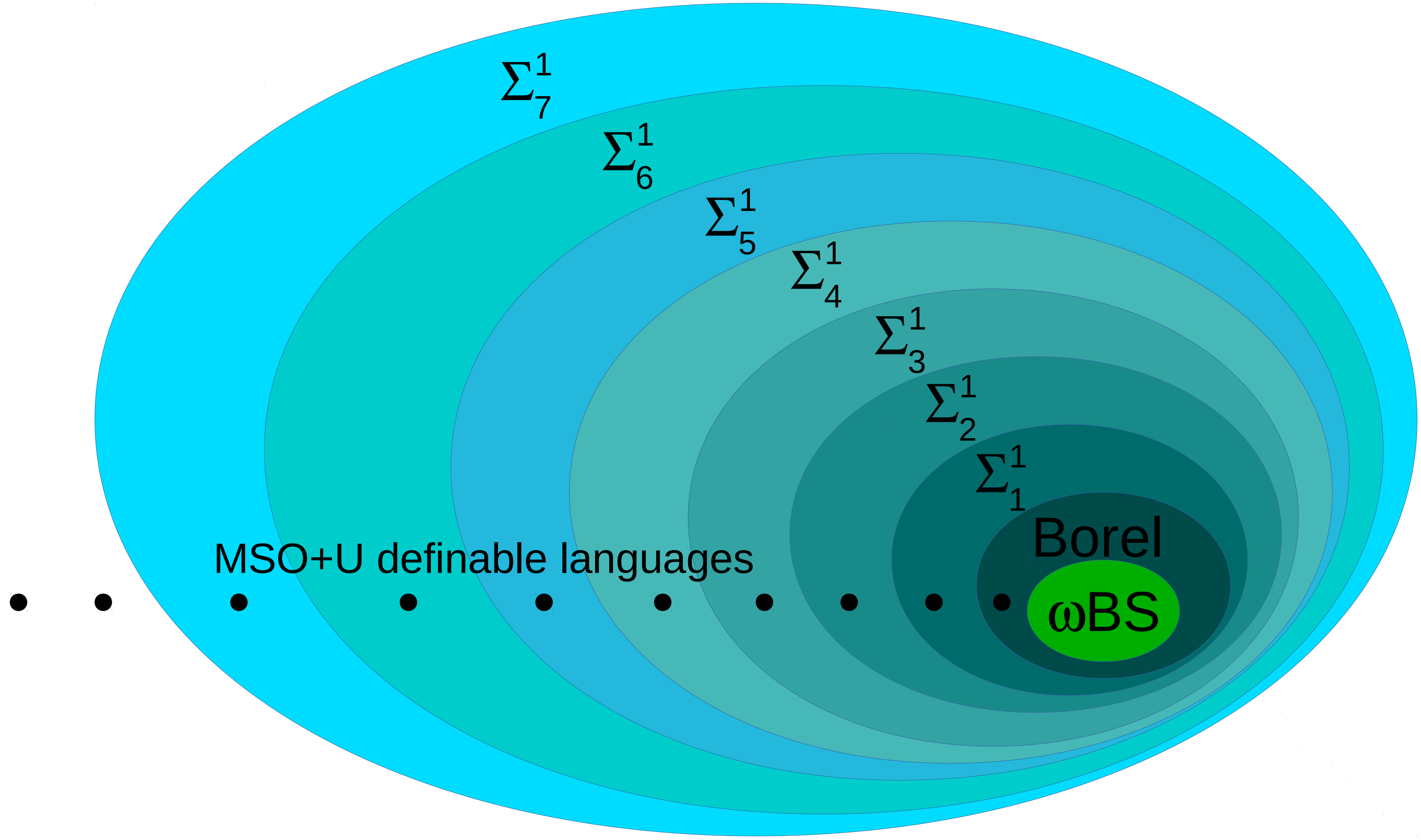
WMSO+U+P on infinite trees (Bojańczyk, 2014)

P = exists path

Solution: equivalent automata models

# Undecidability of MSO+U – earlier work

**Thm.** (Hummel & Skrzypczak 2010/2012) - topology
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.



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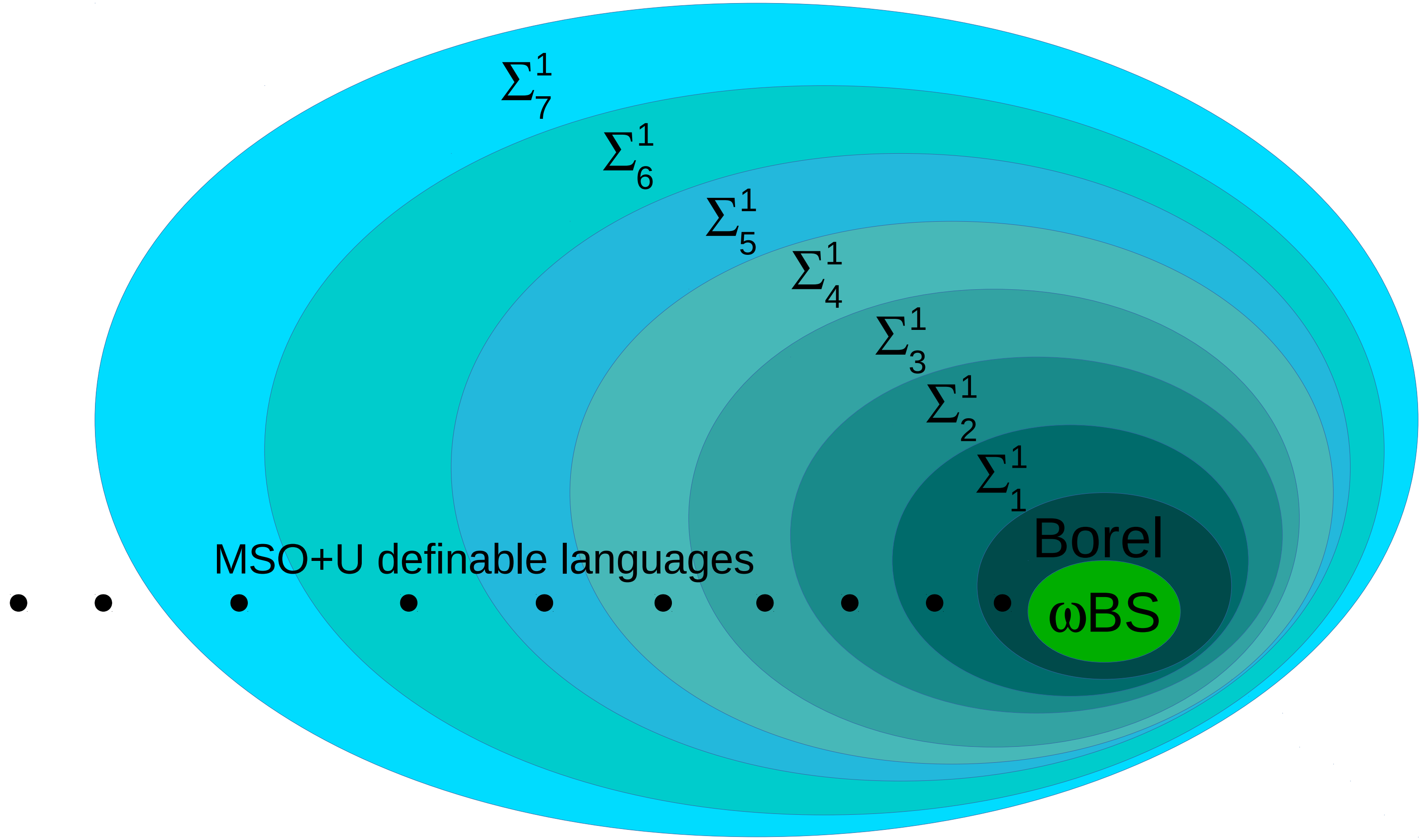


Corollary: MSO+U is not covered by any automata model
(alternating/nondeterm./determ., acceptance condition of bounded projective complexity)

# Undecidability of MSO+U – earlier work

**Thm 1.** (Hummel & Skrzypczak 2010/2012)
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.

**Thm 2.** (Bojańczyk, Gogacz, Michalewski, Skrzypczak 2014)
MSO+U is not decidable over infinite trees...

...assuming that there exists a projective ordering on the Cantor set $2^\omega$.

assumption of set theory consistent with ZFC

Corollary: No algorithm can decide MSO+U over infinite trees and have a correctness proof in ZFC.

Proof:

Bases on Thm 1 & the proof of Shelah that MSO is undecidable in $2^\omega$.
Altogether rather complicated.

# Undecidability of MSO+U

**Thm.** (Bojańczyk, P., Toruńczyk – this paper)
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Proof – short & elementary.

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Proof sketch

Step 1: words = trees (forests) of bounded depth

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## Proof sketch

Step 1: words = trees (forests) of bounded depth

Step 2 - Key Lemma:
There is an MSO+U formula defining the set of depth-3 forests s.t.
a) the degree of depth-2 nodes tends to infinity
b) all but finitely many nodes of depth 1 have the same degree.

equality!!!!

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Step 3:
Having equality, it is easy to encode e.g. runs of a Minsky machine.
Equality of „all but finitely many” (=„from some moment”) is enough - we can repeat the finite run of the M.M. infinitely many times.

Step 2 - Key Lemma:

There is an MSO+U formula defining the set of depth-3 forests s.t.

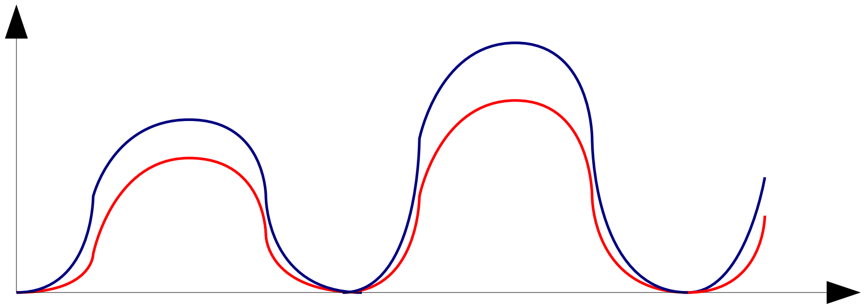
(a) the degree of depth-2 nodes tends to infinity

(b) all but finitely many depth-1 nodes have the same degree.

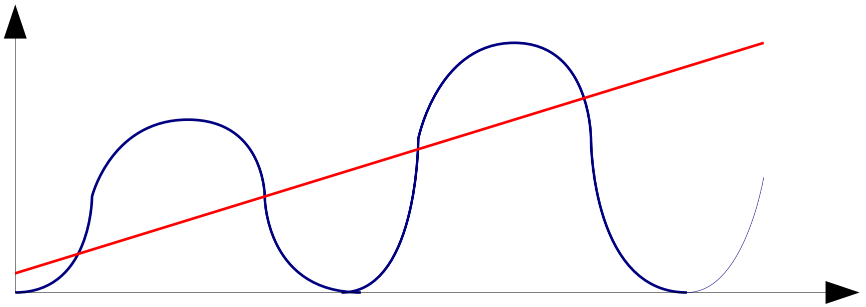
Proof. We use number sequences and vector sequences.

Def. *f*~*g* ⇔ *f* and *g* are bounded on the same sets of positions.

(where f, g – sequences of numbers)

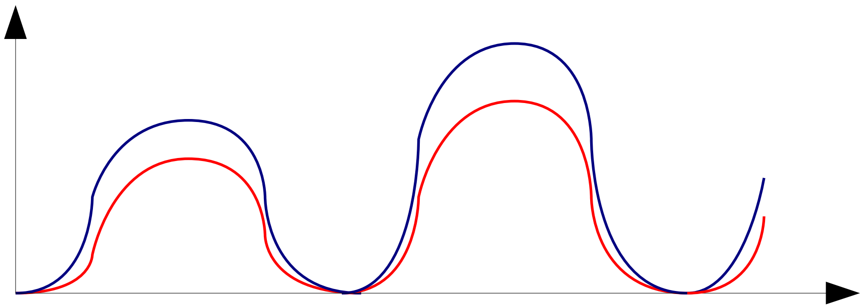


equivalent



non-equivalent

Def. $f \sim g \Leftrightarrow f$ and $g$ are bounded on the same sets of positions.



Def. A vector sequence **f** is an *asymptotic mix* of a vector sequence **g**
if $\forall$ f$\in$**f** . $\exists$ g$\in$**g** . f~g

**f** = (3, 1, 2), (1), (7, 1), (1, 2, 5), (1, 4, 1, 3), (5, 1), …

**g**= (2, 8), (9, 2, 3), (8, 2), (2, 2, 2), (2, 7), (8, 1, 2), ...

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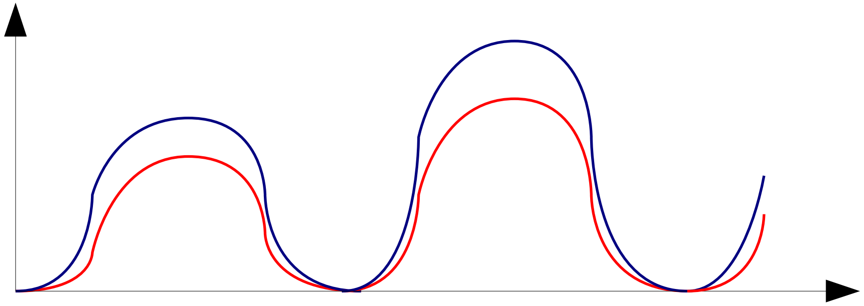

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## Lemma

$\exists\ \mathbf{f} : \mathbb{N} \to \mathbb{N}^d$ . **f** is not a asymptotic mix of any $\mathbf{g} : \mathbb{N} \to \mathbb{N}^{d-1}$

## Proof

For **f** we take all vectors from $\mathbb{N}^d$.

## Lemma

$\exists\ \mathbf{f}:\mathbb{N}\to\mathbb{N}^d$. $\mathbf{f}$ is not a asymptotic mix of any $\mathbf{g}:\mathbb{N}\to\mathbb{N}^{d-1}$

## Corollary

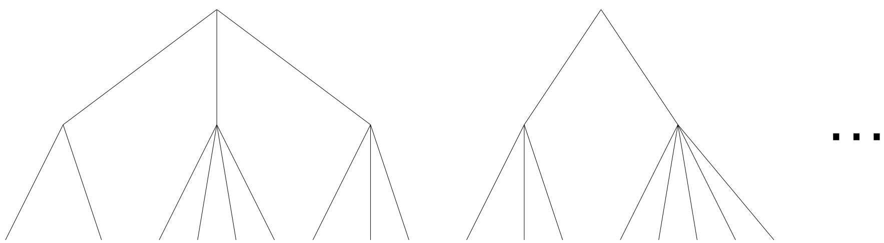
Let $\mathbf{f}_1$, $\mathbf{f}_2$ be vector sequences of bounded dimension, whose entries tend to infinity. Then

| | | |
|---|---|---|
| on infinitely many positions $\mathbf{f}_1$ has vector of higher dimension than corresponding vector in $\mathbf{f}_2$ | $\Leftrightarrow$ | some $\mathbf{g}_1 < \mathbf{f}_1$ is not an asymptotic mix of any $\mathbf{g}_2 < \mathbf{f}_2$ |

We prove the Key Lemma (step 2):

There is an MSO+U formula defining the set of depth-3 forests s.t.
(a) the degree of depth-2 nodes tends to infinity
(b) all but finitely many depth-1 nodes have the same degree.

Forests of depth 3 encode vector sequences:



( 2, 4, 3 ) ( 3 , 5 ) ...

tree = vector
degree of depth-1 node = dimension of vector
degrees of depth-2 nodes = numbers in the vector

It is easy to express (a), and that depth-1 nodes have bounded degree, i.e. dimensions of vectors are bounded, and entries tend to infinity.

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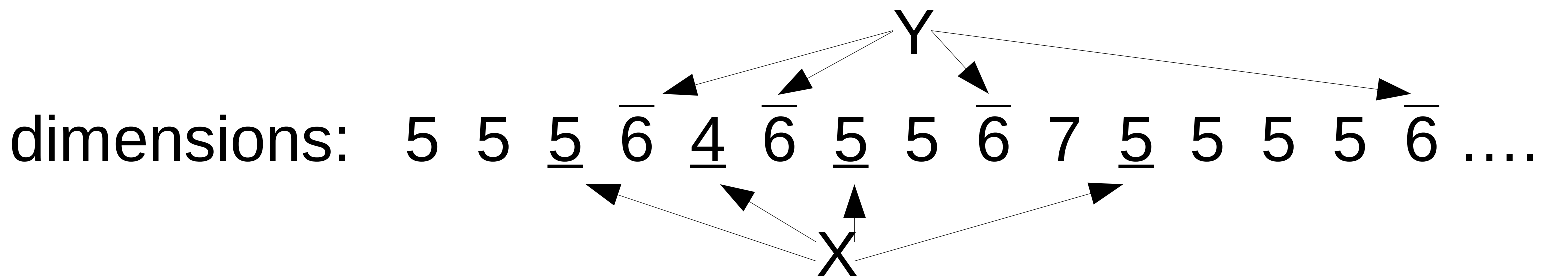
It remains to say that one cannot find two alternating sets X, Y of dimensions (order-1 nodes) such that dimensions in X are smaller than in Y.



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To say this, we use the corollary:

Corollary

Let $\mathbf{f}_1$, $\mathbf{f}_2$ be vector sequences of bounded dimension, whose entries tend to infinity. Then

on infinitely many positions $\mathbf{f}_1$ has vector of higher dimension than $\mathbf{f}_2$ $\Leftrightarrow$ some $\mathbf{g}_1 < \mathbf{f}_1$ is not an asymptotic mix of any $\mathbf{g}_2 < \mathbf{f}_2$

Thank you!