The MSO+U Theory of $(\mathbb{N},<)$ Is Undecidable

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Highlights of Logic, Games and Automata 2016

MSO+U logic

MSO+U extends MSO by the following „U" quantifier:

$$UX.\phi(X)$$

$\phi(X)$ holds for arbitrarily large finite sets

This construction may be nested inside other quantifiers,
and ϕ may have free variables other than X.

MSO+U was introduced by Bojańczyk in 2004

MSO+U logic – separating example

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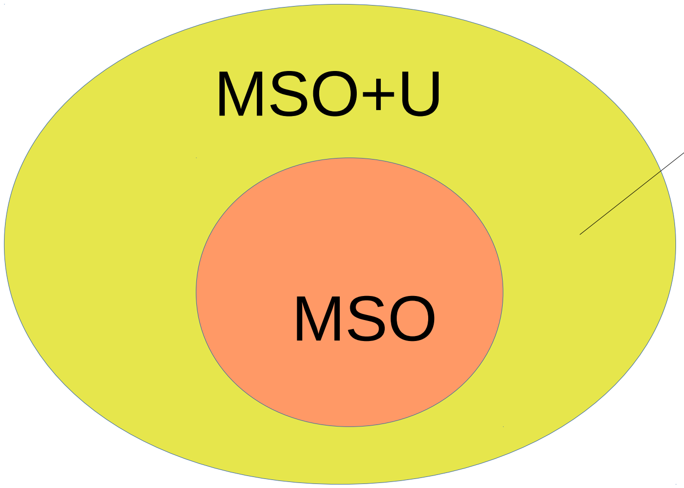


words in $(b^*a)^\omega$ where b^* blocks have unbounded size

$$\{ b^{n_1} a\, b^{n_2} a \ldots \mid \limsup n_i = \infty \}$$

(contains no ultimately periodic word)

MSO+U logic is close to MSO

Fix ϕ and consider the following “Myhill-Nerode” relation:

$v \sim v'$ when for all words $u \in A^*$, $w \in A^\omega$ ($uvw \models \phi \Leftrightarrow uv'w \models \phi$)

This relation has finitely many equivalence classes.

Slogan: The non-regularity of MSO+U is seen only in the asymptotic behavior.

Considered problem

satisfiability for ω-words

Input: formula $\phi \in$ MSO+U

Question: does $w \models \phi$ for some $w \in A^\omega$?

Equivalently:

evaluation over $(\mathbb{N},<)$

Input: formula $\phi \in$ MSO+U

Question: $a^\omega \models \phi$?

Our result: Satisfiability of MSO+U for ω-words is undecidable!

MSO+U logic

Plan of the talk:

1) Some fragments of MSO+U are decidable – *earlier work*

   a) BS-formulas

   b) WMSO+U

2) MSO+U is not decidable over ω-words – ***this work***

Decidable fragments of MSO+U

negation allowed

BS-formulas: boolean combinations of
formulas in which U appears positively

(+ existential quantification outside)

Theorem (Bojańczyk & Colcombet, 2006):
Satisfiability of BS-formulas is decidable over ω-words.

Proof: Uses ωBS-automata.

These are nondeterministic automata with counters that can be incremented and reset to 0, but cannot be read.
Accepting condition: counter is bounded/unbounded.

Decidable fragments of MSO+U

Weak logics: ∃/∀ quantifiers range only over finite sets.

Satisfiability is decidable for:

WMSO+U on infinite words (Bojańczyk, 2009)

WMSO+R on infinite words (Bojańczyk & Toruńczyk, 2009)

R = exists infinitely many sets of bounded size

WMSO+U on infinite trees (Bojańczyk & Toruńczyk, 2012)

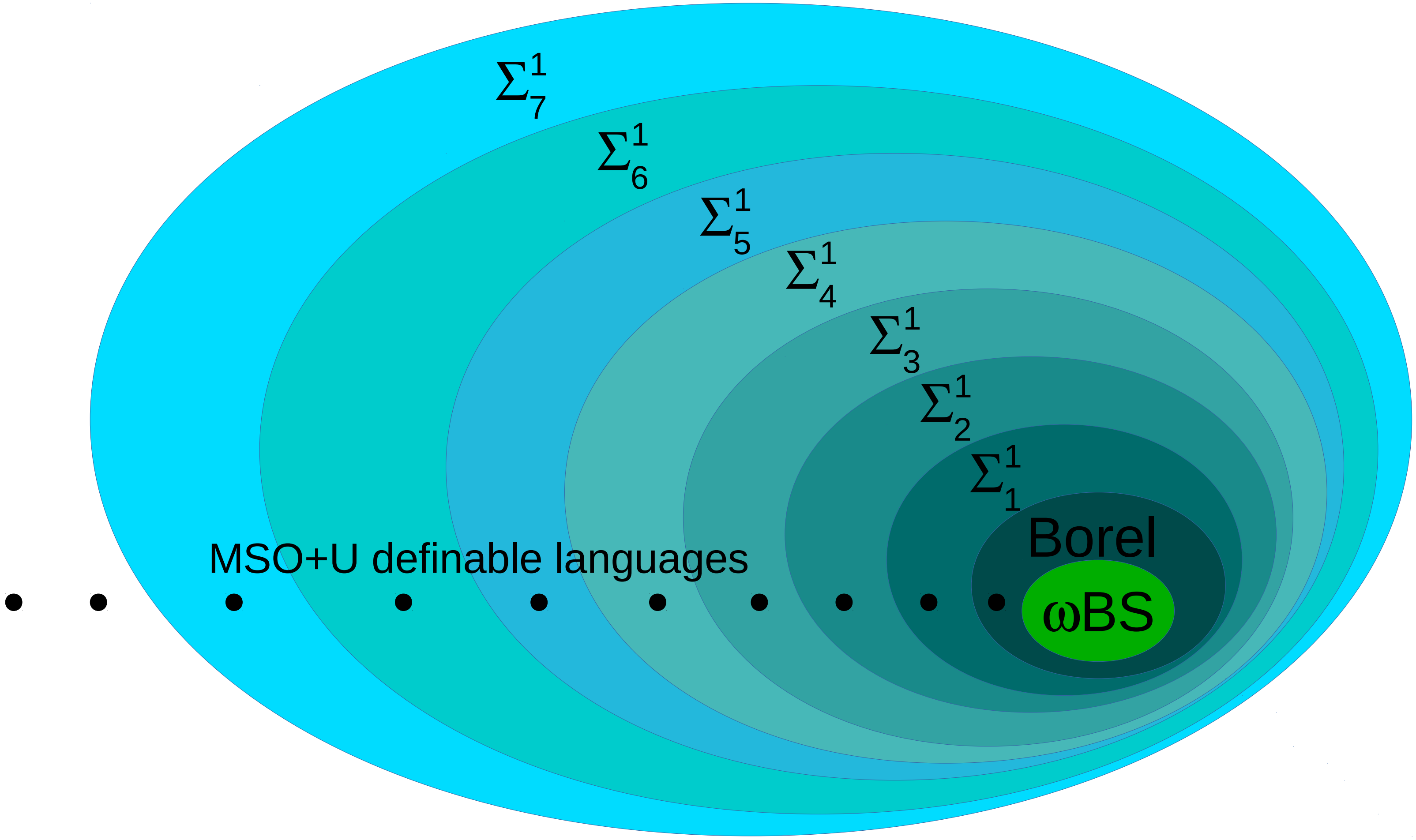
WMSO+U+P on infinite trees (Bojańczyk, 2014)

P = exists path

Proof: equivalent automata models

Undecidability of MSO+U – earlier work

Theorem. (Hummel & Skrzypczak 2010/2012) - topology
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.



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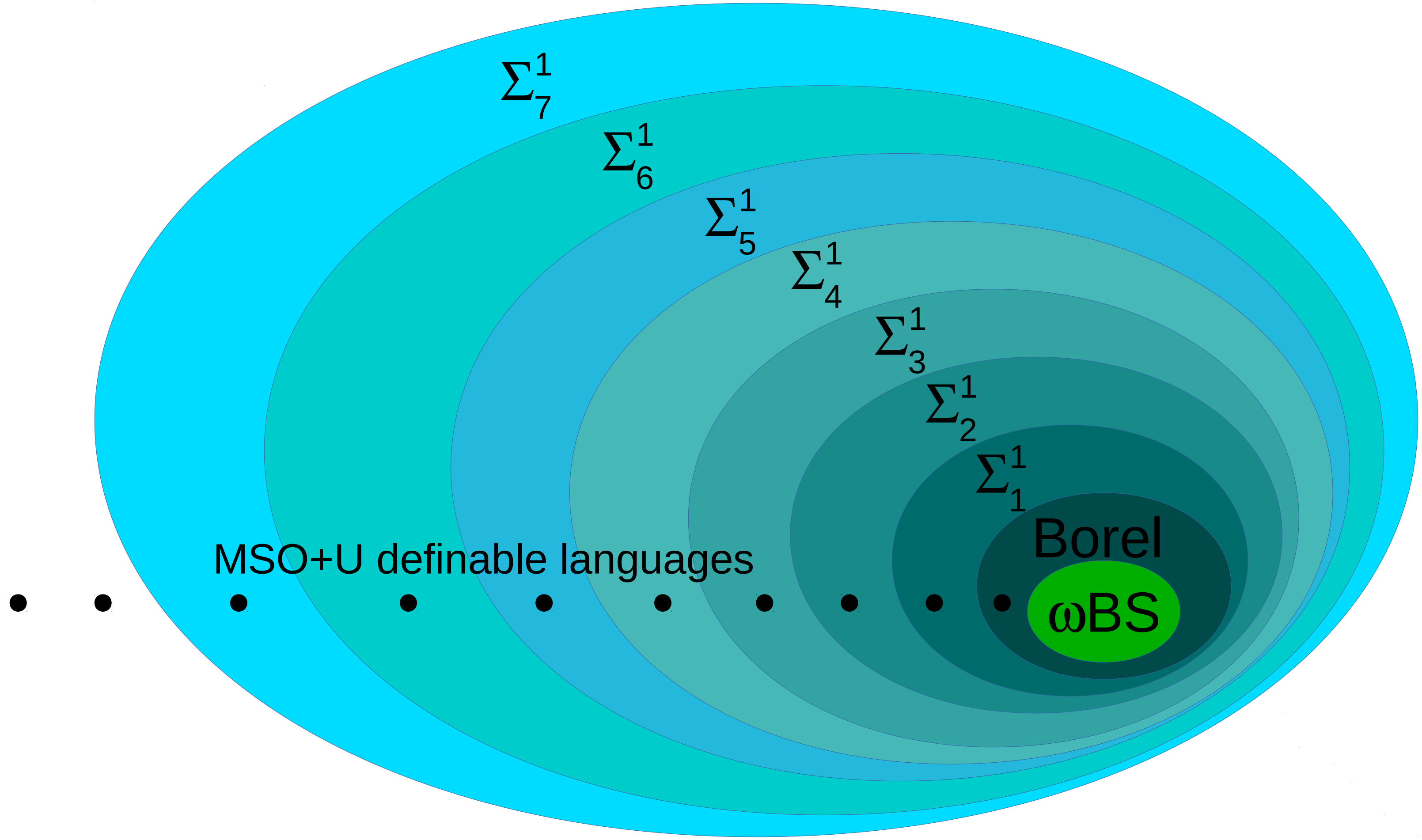


Corollary: MSO+U is not covered by any automata model
(alternating/nondeterm./determ., acceptance condition of bounded projective complexity)

Undecidability of MSO+U – earlier work

Theorem 1. (Hummel & Skrzypczak 2010/2012)
On every level of the projective hierarchy for infinite words, there is a complete language that is definable in MSO+U.

Theorem 2. (Bojańczyk, Gogacz, Michalewski, Skrzypczak 2014)
Satisfiability of MSO+U is not decidable over infinite trees...

...assuming that there exists a projective ordering on the Cantor set 2^{ω}.

assumption of set theory consistent with ZFC

Corollary: No algorithm can decide satisfiability of MSO+U over infinite trees and have a correctness proof in ZFC.

Proof:

Bases on Theorem 1 & the proof of Shelah that MSO is undecidable in 2^{ω}.
Altogether rather complicated.

Undecidability of MSO+U

Thm. (Bojańczyk, P., Toruńczyk 2016)
MSO+U is not decidable over infinite words.

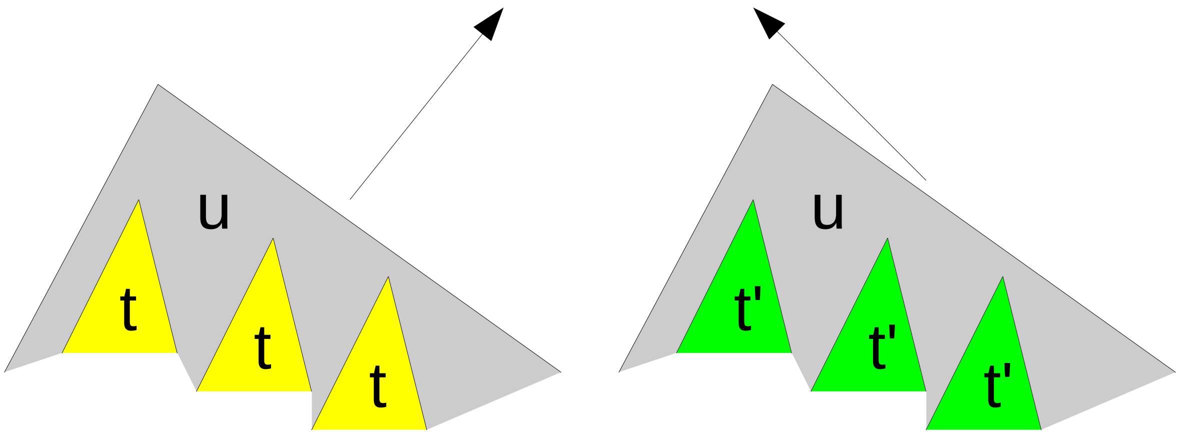
Proof – short & elementary (but too long for 10 minutes).

Details: „The MSO+U Theory of (N, <) Is Undecidable" at STACS 2016

Undecidability of MSO+U – core observation

Consider the following "Myhill-Nerode" relation for trees:

$$t \sim t' \quad \text{when} \quad \text{for all contexts } u \quad (ut \models \phi \Leftrightarrow ut' \models \phi)$$



For some ϕ, this relation has **infinitely** many equivalence classes.

Thank you!