On a Fragment of AMSO and Tiling Systems

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<u>Plan of the talk</u>

1) Tiling Systems
2) Asymptotic Monadic Second-Order Logic (AMSO)

(a fragment of AMSO can be reduced to appropriate tiling systems)

Tiling systems

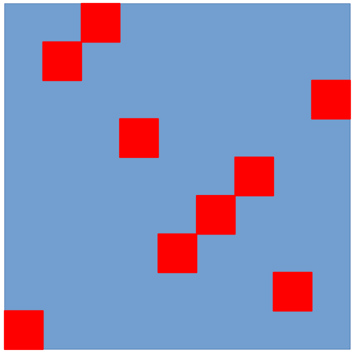
Problem:

Input: regular languages K, L

Question: $\forall\, n \in \mathbb{N}$, there exists a rectangle of height n with all Kolumns in K and all Lines in L?

Example: K = L = {words with at exactly one 'a'}

Answer: yes



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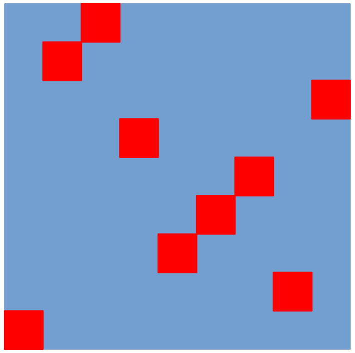
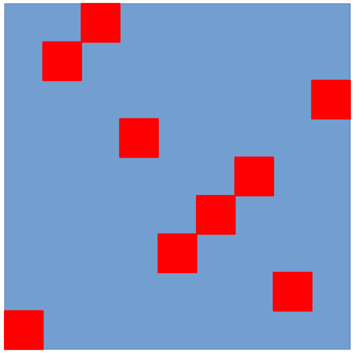


Observation: This problem is undecidable.

Lossy tiling systems

Problem:
Input: regular languages K, L where **K is closed under letter removal**
Question: ∀ $n \in \mathbb{N}$, there exists a rectangle of height n
with all Kolumns in K and all Lines in L?

Example: L = {words with exactly one 'a'}
K = {words with **at most** one 'a'}
Answer: yes



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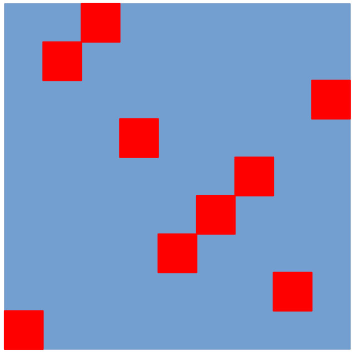
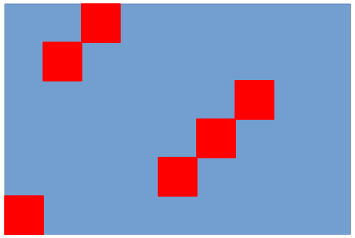




Observation: removing lines from a solution gives a solution,

In this example: every solution of height n has width ≥n.

Symmetric lossy tiling systems

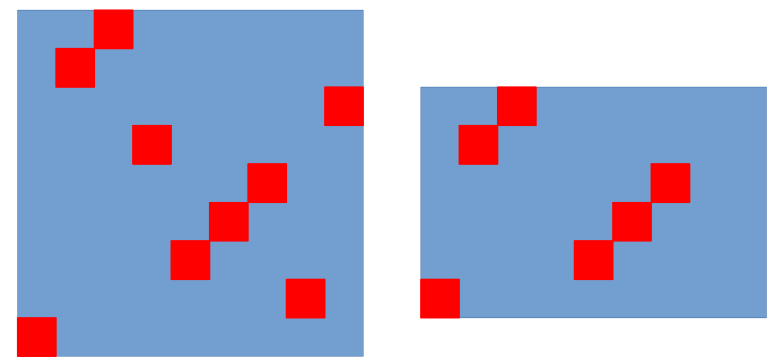
Problem:

Input: regular languages K, L where K is closed under letter removal and **under permutations of letters**

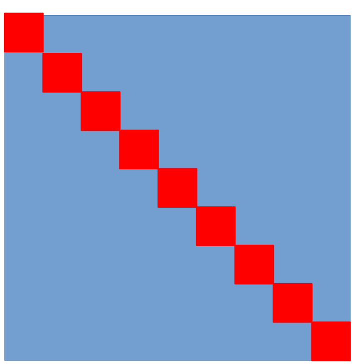
Question: $\forall\ n \in \mathbb{N}$, there exists a rectangle of height n with all Kolumns in K and all Lines in L?

Example: L = {words with exactly one 'a'}
K = {words with at most one 'a'}

Answer: yes



Observation: removing lines from a solution gives a solution,
permuting lines in a solution gives a solution.



In this example: every solution of height n has width $\geq$n.

Contribution

Thm.
Symmetric lossy tiling problem is decidable.

Is the (non-symmetric) lossy tiling problem decidable? - open

Symmetric lossy tiling systems

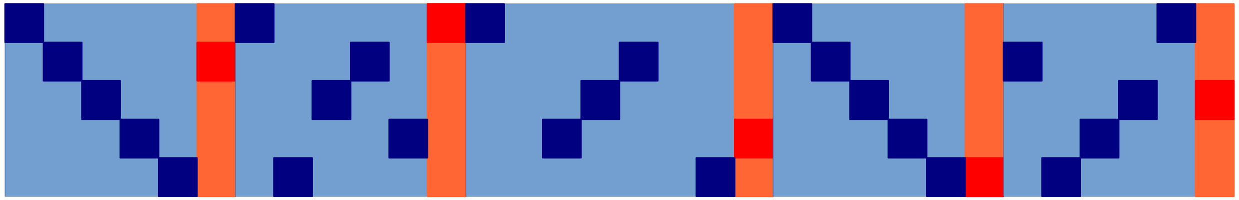
Another example:

L = $((d^*cd^*)^*(a+b))^* \cap (b+c+d)^*a(b+c+d)^*$

exactly one c between any two a / b & exactly one a

K = $d^*c^?d^* \cup b^*a^?b^*$

either many d and at most one c, or many b and at most one a



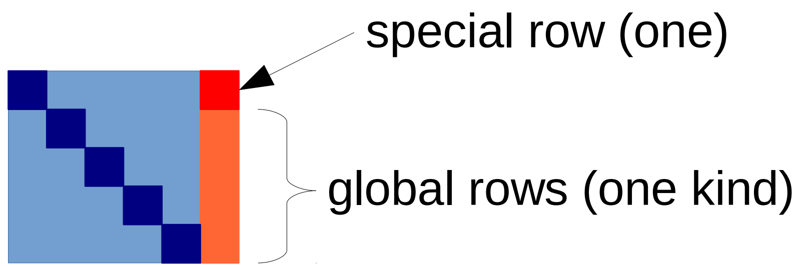
In this example: every solution of height n has width $\geq n^2$

Symmetric lossy tiling systems – decision procedure

General idea

Solution to every instance is a „generalization” of our examples.

We generate some images that can be part of a solution.
They are of this form:



We have:
- some number of *special rows*
- some number of kinds of global rows,
  global rows of each kind can be repeated as many times as we want

We use monoid for L – every row is characterized by its value in this monoid

Symmetric lossy tiling systems – decision procedure

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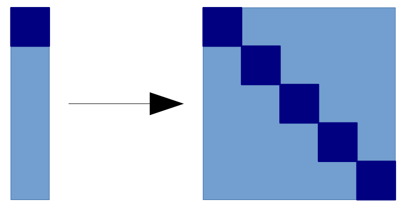
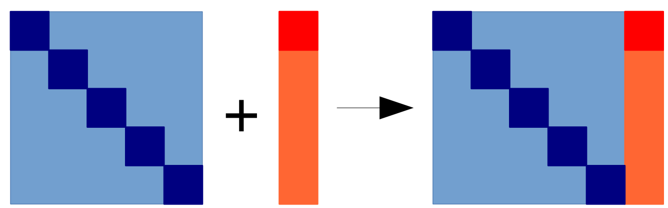
Possible operations:

- diagonal schema



(assumption: )

- product schema



Thm. If a solution exists ∀n, it can be generated in at most C steps,
using in meantime images with at most C special rows, and at most C kinds of global rows.

Symmetric lossy tiling systems – decision procedure

General idea

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We generate some images that can be part of a solution.
Possible operations:

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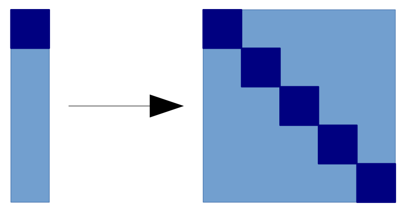

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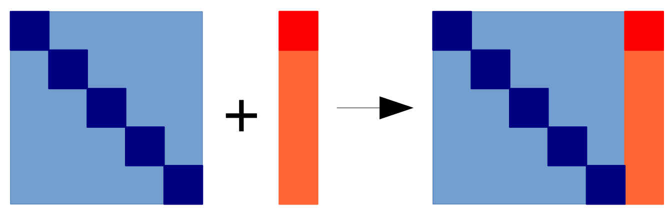
- product schema



Thm. If a solution exists ∀n, it can be generated in at most C steps,
using in meantime images with at most C special rows, and at most C kinds of global rows.

Proof. We develop a new generalization of the factorization forests theorem of Simon.

Non-symmetric lossy tiling systems (decidability open)

Example:

L = a1*+(b1*a1*)*

a and b are alternating after ignoring all 1 & at least one a

K = b*$a^{?}$1*

first some b, then at most one a, then some 1

In this example: every solution of height *n* has width ≥2^{n}-1

(not covered by our algorithm)

Asymptotic Monadic Second-Order Logic

(introduced by Blumensath, Carton & Colcombet, 2014)

Logic	MSO+U	AMSO
Idea	verification of asymptotic behavior (something is bounded / unbounded)	
Structure	ω-words	weighted ω-words (a number is assigned to every position)
Quantities to be measured	set sizes (arbitrary quantities)	weights

Asymptotic Monadic Second-Order Logic

Def. AMSO = MSO extended by:

- quantification over number variables $\exists s\ \forall r$
- construction $f(x) \leq s$ appearing positively if s quantified existentially (negatively if s quantified universally)

Examples:

- weights are bounded: $\exists s \forall x\, (f(x) \leq s)$
- weights $\rightarrow \infty$: $\forall s \exists x\, (\forall y > x)\, (f(y) > s)$
- ∞ many weights occur ∞ often: $\forall s \exists r \forall x\, (\exists y > x)(s < f(y) \leq r)$

Considered problem – satisfiability

Input: $\phi \in$ AMSO

Question: $\exists w\, (w \models \phi)$?

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undecidable for MSO+U $\Rightarrow$ undecidable for AMSO

Asymptotic Monadic Second-Order Logic

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undecidable for MSO+U $\Rightarrow$ undecidable for AMSO

What about fragments of AMSO?

We have reductions: (no number quantifiers in ψ)

decidable!!!

$\exists r \forall s \exists t\ \psi(r,s,t)$
only $s<f(y)\leq t$ allowed → symmetric lossy tiling system

$\exists r \forall s \exists t\ \psi(r,s,t)$ → lossy tiling system

number quantifiers $\psi(\ldots)$ → multi-dimensional lossy tiling system

Conjecture: satisfiability decidable for these fragments.

Thank you!