A Pumping Lemma for Pushdown Graphs of Any Level

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Higher order pushdown systems/automata [Maslov 74, 76]

A 1-stack is an ordinary stack. A 2-stack (resp. (n + 1)-stack) is a stack of 1-stacks (resp. n-stack).

Operations on 2-stacks: s_i are 1-stacks. Top of stack is on right.

$push_2$	:	$[s_1 \ldots s_{i-1} s_i]$	->	$[s_1 \ldots s_{i-1} s_i\, s_i]$
pop_2	:	$[s_1 \ldots s_{i-1} s_i]$	->	$[s_1 \ldots s_{i-1}]$
$push_1 x$	:	$[s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j]]$	->	$[s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j\, x]]$
pop_1	:	$[s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j]]$	->	$[s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1}]]$

An **order-n PDA** has an order-n stack, and has $push_i$ and pop_i for each $1 \leq i \leq n$.

Higher order pushdown systems

Higher order pushdown systems can be used as:

- word language recognizers
- tree generators
- graph generators

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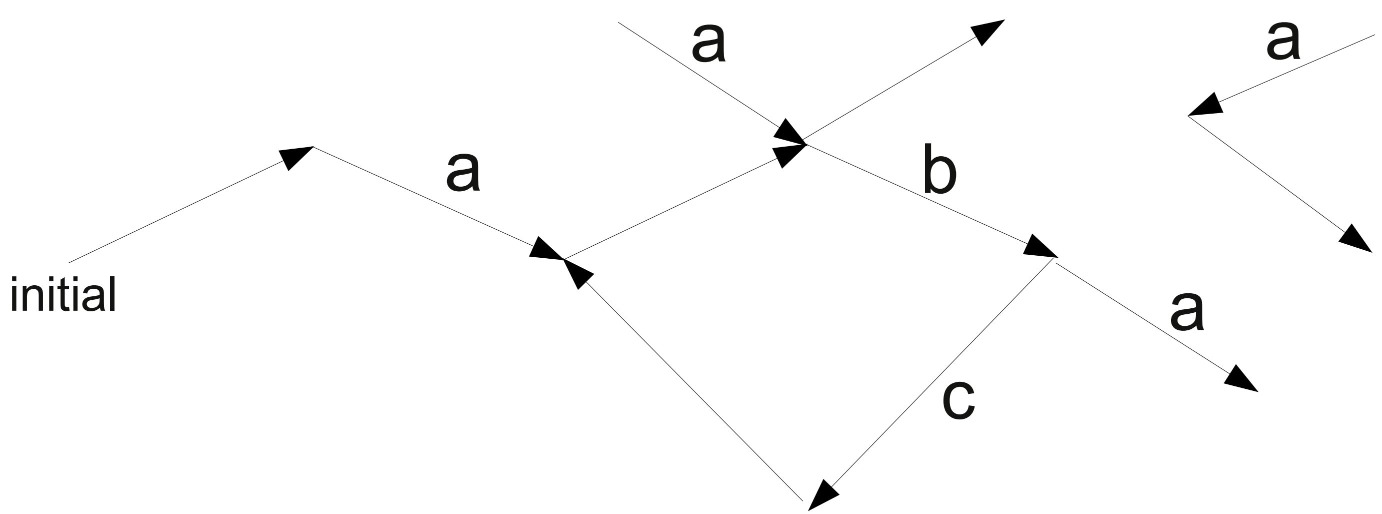
We concentrate here on graph generators.

- The same results can be used for tree generators and deterministic word language recognizers,
- but NOT for (nondeterministic) word language recognizers.

Higher order pushdown graphs

How higher order pushdown systems generate graphs?

We consider ε-contractions of configuration graphs.



- drop unreachable configurations
- nodes = configurations after letter-edges
- edges = any number of epsilons, one letter

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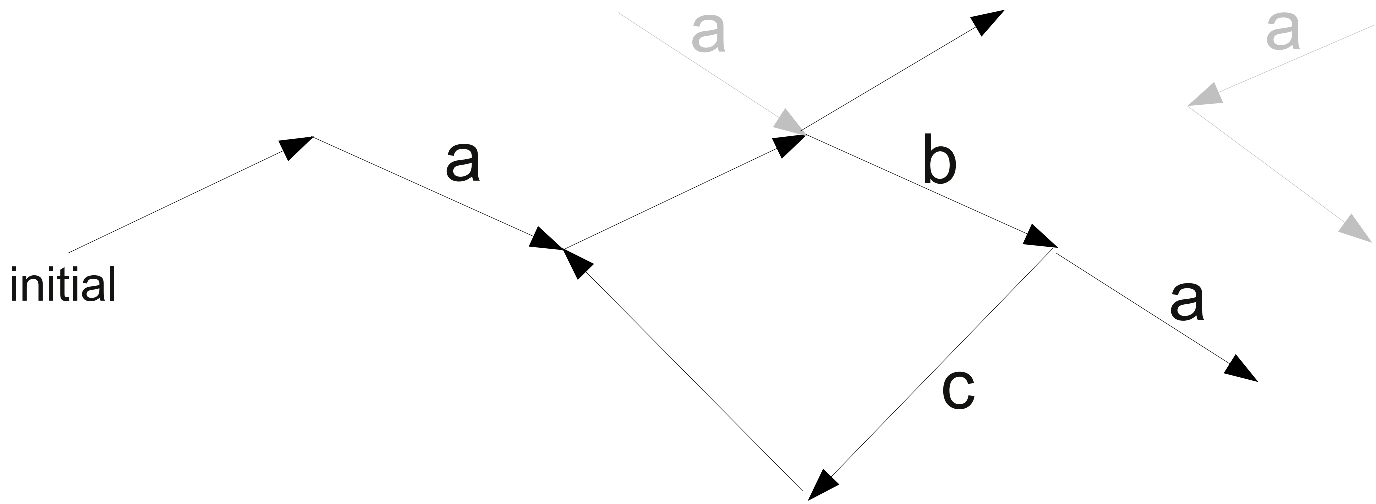

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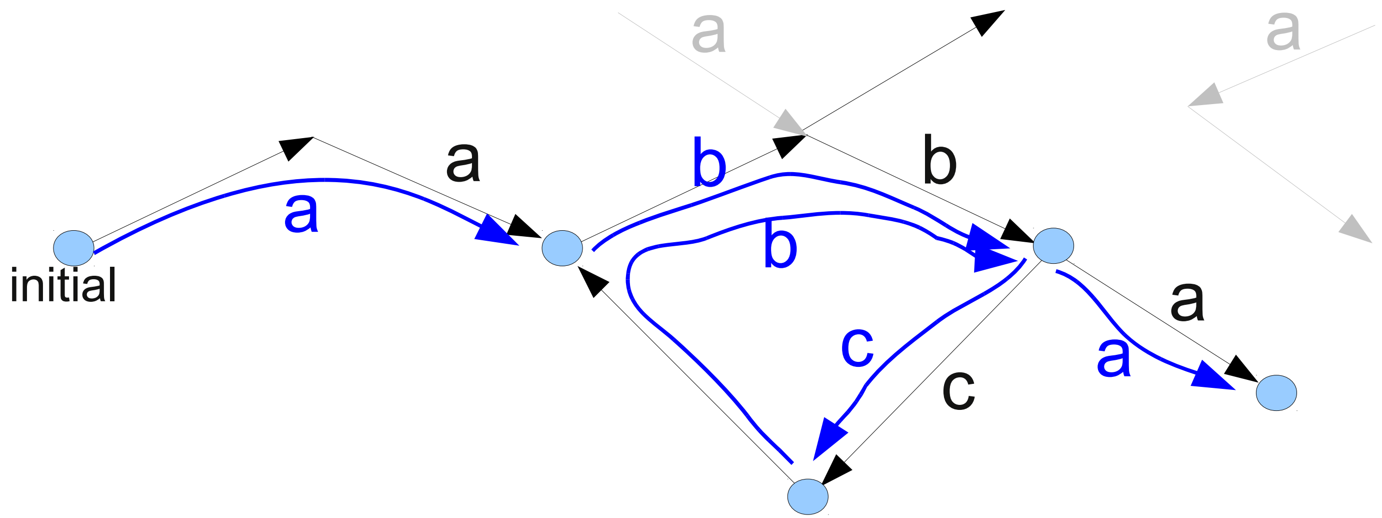

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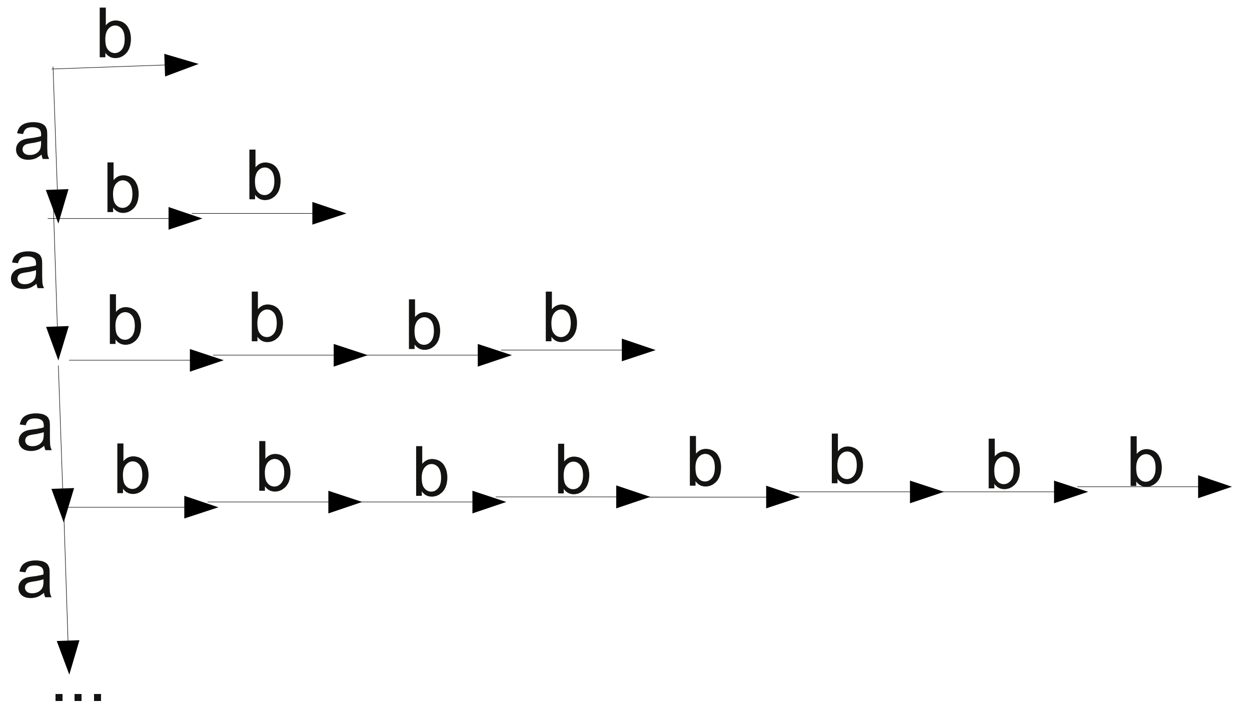
Example: $\{a^k b^m : m \leq 2^k\}$



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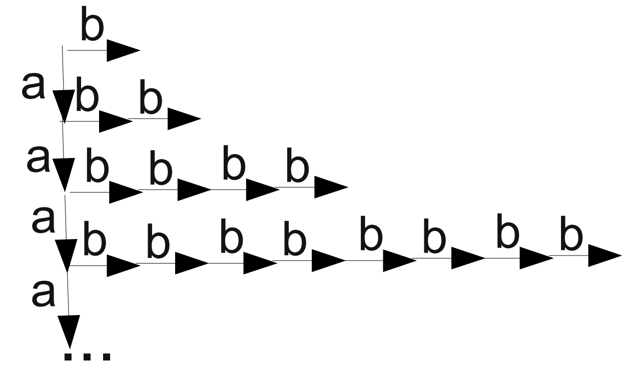
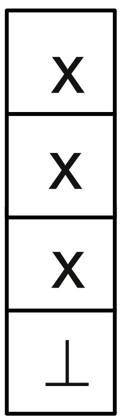
Example: $\{a^k b^m : m \leq 2^k\}$

- level 2
- 3 stack symbols: ⊥, x, #



$(?, q_1, a, push_1(x), q_1)$



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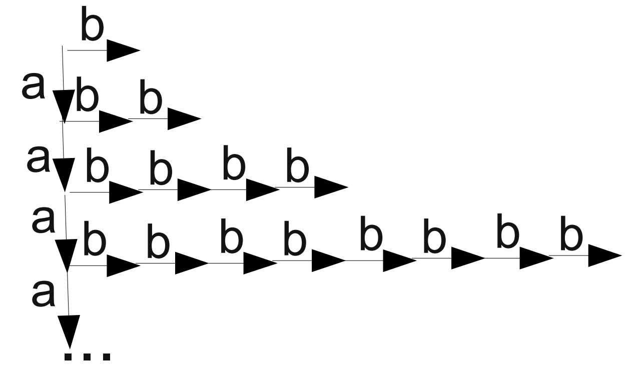

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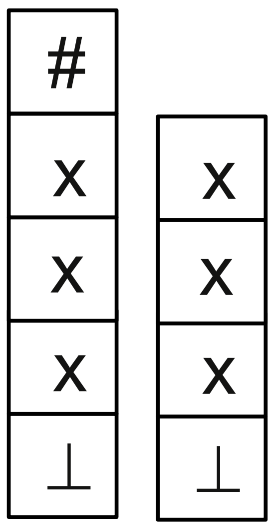

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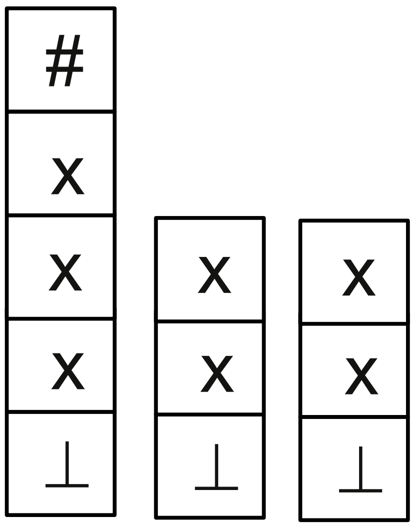

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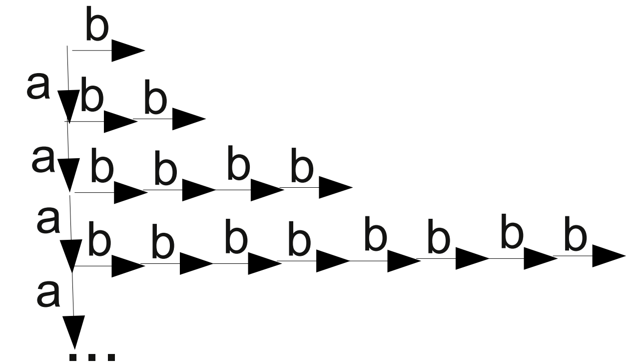

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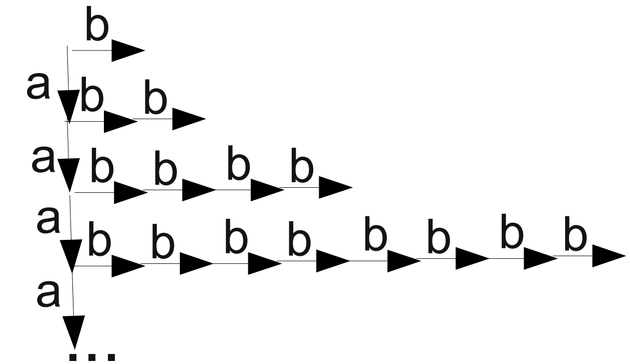

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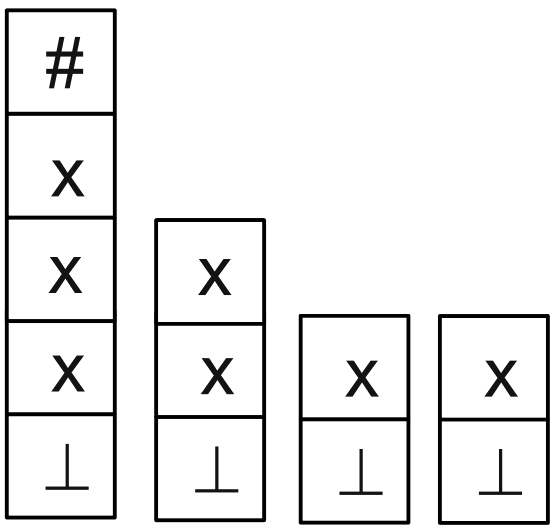

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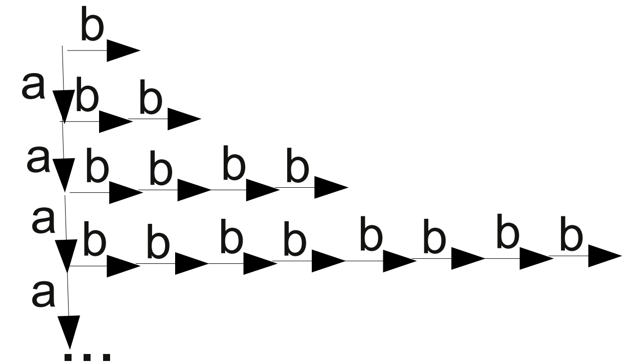

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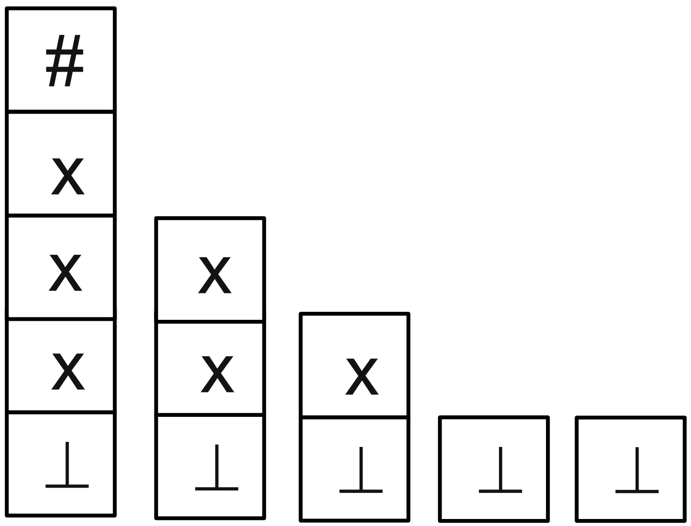

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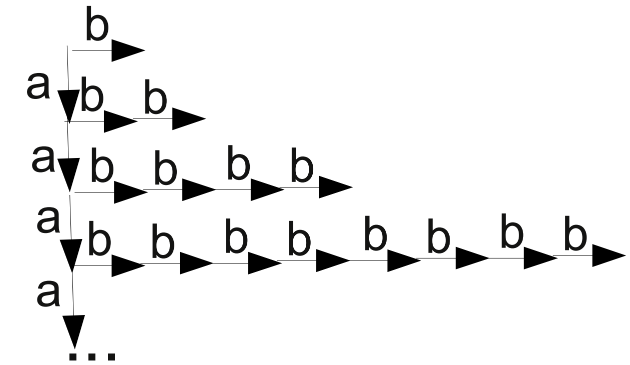

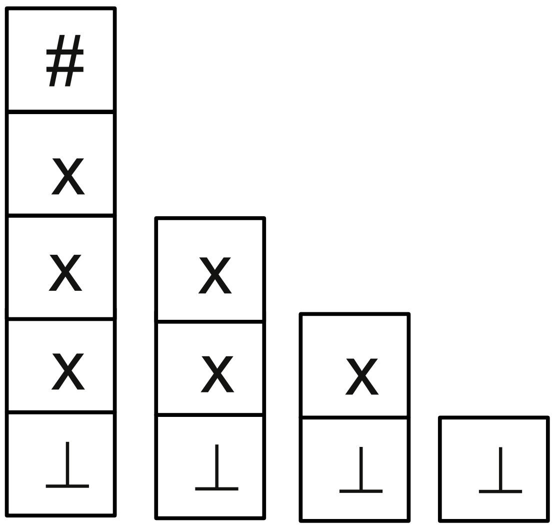

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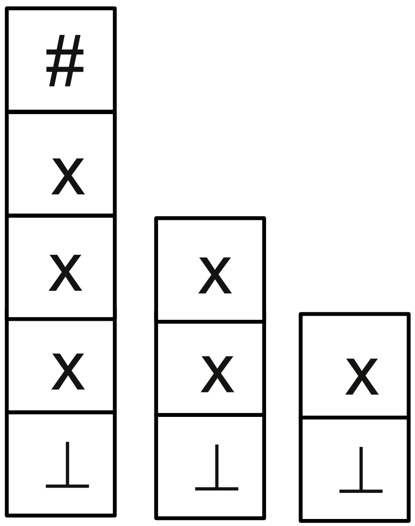

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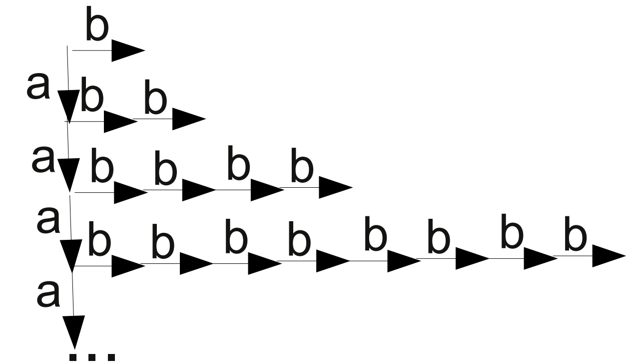

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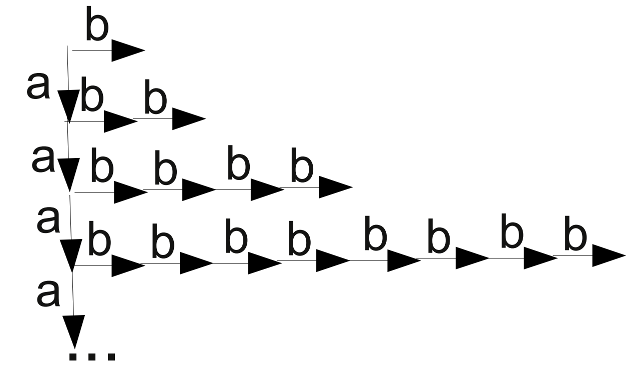

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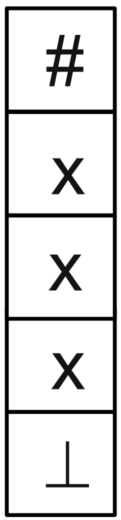

Input: a a a b b b b b b b b

stack with k letters x $\Rightarrow$ 2^k letters b

proof: stack with 0 letters x $\Rightarrow$ 2^0 letters b

stack with k letters x $\Rightarrow$ 2 stacks with k-1 letters x

Higher order pushdown graphs

Example:

$\{a^k b^m : m \leq 2^k\}$ - system of level 2

Similarly:

$\{a^k b^m : m \leq 2^{2^k}\}$ - system of level 3

$\{a^k b^m : m \leq 2^{2^{2^k}}\}$ - system of level 4

....

Higher order pushdown systems

Other characterizations:

- trees generated by safe recursion schemes of level n (Knapik, Niwiński, Urzyczyn 2002)

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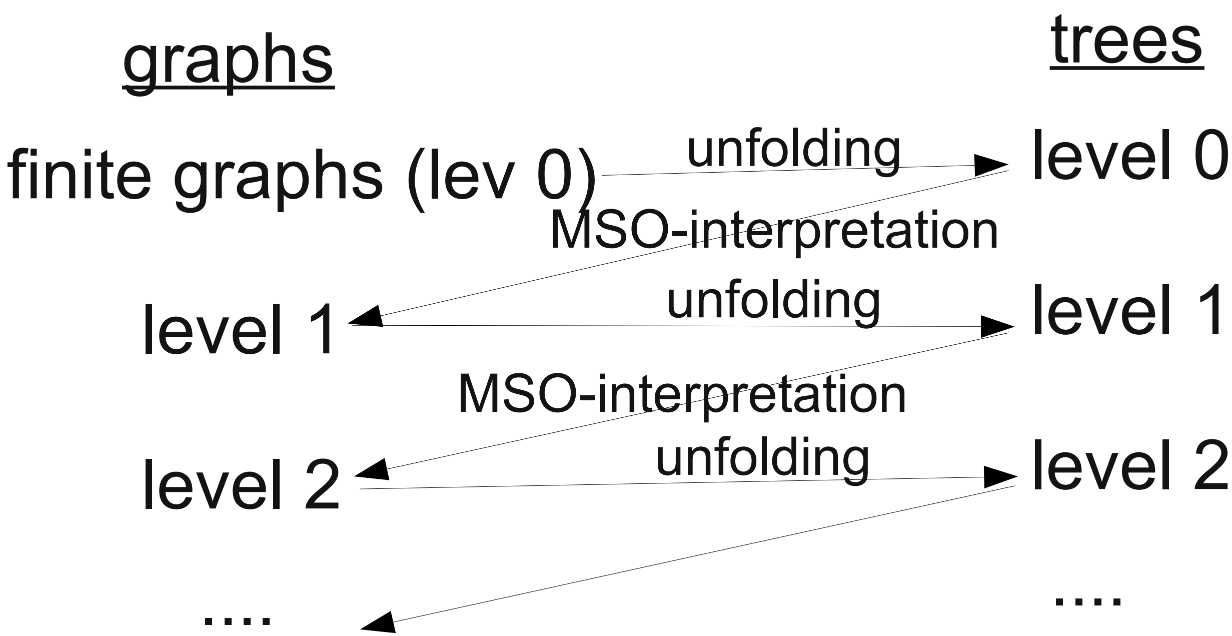
- Caucal hierarchy:



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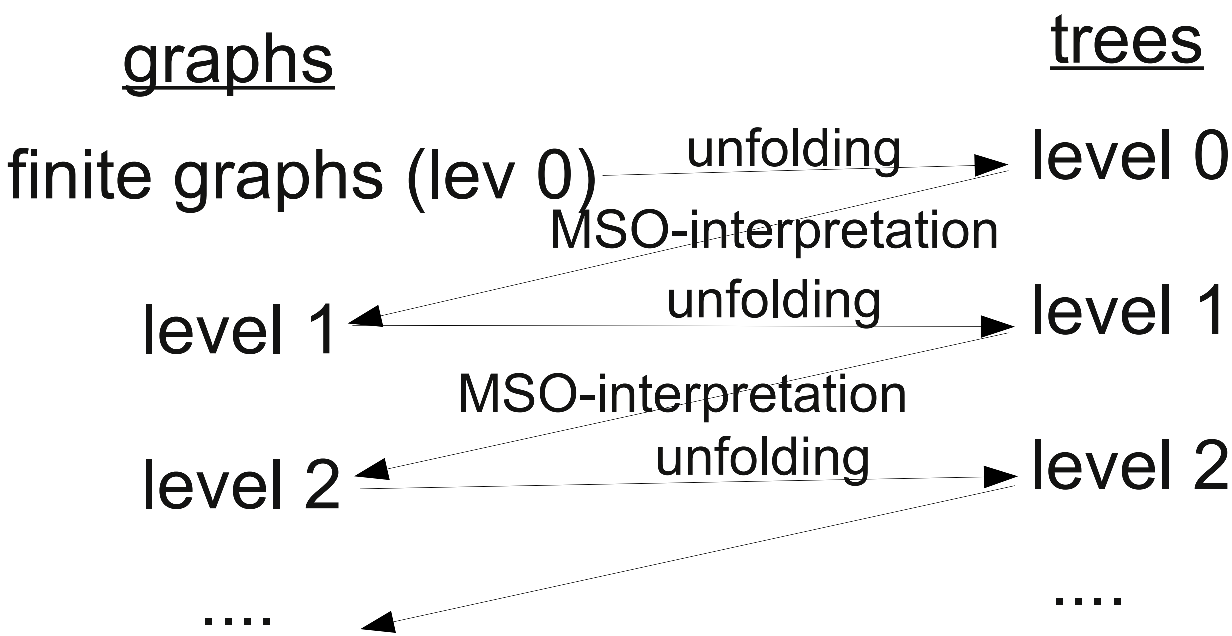

- they have decidable MSO theory (both trees and graphs)

Higher order pushdown graphs

But is a given graph generated by a level-n pushdown system?

YES
give example system

NO
a pumping lemma
would be useful

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Known pumping lemmas for level 2:

- Hayashi (1973) – pumping lemma for word languages
- Gilman (1996) – shrinking lemma for word languages
- Kartzow (2011) – pumping lemma for (collapsible) graphs

For arbitrary level:

- ~~Blumensath (2008) – pumping lemma for graphs~~

incorrect proof

Higher order pushdown graphs

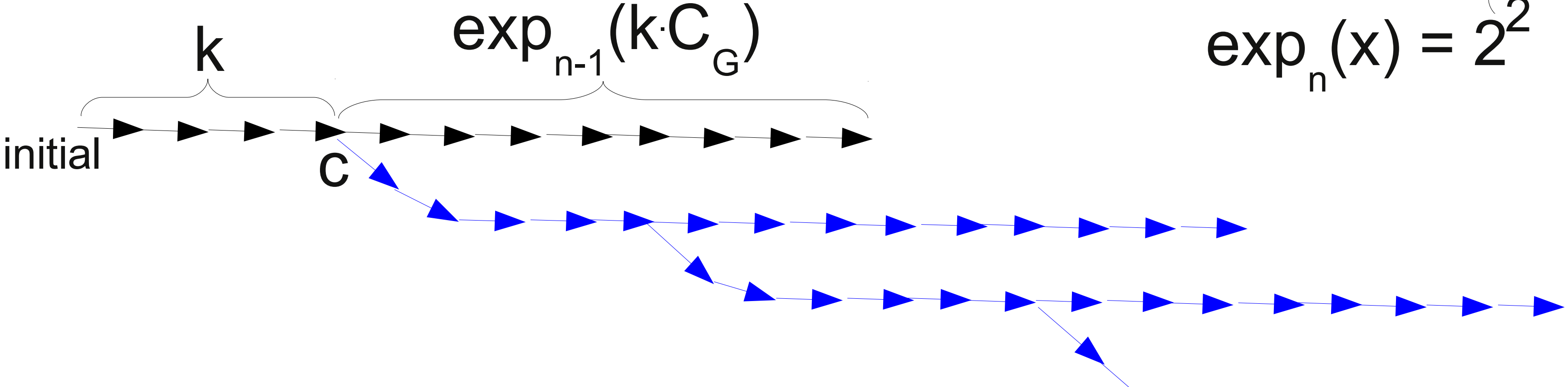
Our pumping lemma:

G – finitely-branching pushdown graph of level n

Then there exists a constant C_G such that:

if c – configuration reachable by k edges from the initial one such that a path of length $exp_{n-1}(k \cdot C_G)$ starts in c

then there exist arbitrary long paths starting in c



$$exp_n(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}} \text{ (n)}$$

Higher order pushdown graphs

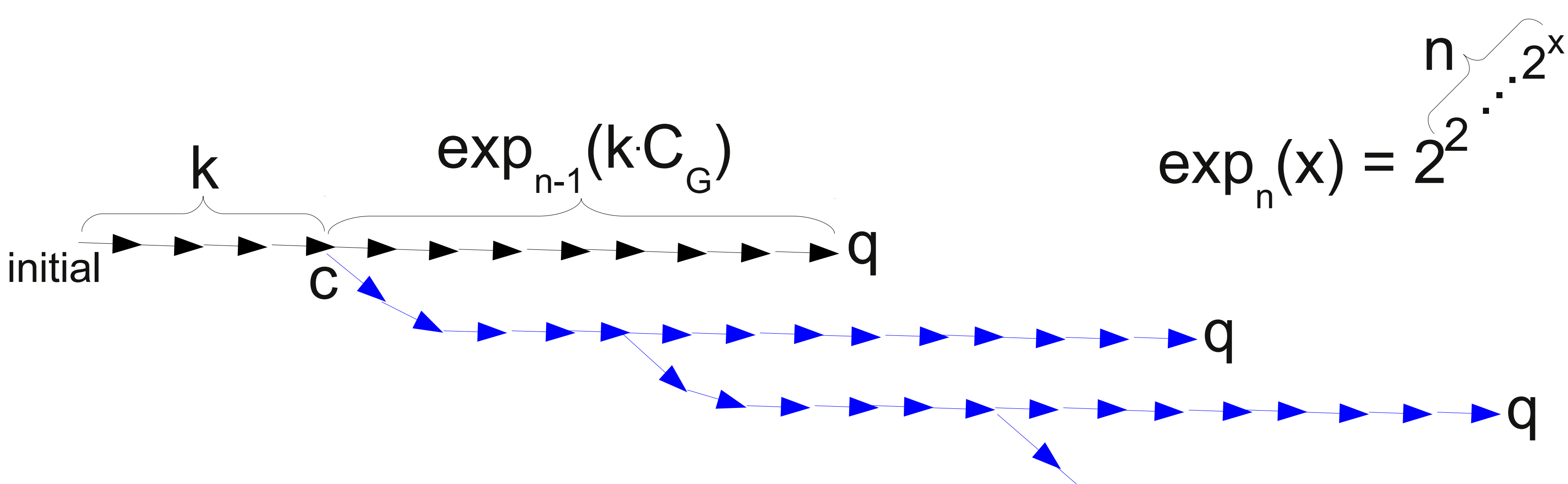
Our pumping lemma:

G – finitely-branching pushdown graph of level n

Then there exists a constant C_G such that:

if c – configuration reachable by k edges from the initial one such that a path of length $\exp_{n-1}(k \cdot C_{GL})$ starts in c

then there exist arbitrary long paths starting in c, ending in the same state



Higher order pushdown graphs

Our pumping lemma:

G – finitely-branching pushdown graph of level n

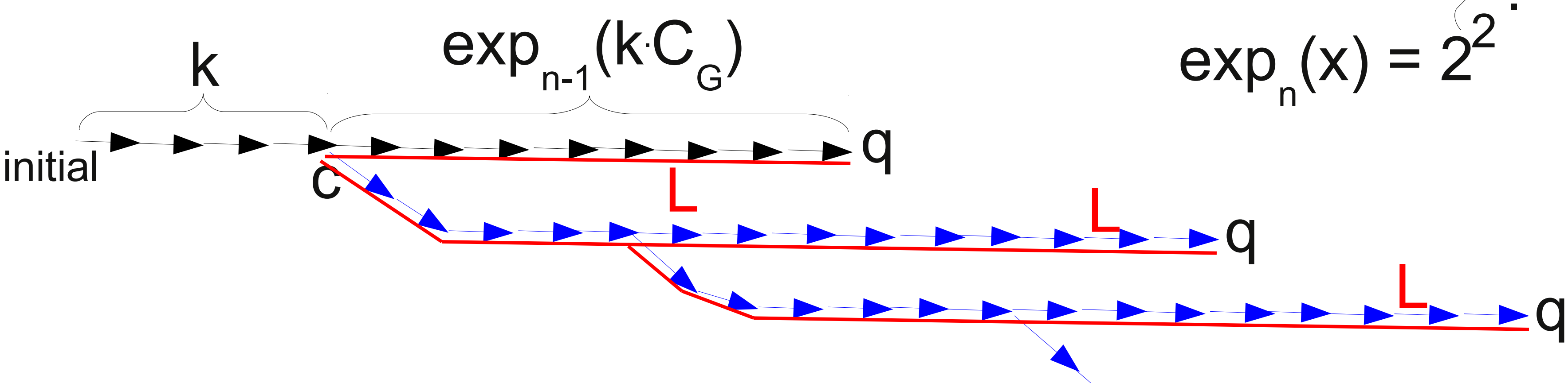
L – regular language

Then there exists a constant C_{GL} such that:

if c – configuration reachable by k edges from the initial one such that a path from L of length $\exp_{n-1}(k \cdot C_{GL})$ starts in c

then there exist arbitrary long paths from L starting in c, ending in the same state

$$\exp_n(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}} \quad (n \text{ twos})$$



Higher order pushdown graphs

Example:

$\{a^k b^m : m \leq exp_{n-1}(k)\}$ - is a graph of level n

$\{a^k b^m : m \leq exp_{n-1}(f(k) \cdot k)\}$ - is NOT a graph of level n if f(k) unbounded

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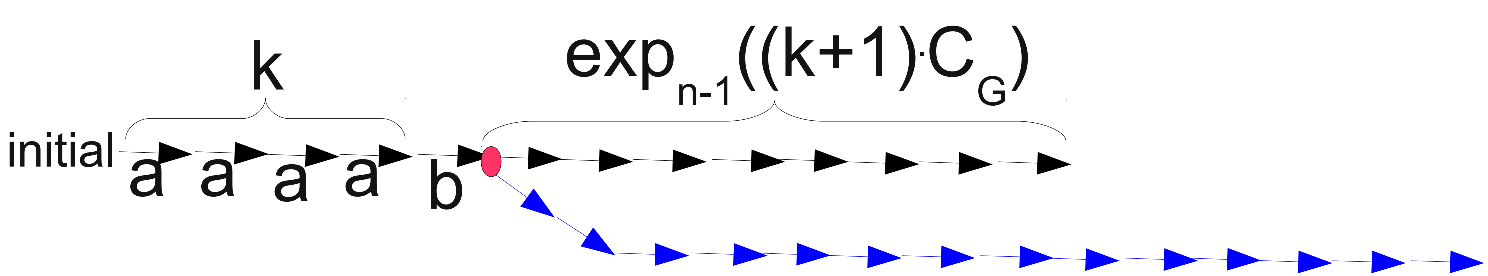
Proof: Choose k such that

$$exp_{n-1}(f(k) \cdot k) - 1 \geq exp_{n-1}((k+1)C_G)$$



Higher order pushdown graphs

Our pumping lemma – more insight:

G – finitely-branching pushdown graph of level n

part 1

c – configuration reachable by m edges from the initial one.

Then the size of every k-stack of c is at most $\exp_{k-1}(m \cdot C_{GL})$

part 2

c – configuration such that the size of every k-stack of c is at most $\exp_{k-1}(m \cdot C_{GL})$, and a path of length $\exp_{n-1}(k \cdot C_G)$ starts in c

Then there exist arbitrary long paths starting in c.

“Types” of stacks

We define a homomorphism from stacks to a finite algebra having:

- n+1 sorts (for levels 0, 1, ..., n)
- operations: empty_k : level-k
  compose_k : level-k $\times$ level-(k-1) $\rightarrow$ level-k

putting level-(k-1) stack on top of level-k stack

(for level-1 systems this homomorphism is a finite automaton)

type(c) says (for example):

- can we reach from c to a configuration with state q ?
- is there a run from c reading letter 'a' ?
- in which state can we remove the topmost k-stack ?
- can we reach a “bigger” configuration having the same type ?

Furure work (together with A.Kartzow)

The same results hold for collapsible pushdown graphs.

This implies that the hierarchy of collapsible pushdown graphs is strict (a new result).

Collapsible PDS are an extension of a higher-order PDS

$push_1(x)$ pushes not only the x symbol, but also a fresh marker

new operation: $collapse_k$ – removes all those (k-1)-stack from the topmost k-stack, which contain the marker present in the topmost symbol

Open problems

1) Describe more precisely how the arbitrarily long paths are created from the input path.

2)
- A shrinking lemma.
- A lemma applicable to infinitely-branching graphs.
- A lemma applicable to word languages.