The pumping lemma is incorrect?

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We will show that Lemma 9.5 in [1] is false. This lemma says that in each long enough run r of any automaton there exists a *pumping pair* of configurations u, v. From the definition of a pumping pair we use only the following:

- u is (strictly) before v in the run,
- $\rho r(u) = \rho r(v)$ (the state in u and in v is the same),
- $\pi r(u) \lhd_1 \pi r(v)$.

Consider an automaton $\mathcal{A}$ of level 3, which realizes the following program:

```
repeat forever
    push_2
    while topmost symbol is a do
        pop_1
        push_3
    pop_2
    push_1(a)
    push_3
```

The stack alphabet is $\{a, b\}$, where b is used only to mark the bottom of the stack. The automaton does not read any input (it has only ϵ-transitions). Take the initial configuration $[[[ba]]]$ (one order 1 stack with b at the bottom and a above). Started from it, the automaton has exactly one infinite run. Hence from Lemma 9.5 there is a pumping pair u, v in it. We will show that this is not true.

How our automaton works? First observe that it never makes any pop_3 operation. Hence only the topmost order 2 stack is accessed. By making a push_3 operation we keep a history of the current contents of the topmost order 2 stack.

Now observe how the topmost order 2 stack changes. It always contains either one or two order 1 stacks. The first of them is only increased (once per iteration of the big loop). Then it is copied, and the second stack is decreased until it becomes empty.

Assume we have a pumping pair u, v (u is before v). Let $\pi r(u) = \xi_1 \dots \xi_k$ and $\pi r(v) = \zeta_1 \dots \zeta_l$. The configurations u, v have the same state, which means that we are in the same point of the program. Assume first that this is after the push_2 operation but before the pop_2 operation. Thus, there are two order 1 stacks in ξ_k and in ζ_l. If u and v come from one iteration of the big loop, then the second (topmost) order 1 stack of ξ_k is bigger than the second order 1 stack of ζ_l, hence they are not in the $\lhd_1$ relation. Otherwise some ζ_i for $k \leq i < l$ contains just one order 1 stack (as the push_3 from the last line was executed), hence $\xi_k \lhd_1 \zeta_i$ is false.

The other possibility is that both ξ_k and ζ_l contain just one order 1 stack (we are after pop_2, but before push_2). Then ζ_{l-1} contains two stacks, and the second of them contains only the b letter, while the order 1 stack in ξ_k contains also a letters, hence we do not have $\xi_k \lhd \zeta_{l-1}$. This shows that a pumping pair does not exists.

References

[1] A. Blumensath. On the structure of graphs in the caucal hierarchy. *Theor. Comput. Sci.*, 400(1-3):19–45, 2008.