The pumping lemma is incorrect?

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We will show that Lemma 9.5 in [1] is false. This lemma says that in each long enough run r of any automaton there exists a *pumping pair* of configurations u, v. From the definition of a pumping pair we use only the following:

- u is (strictly) before v in the run,
- $\rho r(u) = \rho r(v)$ (the state in u and in v is the same),
- $\pi r(u) \lhd_1 \pi r(v)$.

Consider an automaton $\mathcal{A}$ of level 3, which realizes the following program:

repeat forever
push$_2$
push$_3$
pop$_1$
push$_3$
pop$_2$
push$_3$

Thus it has 6 states and one loop of transitions between them. The stack alphabet contains only the a symbol. The automaton does not read any input (it has only ϵ-transitions). Take the initial configuration $[[[aa]]]$ (one order 1 stack with two symbols). Started from it, the automaton has exactly one infinite run. Hence from Lemma 9.5 there is a pumping pair u, v in it. We will show that this is not true.

How our automaton works? First observe that it never makes any pop$_3$ operation. Hence only the topmost order 2 stack is accessed. By making a push$_3$ operation we keep a history of the current contents of the topmost order 2 stack.

Now observe how the topmost order 2 stack changes. It has tree possible contents, between which we loop:

$$\begin{aligned} x =& [[aa]], \\ y =& [[aa][aa]], \\ z =& [[aa][a]]. \end{aligned}$$

We have $x \lhd_1 y$ and $z \lhd_1 y$, but x and z are $\lhd_1$-incomparable.

Assume we have a pumping pair u, v (u is before v). Let $\pi r(u) = \xi_1 \dots \xi_k$ and $\pi r(v) = \zeta_1 \dots \zeta_l$. The configurations u, v have the same state, which means that $\xi_k = \zeta_l$. As v is strictly after u, there is $l \geq k + 3$. Because $\pi r(u) \lhd_1 \pi r(v)$, it has to be

$$\xi_k \lhd_1 \zeta_k \qquad \text{and} \qquad \xi_k \lhd_1 \zeta_{k+1} \qquad \text{and} \qquad \xi_k \lhd_1 \zeta_{k+2}.$$

We know that $\zeta_1, \zeta_2, \ldots, \zeta_{l-1} = x, y, z, x, y, z, \ldots$ (and ζ_l is either equal to ζ_{l-1}, or is the next symbol). This means that

$$\xi_k \lhd_1 x \qquad \text{and} \qquad \xi_k \lhd_1 y \qquad \text{and} \qquad \xi_k \lhd_1 z.$$

But none of x, y, z satisfies this. Hence there is no pumping pair.

References

[1] A. Blumensath. On the structure of graphs in the caucal hierarchy. *Theor. Comput. Sci.*, 400(1-3):19–45, 2008.