The problems of reachability in a Petri and emptiness of intersection of commutative languages are equivalent

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As a Petri net we understand a Petri net with given initial and final configuration, and with transitions labeled by letters from some alphabet Σ. A word $w \in \Sigma^*$ is accepted by a Petri net when there is a sequence of transitions leading from the initial to the final configuration such that the letters on that transitions form the word w. A language accepted by a Petri Net N, denoted $L(N)$, consists of all such words. The well known reachability problem can be stated as: ,,is $L(N)$ empty?" (a typical presentation does not use the language, but the equivalence is obvious).

On the other hand we have commutative languages. We will not go into the original definition; it is easy to see that equivalently they can be defined as languages of the form $L(N)$ for a Petri net N in which each transition consumes exactly one token (it has only one incoming place with arity one; the number of outgoing places is arbitrary).

Theorem 1. *The following two problems are equivalent:*

(a) for given Petri net N, is $L(N) = \emptyset$? (reachability in a Petri net);

(b) for given two Petri nets N_1 and N_2 in which each transition consumes exactly one token, is $L(N_1) \cap L(N_2) = \emptyset$? (emptiness of intersection of two commutative languages).

The reduction of problem (b) to problem (a) is easy. We produce a net N with $L(N_1) \cap L(N_2) = L(N)$. To N we take places of both N_1 and N_2. For each pair of transitions t_1 from N_1 and t_2 from N_2, both labeled by the same letter $a \in \Sigma$, we create a transition in N labeled by a, which consumes tokens from all (namely: two) places from which any of t_1 and t_2 consumes a token, and produces tokens to all places to which any of t_1 and t_2 produces a token. Directly from the definition follows that N accepts exactly words from the intersection.

Now we come to the reduction from (a) to (b). We use here an easy folklore fact that having any net N we may create a net N' in which each transition consumes exactly two tokens such that $L(N) = \emptyset \Leftrightarrow L(N') = \emptyset$ (even more: $L(N) = L(N')_{|\Sigma}$, where $L(N')_{|\Sigma}$ consists of words from $L(N')$ with removed letters not being in the original alphabet Σ). Roughly speaking, each transition consuming n tokens is replaced by n transitions consuming 2 tokens: one as in the original transition and one from an added control place. The special control places are organized in such a way that all n transitions corresponding to the original one have to fire one after another. We also need to convert transitions consuming one or zero tokens; but it is enough to add a new place from which one or two tokens are consumed an then produced there again.

So we may assume that in N each transition consumes exactly two tokes. Now we describe how N_1 and N_2 are created. They both have (an own copy of) the same set of places as N, plus one additional place p_0 used for initialization. Assume N also has the place p_0, but it is never used. So all the three nets have the same set of n places, assume they are numbered from 1 to n.

A configuration of a net may be described by a vector $v \in \mathbb{Z}_+^n$ of n nonnegative integers, where v_p is the number of tokens on place p. On the other hand a transition may be described by $t = (p_1, p_2, v)$ where p_1, p_2 are numbers of places from which tokens are consumed and $v \in \mathbb{Z}_+^n$ is a vector of produced tokens. Let $\mathbf{1}_p$ be the vector consisting of 1 on position p and zeroes on all other positions. A transition $t = (p_1, p_2, v)$ may be used in a configuration u when $u - \mathbf{1}_{p_1} - \mathbf{1}_{p_2} \in \mathbb{Z}_+^n$; as a result we get a configuration $u - \mathbf{1}_{p_1} - \mathbf{1}_{p_2} + v$. Transitions in N_1 and N_2 will be of the form (p, v) as they consume just one token.

For each transition $t = (p_1, p_2, v)$ of N several transitions are produced. For each vector $v_1 \leq v, v_1 \in \mathbb{Z}_+^n$ we produce a pair of transitions: in N_1 a transition (p_1, v_1) and in N_2 a transition $(p_2, v - v_1)$. We label both transitions in the pair by the same new unique label (the label is different for each v_1 and for each t; each label is used only twice: one in N_1 and once in N_2).

Let v^0 be the initial configuration of N. For each $v_1 \leq v^0, v_1 \in \mathbb{Z}_+^n$ we generate a pair of transitions: (p_0, v_1) in N_1 and $(p_0, v^0 - v_1)$ in N_2, where p_0 is the additional place, never used in N. Both these transitions are labeled by the same new unique letter (different for each pair). The initial configurations of N_1 and N_2 are $\mathbf{1}_{p_0}$ (one token on the additional place p_0). The final configuration of N_1 is the same as of N, while the final configuration of N_2 contains no tokens.

Now we need to prove that $L(N_1) \cap L(N_2) \neq \emptyset$ iff $L(N) \neq \emptyset$. First we consider the implication from left to right. Let w be a word accepted by both N_1 and N_2. Then we have two sequences of transitions, one in N_1, one in N_2, leading from the initial to the final configurations. Both are labeled by the same word w, so they are of the same length and the corresponding transitions belong to some of our pairs of transitions, since only such pairs are labeled by the same letter. Look at the configurations v_1 and v_2 after the first transition (at least one transition has to exist, since the initial and the final configurations of N_1 or N_2 are different). Notice that $v_1 + v_2 = v^0$, the initial configuration of N. This is because in N_1 only the transition (p_0, v_1) could be used (as only on p_0 we had a token), which is in pair with $(p_0, v^0 - v_1)$.

We will prove by induction on the number of used transitions that whenever in N_1 and N_2 we reach configurations v_1 and v_2, then in N we can reach $v_1 + v_2$. Then, since word w leads to the final configurations of N_1 and N_2 and their sum gives the final configuration of N, the final configuration of N can be reached. The thesis is true after the first step, as $v_1 + v_2 = v^0$ is the initial configuration in N. Now assume that $v_1 + v_2$ can be reached in N and consider transitions (p_1, u_1) in N_1 and (p_2, u_2) in N_2, going from v_1 and v_2 labeled by the same letter. These transitions are from one pair, corresponding to some transition $(p_1, p_2, u_1 + u_2)$ of N. This transition can be used in N from $v_1 + v_2$, and by applying it we get configuration $v_1 + v_2 - \mathbf{1}_{p_1} - \mathbf{1}_{p_2} + u_1 + u_2$, while in N_1 and in N_2 the next configurations are, respectively, $v_1 - \mathbf{1}_{p_1} + u_1$ and $v_2 - \mathbf{1}_{p_2} + u_2$, so the thesis holds.

Now consider the implication from right to left. Take a sequence of transitions leading from the initial to the final configuration of N. Appropriate sequences in N_1 and N_2 will be constructed starting from the end. Namely, the following thesis will be proved by induction on the number of used transitions: whenever from a configuration v the final configuration may be reached in N, then there are configurations v_1, v_2 of N_1, N_2 such that $v_1 + v_2 = v$ and from which there are two sequences of transitions (one in N_1, one in N_2) generating the same word and leading to the final configurations of N_1 and N_2.

For v being the final configuration the thesis is true, since the final configuration of N is the sum of final configurations of N_1 and N_2. Now take any configuration v of N satisfying the thesis and a preceding configuration from which a transition (p_1, p_2, u) leads to v. The preceding configuration is $v - u + \mathbf{1}_{p_1} + \mathbf{1}_{p_2}$. Because the transition was possible, we have $v - u \in \mathbb{Z}_+^n$, in other words $u \leq v$ (each element of u is $\leq$ than the corresponding element of v). Let v_1, v_2 be (from the induction assumption) the configurations of N_1, N_2 such that $v_1 + v_2 = v$ and from v_1 and v_2 we may reach the final configurations using a common word. Then we may distribute u into $u_1, u_2 \in \mathbb{Z}_+^n$, $u_1 + u_2 = u$ such that $u_1 \leq v_1$ and $u_2 \leq v_2$. We have transitions (p_1, u_1) in N_1 and (p_2, u_2) in N_2, labeled by the same letter. Since $u_1 \leq v_1$ and $u_2 \leq v_2$, they may be applied from configurations $v_1 - u_1 + \mathbf{1}_{p_1}$ in N_1 and $v_2 - u_2 + \mathbf{1}_{p_2}$ in N_2, which leads to v_1 in N_1 and v_2 in N_2. Moreover the sum of these configurations is $v - u + \mathbf{1}_{p_1} + \mathbf{1}_{p_2}$, as we wanted.

From the above we conclude that there are configurations v_1 in N_1 and v_2 in N_2 from which the final configuration may be reached using a common word, such that $v_1 + v_2 = v^0$, the initial configuration in N. But then there exists a pair of transitions (p_0, v_1) in N_1 and (p_0, v_2) in N_2, reading the same letter. They may be used to get v_1 in N_1 and v_2 in N_2 from the initial configuration $\mathbf{1}_{p_0}$.