

Computational complexity

lecture 9

Polynomial hierarchy

The following problem is in **NP**:

INDSET = $\{(G,k) : \text{in graph } G \text{ there is an independent set of size } \geq k\}$

Consider now a slightly more difficult problem:

EXACT-INDSET = $\{(G,k) : \text{the largest independent set in } G \text{ is of size } k\}$

We see no reason for this problem to be in **NP**...

What would be a witness?

Polynomial hierarchy

EXACT-INDSET = $\{(G,k) : \text{the largest independent set in } G \text{ is of size } k\}$

A similar problem:

MIN-DNF = $\{ \phi : \phi \text{ is a formula in the DNF form, not equivalent to any smaller formula in the DNF form} \}$
 $= \{ \phi : \forall \psi, |\psi| < |\phi| \Rightarrow \exists \text{ valuation } s \text{ such that } \phi(s) \neq \psi(s) \}$

In order to describe these problems, it is not enough to use one „exists” quantifier (as in **NP**), neither one „for all” quantifier (as in **coNP**). We have here a combination of two quantifiers.

Polynomial hierarchy

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In order to describe these problems, it is not enough to use one „exists” quantifier (as in **NP**), neither one „for all” quantifier (as in **coNP**). We have here a combination of two quantifiers.

Class Σ_2^P contains languages L for which there is a machine M working in polynomial time, and a polynomial q such that:

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{q(|x|)} \forall v \in \{0,1\}^{q(|x|)} M(x,u,v)=1$$

The language EXACT-INDSET is of this form:

$\exists S \forall S' . S$ is an independent set of size k and
 S' is not an independent set of size $>k$

Polynomial hierarchy

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Class Π_2^P contains complements of languages from Σ_2^P ; it is easy to see that it contains languages L for which there is a machine M working in polynomial time, and a polynomial q such that:

$$x \in L \Leftrightarrow \forall u \in \{0,1\}^{q(|x|)} \exists v \in \{0,1\}^{q(|x|)} M(x,u,v)=1$$

The language EXACT-INDSET is of this form as well:

$\forall S' \exists S . S$ is an independent set of size k and

S' is not an independent set of size $>k$

Also the language MIN-DNF is of this form:

$$\forall \psi \exists s . |\psi| < |\phi| \Rightarrow \phi(s) \neq \psi(s)$$

However, it is believed that MIN-DNF does not belong to Σ_2^P

Polynomial hierarchy

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Class Π_k^p contains complements of languages from Σ_k^p , i.e., languages L for which there is a machine M working in polynomial time, and a polynomial q such that:

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We also define $\mathbf{PH} = \bigcup_k \Sigma_k^p$

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Fact 1

Class Σ_k^p contains precisely languages recognizable in polynomial time by nondeterministic Turing machines with an oracle for a problem from Σ_{k-1}^p , and Π_k^p contains their complements.

Polynomial hierarchy

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Proof

Let $L \in \Sigma_k^p$. By definition there is a machine M working in polynomial time, and a polynomial q such that:

$$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{q(|x|)} \forall u_2 \in \{0,1\}^{q(|x|)} \dots \forall u_k \in \{0,1\}^{q(|x|)} . M(x, u_1, \dots, u_k) = 1$$

Consider the language L' defined by

$$(x, u_1) \in L' \Leftrightarrow \forall u_2 \in \{0,1\}^{q(|x|)} \dots \forall u_k \in \{0,1\}^{q(|x|)} . M(x, u_1, \dots, u_k) = 1$$

The complement of L' is in Σ_{k-1}^p .

It is easy to recognize L by a nondeterministic machine with oracle for (the complement of) L' .

Polynomial hierarchy

Fact 1

Class Σ_k^p contains precisely languages recognizable in polynomial time by nondeterministic Turing machines with an oracle for a problem from Σ_{k-1}^p , and Π_k^p contains their complements.

Proof

Let L be recognized by a nondet. machine N with oracle for $L' \in \Sigma_{k-1}^p$. By definition there is a machine M' working in polynomial time, and a polynomial q' such that:

$$y \in L' \Leftrightarrow \exists v_1 \in \{0,1\}^{q'(|y|)} \forall v_2 \in \{0,1\}^{q'(|y|)} \dots \overline{Q}v_{k-1} \in \{0,1\}^{q'(|y|)} . M'(y, v_1, \dots, v_{k-1}) = 1$$

We observe that (for an appropriate polynomial q)

$$x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{q(|x|)} \forall u_2 \in \{0,1\}^{q(|x|)} \dots Qu_k \in \{0,1\}^{q(|x|)} . M(x, u_1, \dots, u_k) = 1$$

where M checks that:

- a prefix of u_1 is of the form $R, v_{1,1}, \dots, v_{1,n}$, where R is a run of N
- if y is the i -th query to L' in R with answer yes, $M'(y, v_{1,1}, u_2, \dots, u_{k-1}) = 1$
- if y is a query to L' in R with answer no, $M'(y, u'_2, \dots, u'_k) = 0$
(where u'_2, \dots, u'_k are prefixes of u_2, \dots, u_k of length $q'(y)$)

Thus $L \in \Sigma_k^p$.

Polynomial hierarchy

Fact 1

Class Σ_k^p contains precisely languages recognizable in polynomial time by nondeterministic Turing machines with an oracle for a problem from Σ_{k-1}^p , and Π_k^p contains their complements.

In particular:

- $\Sigma_1^p = \mathbf{NP}$
- $\Pi_1^p = \mathbf{coNP}$
- Σ_2^p is sometimes denoted $\mathbf{NP}^{\mathbf{NP}}$ (\mathbf{NP} with oracle in \mathbf{NP})
- Σ_2^p contains in particular all languages from \mathbf{NP} and from \mathbf{coNP}

Polynomial hierarchy

Class Σ_k^p contains languages L for which there is a machine M working in polynomial time, and a polynomial q such that:

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Class Π_k^p contains complements of languages from Σ_k^p , i.e., languages L for which there is a machine M working in polynomial time, and a polynomial q such that:

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We also define $\mathbf{PH} = \bigcup_k \Sigma_k^p$

How are these classes related?

Polynomial hierarchy

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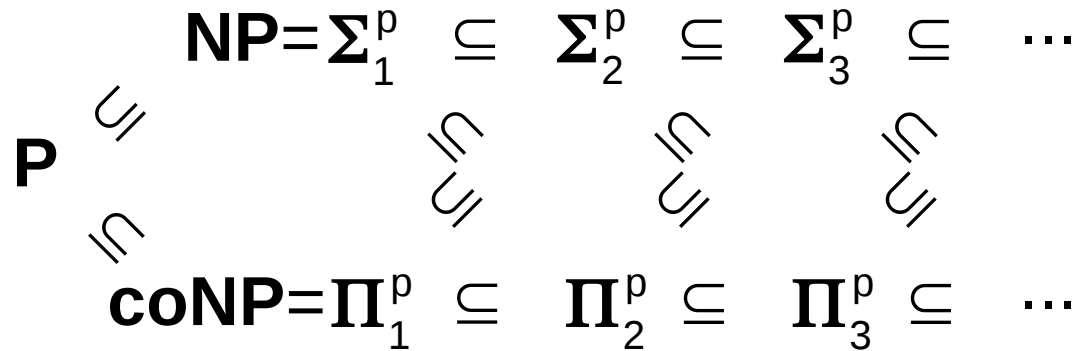
How are these classes related?

Fact 2: $\Sigma_k^p \subseteq \Sigma_{k+1}^p$, $\Sigma_k^p \subseteq \Pi_{k+1}^p$, $\Pi_k^p \subseteq \Sigma_{k+1}^p$, $\Pi_k^p \subseteq \Pi_{k+1}^p$

Proof: Obvious (follows from Fact 1)

Polynomial hierarchy

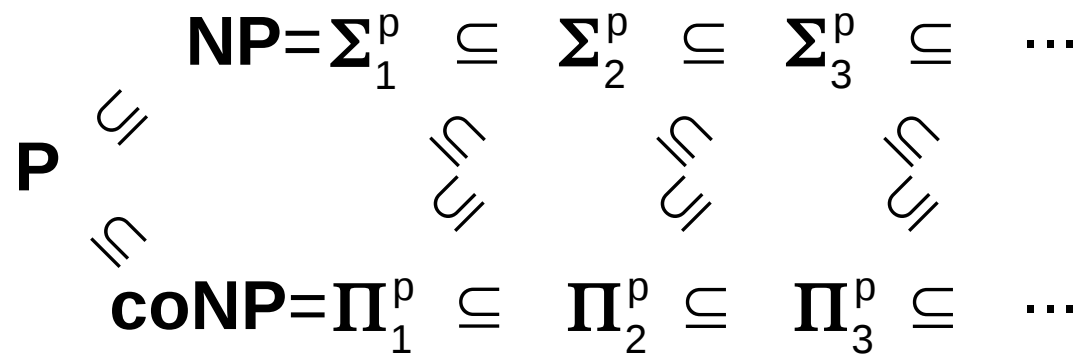
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Are these inclusions strict? And how are Σ_k^p and Π_k^p related?

Polynomial hierarchy

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Are these inclusions strict? And how are Σ_k^p and Π_k^p related?

We don't know (it is believed that all these classes are different).

But there are only two possibilities:

- either all the classes are different, or
- they are different to some point, and then they start to be equal

Fact 3:

If $\Sigma_k^p = \Pi_k^p$, then $\Sigma_k^p = \Sigma_{k+1}^p = \dots = \Pi_k^p = \Pi_{k+1}^p = \dots = \mathbf{PH}$.

If $\mathbf{P} = \mathbf{NP}$, then $\mathbf{P} = \Sigma_1^p = \Sigma_2^p = \dots = \Pi_1^p = \Pi_2^p = \dots = \mathbf{PH}$.

Polynomial hierarchy

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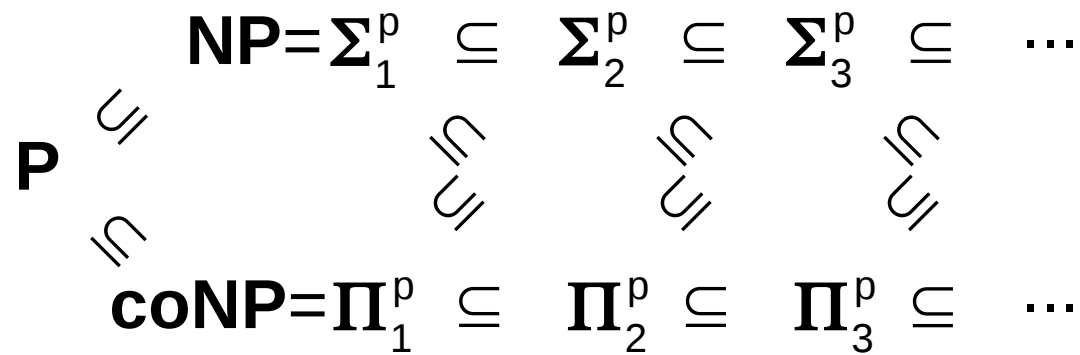
Proof (first part, the second part is analogous):

Suppose that $\Sigma_k^p = \Pi_k^p$, and take $L \in \Sigma_{k+1}^p$. Then L is recognized by a nondeterministic machine M with oracle for $L' \in \Sigma_k^p = \Pi_k^p$, and L' is recognized by a nondeterministic machine M_+ with oracle for $L_+ \in \Sigma_{k-1}^p$, and the complement of L' is recognized by a nondeterministic machine M_- with oracle for $L_- \in \Sigma_{k-1}^p$. We can assume that both M_+ and M_- use the same oracle $L_{\pm} = \{(i, x) : x \in L_i\} \in \Sigma_{k-1}^p$.

We modify machine M to a machine with oracle L_{\pm} – instead of asking a query to L' , it guesses an accepting run of M_+ or an accepting run of M_- . Thus $L \in \Sigma_{k+1}^p$.

Other equalities follow easily.

Polynomial hierarchy



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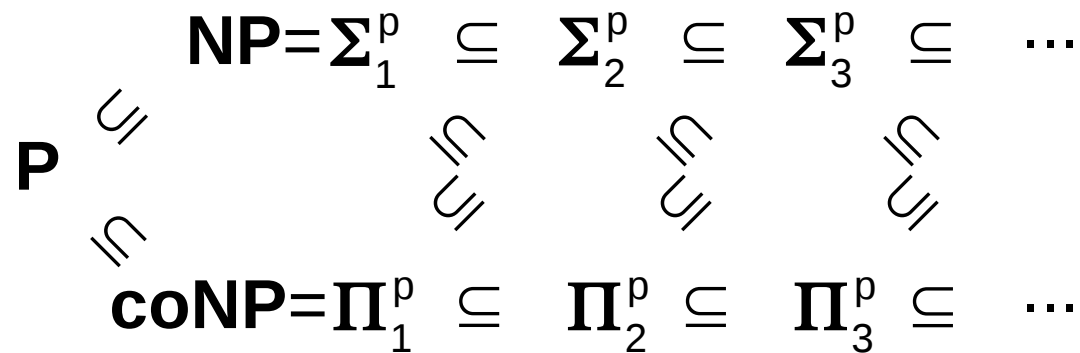
Complete language in Σ_k^p ?

Input: a sentence of the following form (with k blocks of quantifiers)

$$\exists x_{11}, \dots, x_{1n} \forall x_{21}, \dots, x_{2n} \exists x_{31}, \dots, x_{3n} \dots Q x_{k1}, \dots, x_{kn} \phi(x_{11}, \dots, x_{kn})$$

Question: is the sentence true?

Polynomial hierarchy



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Question: is the sentence true? (similarly for Π_k^p)

Complete language in **PH**?

Fact 4:

If there exists a **PH**-complete language, then $\mathbf{PH} = \Sigma_k^p$ for some k

Proof – The **PH**-complete language belongs to some Σ_k^p , and Σ_k^p is closed under reductions in polynomial time.

Polynomial hierarchy

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Fact 4:

If there exists a \mathbf{PH} -complete language, then $\mathbf{PH} = \Sigma_k^p$ for some k

Fact 5: $\mathbf{PH} \subseteq \mathbf{PSPACE}$

Proof: The Σ_k^p -complete language mentioned above is a special case of QBF, which belongs to \mathbf{PSPACE} .

Polynomial hierarchy

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If there exists a **PH**-complete language, then $\mathbf{PH} = \Sigma_k^p$ for some k

Fact 5: **PH** \subseteq **PSPACE**

Fact 6: If the classes Σ_k^p are all different, then **PH** \neq **PSPACE**

Proof: Follows from Fact 4 – in **PSPACE** there is a complete language.

Alternating machines

- Alternating Turing machines (ATM) generalize nondeterministic ones (NTM)
- Both NTM and ATM are not a realistic model of computation (we cannot build such machines). But NTM help us to observe a very natural phenomenon: a difference between finding a solution and verifying a solution.
- ATMs have a similar role for some languages, for which there are no short witnesses, i.e., which cannot be characterized using nondeterminism.

Alternating machines

Definition of ATM:

- a configuration can have multiple successors (as in NTM)
- additionally states of the machine (and in effect its configurations) are divided to existential and universal ones

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The set of winning configurations is defined as the smallest set s.t.:

- accepting configurations are winning
- every existential configuration, whose some successor is winning, is also winning
- every universal configuration, whose all successors are winning, is also winning

We accept a word w , if the initial configuration for this word is winning.

M works in time $T(n)$ / in space $S(n)$, if every computation fits in this time / space.

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Observation:

NTM is a special case of an ATM – only existential states

Alternating machines

Equivalently: acceptance can be defined using a game:

- we consider the configuration graph (edges = possible transitions)
- players \exists and \forall alternately move a pawn (common to both player) around the graph
- in existential states player \exists decides, in universal states player \forall decides (player \exists wants to accept, player \forall wants to reject)
- we accept a word, if player \exists has a winning strategy – he can reach an accepting configuration regardless moves of player \forall

Alternating machines

Classes **ATIME**($T(n)$), **ASPACE**($S(n)$),
AP = $\bigcup_k \mathbf{ATIME}(n^k)$, **AL** = **ASPACE**($\log n$)

Theorem

AL = **P**, **AP** = **PSPACE** (the same can be said more generally)

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Theorem

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Proof **AP** \subseteq **PSPACE**

Backtracking: we browse through all computations of the alternating machine (such a computation can be represented in polynomial space)

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Proof **AL** \subseteq **P**

We construct the graph containing all reachable configurations of the alternating machine – it is of polynomial size. Then in polynomial time we can find all winning configurations, by going backwards (starting from accepting configurations).

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Proof **PSPACE** \subseteq **AP**

It is enough to prove that **QBF** \in **AP**, as **QBF** is **PSPACE**-complete. This is almost obvious – player \exists chooses values of variables quantified existentially, and player \forall chooses values of variables quantified universally; at the end we deterministically compute the value of the formula.
Actually: the algorithm for **AP** is simpler than for **PSPACE**.

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Proof $\mathbf{P} \subseteq \mathbf{AL}$

- For an algorithm in **P** there is an equivalent boolean circuit, and we can construct it in logarithmic space.
- It is easy to give an algorithm in **AL**, which computes the value of a circuit: players walk from the output gate, in OR gates player \exists decides which predecessor is true, and in AND gates player \forall decides which predecessor is supposed to be false.
- We do not generate the whole circuit, only particular fragments, „on demand“.

Alternating machines

Consider alternating machines which:

- work in polynomial time
- the initial state is existential (universal)
- every computation leads to at most $k-1$ changes between existential and universal states

Fact

Such machines recognize languages from $\Sigma_k^p (\Pi_k^p)$

(we skip the formal proof, although it is easy)

Probabilistic machines

Machines with a source of random bits (probabilistic machines):

- a deterministic machine
- an additional read-once tape (the head cannot move left along this tape)

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Notice that **NP** can be defined as follows: a language L is in **NP** iff there is a polynomial $p(n)$ and a machine M with a source of random bits, working in at most $p(n)$ steps, and such that:

- $w \in L \Rightarrow \exists s. (w, s) \in L_M$
- $w \notin L \Rightarrow \nexists s. (w, s) \in L_M$

(a word is in L iff some witness confirms this)

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Class **RP** (randomized polynomial time): as above, but

- $w \in L \Rightarrow \Pr_s[(w, s) \in L_M] \geq 0.5$
- $w \notin L \Rightarrow \nexists s. (w, s) \in L_M$

Intuition: a word is in L , if at least half of possible witnesses confirm this.

Probabilistic machines

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As s we can take sequences of length $p(n)$, or infinite sequences, does not matter.

Intuition: a word is in L , if at least half of possible witnesses confirm this (but there are no witnesses for words not in L)

In other words: if a word is not in L , we will certainly reject; if it is in L , then choosing transitions randomly, we will accept with probability at least 0.5

Probabilistic machines

Class **RP** (randomized polynomial time): a language L is in **RP** iff there is a polynomial $p(n)$ and a machine M with a source of random bits, working in at most $p(n)$ steps, and such that:

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- $w \notin L \Rightarrow \nexists s. (w,s) \in L_M$

Remark: Some machines does not accept any language in the sense of **RP**. It is undecidable whether a machine is correct in the sense of **RP**, even if we know the polynomial $p(n)$

For this reason we do not know any **RP**-complete problem.
Intuition: we cannot reduce from every machine recognizing a language from **RP**, because we do not know how such machines look like.

Probabilistic machines

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Fact: $\mathbf{P} \subseteq \mathbf{RP} \subseteq \mathbf{NP}$ (both inclusions are obvious)

Probabilistic machines

Class **RP** (randomized polynomial time): a language L is in **RP** iff there is a polynomial $T(n)$ and a machine M with a source of random bits, working in at most $T(n)$ steps, and such that:

- $w \in L \Rightarrow \Pr_s[(w,s) \in L_M] \geq 1-p=0.5$
- $w \notin L \Rightarrow \nexists s. (w,s) \in L_M$

Fact (amplification): in the definition of **RP** the number 0.5 can be changed to any number from the interval (0,1), and the class of defined languages will remain the same

Proof: Let \mathbf{RP}_p be the class with error probability p

Obviously $\mathbf{RP}_p \subseteq \mathbf{RP}_q$ when $p \leq q$

We will now prove that $\mathbf{RP}_p \subseteq \mathbf{RP}_{p^2}$

- Out of a machine M with error p we construct a machine M' , which on the same input chooses randomly two witnesses, and accepts if some of them is a correct witness
- The running time doubles, so it remains polynomial
- The error probability decreases to p^2 – M' is wrong only when M made a mistake twice

Probabilistic machines

Is this a realistic model?

- It is more realistic than nondeterministic or alternating machines: we can run a probabilistic machine, give it some sequence of bits as random bits, and obtain a result that is correct with some probability.
- We obtain a result that is correct with some probability (and due to amplification this probability can be arbitrarily high), but we cannot be sure.
- How to generate bits that are really random? There exist physical random number generators (basing e.g. on quantum effects). Problems: they are relatively slow, and can be biased (in particular after some time, when they start to be broken).
- In practice, we use pseudo-random generators, that generate “random” bits using some algorithm. In practice, this works well, as the generated sequence looks like a random one. But theoretically, we cannot be sure about the probability of correctness.