Computational complexity

lecture 3

Announcement

Mid-term exam:

12.12.2017, during the lecture (Tuesday, 12:15)

Universal machines

Theorem:

There exists a universal Turing machine U (an “interpreter”), such that $U(\langle M\rangle,w)=M(w)$. If M works in time $T(|w|)$ and space $S(|w|)$, then U works in time $O(T(|w|)\cdot log(T(|w|)))$ and space $O(S(|w|))$.

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Two possible definitions of time / space complexity:

- T_1/S_1 using machines (“there exists a machine...”)
- T_2/S_2 using programs for the universal machine (“there exists a program...”)

Relation between them:

- $T_1 \leq T_2 \leq T_1 \cdot log\ T_1$
- $S_1=S_2$

only small difference!
we use the definition with machines

Hierarchy theorems

Are there problems, which require very large time / space to be solved?
(Maybe every problem can be solved e.g. in polynomial time?)

Hierarchy theorems

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Space hierarchy theorem:

If:

- function $g(n)$ is space-constructible, and
- $f(n)=o(g(n))$

then DSPACE($f(n)$)≠DSPACE($g(n)$)

Time hierarchy theorem – similar

definition: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

Hierarchy theorems

Space hierarchy theorem:

If:

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then DSPACE($f(n)$)≠DSPACE($g(n)$)

Proof:

- Consider the language

$$L = \{(\langle M\rangle,w) \mid \text{tape alphabet of } M \text{ is } \{0,1,\triangleright,\perp\}, \text{ and } |\langle M\rangle| \leq g(|(\langle M\rangle,w)|), \text{ and } M \text{ rejects } (\langle M\rangle,w) \text{ in space } g(|(\langle M\rangle,w)|)\}$$

Hierarchy theorems

$L = \{(\langle M\rangle,w)$ | tape alphabet of M is $\{0,1,\triangleright,\perp\}$, and $|\langle M\rangle|\leq g(|(\langle M\rangle,w)|)$, and M rejects $(\langle M\rangle,w)$ in space $g(|(\langle M\rangle,w)|)\}$

<u>Part 1</u> – $L\notin$DSPACE($f(n)$)

Suppose that $L\in$DSPACE($f(n)$). Then there is M with tape alphabet $\{0,1,\triangleright,\perp\}$, which recognizes L in space $O(f(n))$.

Because $f(n)=o(g(n))$, for some long word w machine M works on $(\langle M\rangle,w)$ in space $g(|(\langle M\rangle,w)|)$, and $|\langle M\rangle|\leq g(|(\langle M\rangle,w)|)$

We have a contradiction:

(M accepts $(\langle M\rangle,w)$) $\Leftrightarrow$ $(\langle M\rangle,w)\in L$ $\Leftrightarrow$ (M rejects $(\langle M\rangle,w)$)

Remark – for the language

$L' = \{((\langle M\rangle,w)$ | M rejects $(\langle M\rangle,w)\}$

the same argument gives undecidability.

Hierarchy theorems

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Part 2: $L \in$ DSPACE($g(n)$) – i.e., L can be recognized in space $O(g(n))$.

- Generally: simulate the run of M on $(\langle M\rangle,w)$

Hierarchy theorems

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- Reserve working space $g(n)$ (where n = length of input)
  - space $O(g(n))$ is enough (by assumption g is space-constructible)

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- Reserve working space $g(n)$ (where n = length of input)
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- Check that the input is of the form $(\langle M\rangle,w)$, that the alphabet is $\{0,1,\triangleright,\perp\}$, and that $|\langle M\rangle|\leq g(n)$
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- Use the Sipser's theorem (or assume that $g(n)=\Omega(log(n))$, and use the approach with a counter), and check whether M rejects $(\langle M\rangle,w)$ in reserved space $g(n)$.
  - when M rejects → we accept
  - when M accepts or loops or exceeds space → we reject
  - space $O(g(n))$ is enough

Hierarchy theorems

Space hierarchy theorem:

If:

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then DSPACE($f(n)$)≠DSPACE($g(n)$)

Time hierarchy theorem:

If:

- function $g(n)$ is time-constructible,
- $f(n)=o(g(n))$

then DTIME($f(n)$)≠DTIME($g(n)log(g(n))$)

Hierarchy theorems

Time hierarchy theorem:

If:

- function $g(n)$ is time-constructible,
- $f(n)=o(g(n))$

then DTIME($f(n)$)$\neq$DTIME($g(n)log(g(n))$)

Proof

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$$\text{and } M \text{ rejects } (\langle M\rangle,w) \text{ in time } g(|(\langle M\rangle,w)|)\}$$

- Part 1 – $L \notin$DTIME($f(n)$) → exactly as previously

Hierarchy theorems

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<u>Part 2</u> – $L\in$DTIME($g(n)log(g(n))$) – i.e., L can be recognized in time $O(g(n)log(g(n)))$

- Generally: simulate the run of M on $(\langle M\rangle,w)$
- Check that the input is of the form $(\langle M\rangle,w)$, that the alphabet is $\{0,1,\triangleright,\perp\}$, and that $|\langle M\rangle| \leq log(n)$ (where n = length of input)
  - running time: $O(n)$
- Reserve a unary counter of length $g(n)$, on a separate tape
  - g is time constructible
  - running time: $O(g(n))$
- Simulate M on word $(\langle M\rangle,w)$, like the universal machine; increase the counter after every step.
  - running time: $O(g(n)\cdot(log\ g(n)+|\langle M\rangle|)) = O(g(n)log(g(n)))$

simulating tapes (↑ $log\ g(n)$)

reading the description of M, modifying state (↑ $|\langle M\rangle|$)

Hierarchy theorems

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  - when M rejects → we accept
  - when M accepts or exceeds time → we reject

Hierarchy theorems

Are there problems, which require very large time / space to be solved?
(Maybe every problem can be solved e.g. in polynomial time?)

Corollary from hierarchy theorems

- DTIME(n^k)≠DTIME(n^{k+1}), DSPACE(n^k)≠DSPACE(n^{k+1})
- L≠PSPACE, P≠EXPTIME

because P⊆DTIME(2^n)≠DTIME(4^n)⊆EXPTIME

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If a machine M works in time / space precisely $f(n)$, then there exists a problem requiring more time / space to be solved

- e.g. $2^{f(n)}$ *or* $f(n)^2$ – for time & space
- e.g. $f(n) \cdot log(log(n))$ – for space
- Moreover, functions being complexities of problems are distributed "quite densely", especially for space

Gap theorems

- Functions being complexities of problems are distributed "quite densely"
- Simultaneously, we have the following gap theorems:

There is a computable function $f(n) \geq n$ such that DTIME($f(n)$)=DTIME($2^{f(n)}$).

There is a computable function $f(n)$ such that DSPACE($f(n)$)
=DSPACE($2^{f(n)}$).

A contradiction with hierarchy theorems?

No – the function f will not be constructible (it can be computed, but in a larger time / space)

At the same time: we see that in the hierarchy theorems the assumption about constructability is really needed

Gap theorems (✶)

<u>Gap theorem</u> – time

There is a computable function $f(n) \geq n$ such that DTIME($f(n)$)=DTIME($2^{f(n)}$).

<u>Proof</u>

Fix an input alphabet $\Sigma = \{0,1\}$ (another alphabet → time multiplied by a constant)

We construct a function $f(n)$ such that no machine stops between $f(n)$ and $2^{f(n)}$ steps:

- Assign numbers to Turing machines (in a computable way)

Gap theorems (∗)

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- Assign numbers to Turing machines (in a computable way)
- We say that $P(i,k)$ is satisfied iff none among the first i machines on none among inputs of length i stops between k and $i \cdot 2^k$ steps (they stop earlier than k or later than $i \cdot 2^k$ or loop forever)

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- Let $k_1(i)=i$ and $k_{j+1}(i)=i \cdot 2^{k_j(i)}$
- For a fixed i, every pair (input_of_length_i, machine_with_number_$\leq i$) can falsify $P(i,k_j(i))$ for at most one j,

  Thus there exists some $j \leq i \cdot 2^i$ such that $P(i,k_j(i))$ is true.

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  Thus there exists some $j \leq i \cdot 2^i$ such that $P(i,k_j(i))$ is true.
- We put $f(i)=k_j(i)$. This function is computable.

Gap theorems (✶)

Gap theorem – time

There is a computable function $f(n) \geq n$ such that DTIME($f(n)$)=DTIME($2^{f(n)}$).

Proof

- For every n, none among the first n machines on none among inputs of length n stops between $f(n)$ and $n \cdot 2^{f(n)}$ steps.
- Take any machine M with number m running in time $c \cdot 2^{f(n)}$
- For every input of length $n \geq max(m,c)$ the machine stops in $\leq c \cdot 2^{f(n)}$ steps, but not between $f(n)$ and $n \cdot 2^{f(n)}$ steps, hence in $\leq f(n)$ steps

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- For every input of length $n \geq max(m,c)$ the machine stops in $\leq c \cdot 2^{f(n)}$ steps, but not between $f(n)$ and $n \cdot 2^{f(n)}$ steps, hence in $\leq f(n)$ steps
- There are only constantly many inputs of length $< max(m,c)$
- Thus the language can be recognized in time $O(f(n))$

Gap theorems

<u>Remarks</u>

- In the same way we can construct a function f such that DSPACE($f(n)$)=DSPACE($2^{f(n)}$).
- Actually, for every function g such that $g(n)\geq n$ (instead of $g(n)=2^n$) we can find f a such that DTIME($f(n)$)=DTIME($g(f(n))$) or DSPACE($f(n)$)=DSPACE($g(f(n))$).
- The functions f grow very quickly.
- They are not time/space-constructible.
- But they are computable.

Just finished:

Deterministic Turing machines – basic facts

Next topic:

Boolean circuits

Later:

- Nondeterministic Turing machines, reductions
- Probabilistic computations
- Fixed parameter tractability (FPT)
- Interactive proofs
- Alternating Turing machines
- Probabilistically checkable proofs (PCP)
- ...

Nonuniform computation models

- Suppose that P≠NP. Then there is no algorithm which quickly solves all instances of the SAT problem.
- But maybe for every n there is a separate algorithm, which quickly solves all instances of size n?
- Even if these algorithms are difficult to find, this would mean that SAT can be solved in practice.

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- A similar example: breaking the cryptographic algorithm RSA. If there is an algorithm, which quickly breaks the RSA encoding for a fixed (being currently used) key length, in practice we can treat the RSA code as insecure (even if the algorithm works only for one fixed n, not for all n).

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Hence, it makes sense to consider computation models in which for every n we apply a different algorithm.

One has to be careful, though: for every n, the language of instances of size n is regular.

Models of parallel computations

What if we have plenty of processors?

Example: matrix multiplication

- 1 processor: time $O(n^3)$ (the standard algorithm)
- n^2 processors: time $O(n)$
- n^3 processors: time $O(log(n))$ – an exponential speed up!

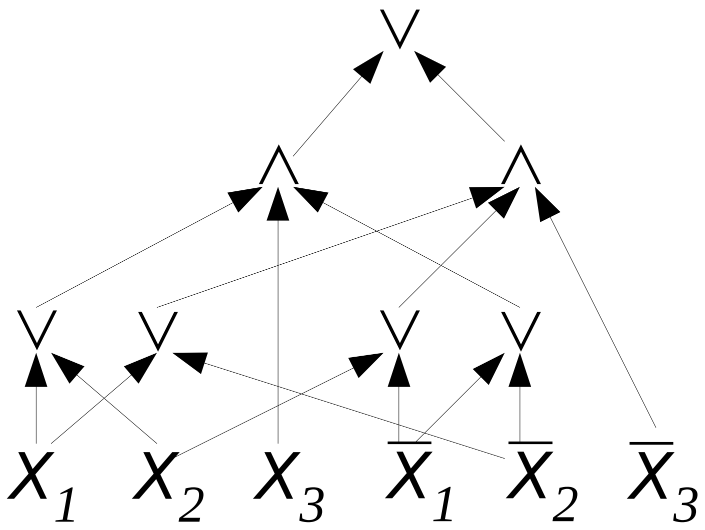
Question: Which algorithms do parallelize well, and which do not?

Boolean circuits

Another computational model: boolean circuits

idea: computing boolean functions using logical gates

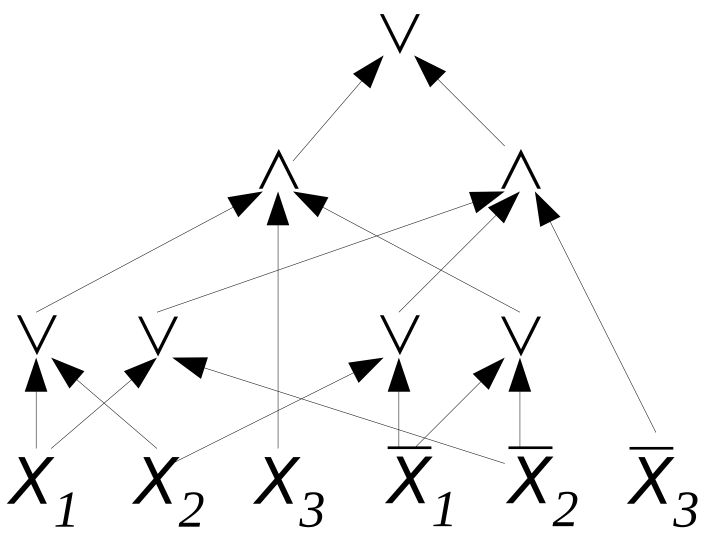
intuition: every gate represents a very simple processor



Boolean circuits

Definition: a boolean circuit having input of size n is given by an acyclic directed graph, in which:

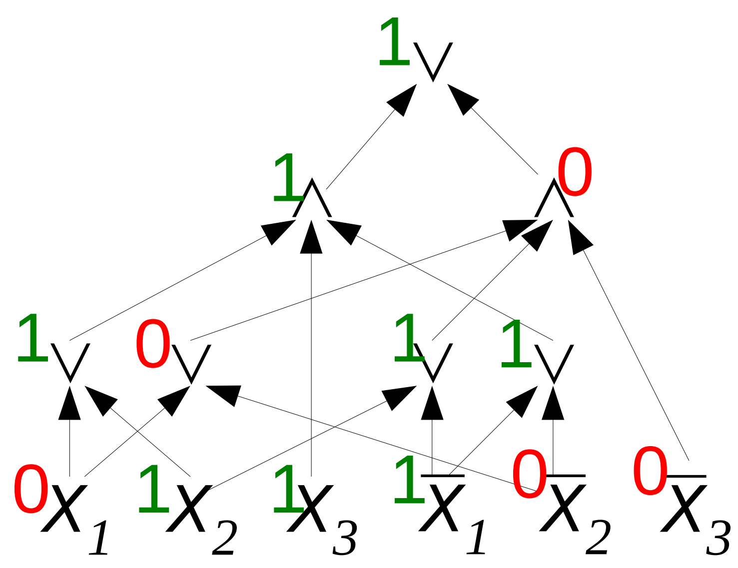
- there are $2n$ gates (nodes) of in-degree 0, denoted $X_1, \overline{X}_1, \ldots, X_n, \overline{X}_n$ (input gates)
- all other gates (having in-degree ≥ 0) are marked by one of the symbols $\wedge$ or $\vee$
- one of the gates (having out-degree 0) is marked as the output gate [another version: multiple outputs – when we compute a function]



Boolean circuits

For a fixed valuation $v:\{X_1,\ldots,X_n\}\rightarrow\{0,1\}$ we define:

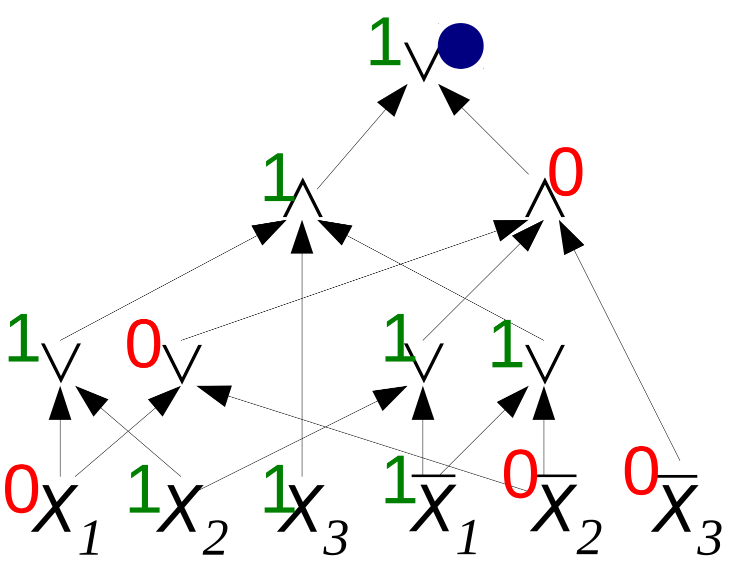
- the gate labeled by X_i gets value $v(X_i)$
- the gate labeled by $\overline{X}_i$ gets value $\neg v(X_i)$
- the value of an OR (AND) gate is computed as the disjunction (conjunction) of values of predecessors of the gate
- the value of the circuit = the value of the output gate
- the definition makes sense, because the graph is acyclic



Boolean circuits

An equivalent definition – a circuit as a game:

- two players (AND and OR) move a pawn over the graph, going back from the output gate
- AND (OR) decides in $\wedge$ nodes ($\vee$ nodes, respectively)
- OR wins, if the game finishes in X_i and $v(X_i)=1$, or in $\overline{X}_i$ and $v(X_i)=0$
- the value of the circuit is 1 if OR has a winning strategy



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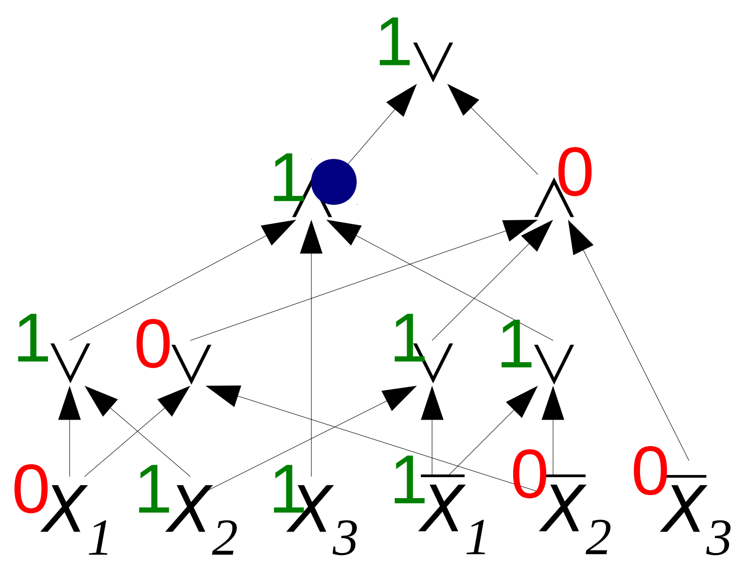

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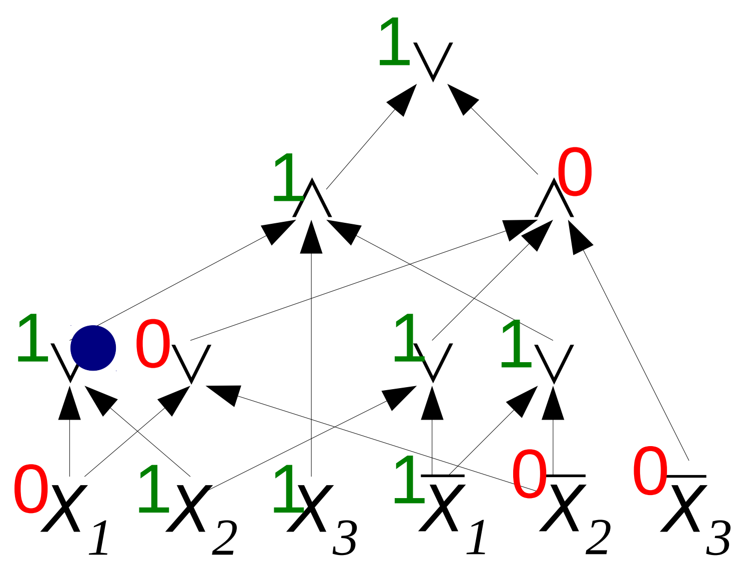

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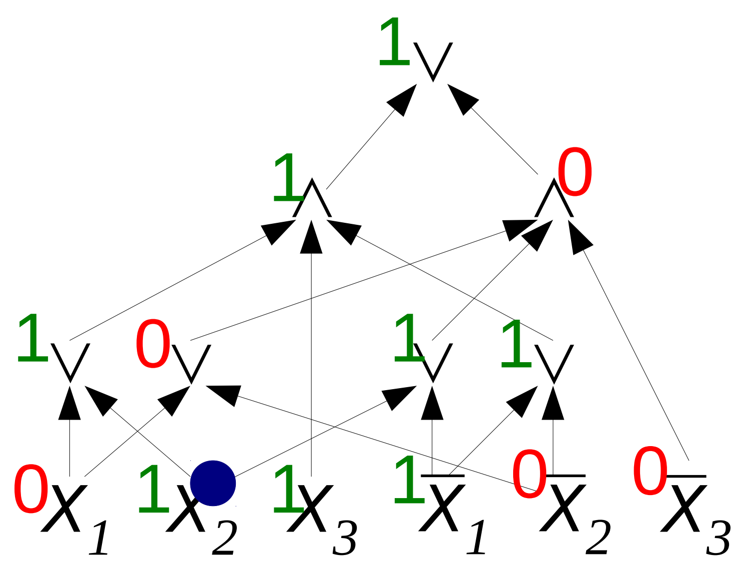


Boolean circuits

Equivalence of the two definitions:

- if the output has value 1, we have a strategy for OR: descend always to a node labeled by 1
- if the output has value 0, we have a strategy for AND: descend always to a node labeled by 0

Boolean circuits

- For a fixed valuation $v:\{X_1,\ldots,X_n\}\to\{0,1\}$ we have defined the value of a circuit
- The input amounts to a word $v\in\{0,1\}^n$
- A circuit C computes a function $\{0,1\}^n\to\{0,1\}$, i.e., it recognizes a subset of $\{0,1\}^n$

Boolean circuits

<u>Size?</u>

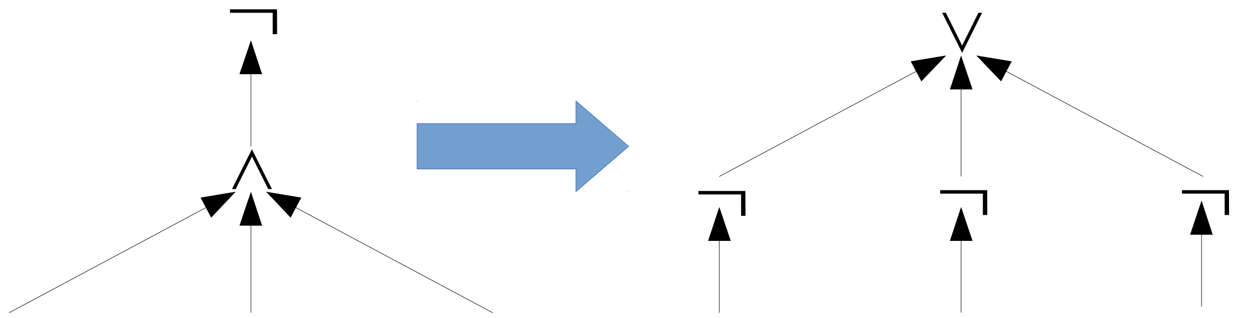
We have several parameters:

- the length of an input n
- the depth of a circuit (the length of the longest path)
- the number of gates B, the number of edges K
- the length of a representation of a circuit: $(B+K)\cdot log(B)$ (because numbers of gates have $log(B)$ bits)
- in-degree of gates (fan-in) – we consider circuits
  - ➔ with arbitrary fan-in
  - ➔ with fan-in ≤ 2

Boolean circuits

Negations?

- in our definition there are no NOT gates, but we have negated input gates
- this does not change anything: negations can be easily moved to leaves (De Morgan laws)



Boolean circuits

Recognizing languages by sequences of circuits:

- A circuit C_n having input of size n recognizes $L(C_n)$ – a subset of $\{0,1\}^n$ [in particular C_0 has no inputs, returns always 1 or always 0]
- Having a sequence of circuits $C_0,C_1,C_2,...$ we can recognize a language containing words of any length:

$$L((C_n)_{n\in\mathbb{N}})=L(C_0)\cup L(C_1)\cup L(C_2)\cup ...$$

- Which languages can be recognized using boolean circuits?

Boolean circuits

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- Which languages can be recognized using boolean circuits?

Fact.
Every laguage can be recognized by some sequence of boolean circuits (having depth 2 and exponential size)

i.e., the size of C_n is exponential in n

Boolean circuits

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- Which languages can be recognized using boolean circuits?

<u>Fact</u>.
Every laguage can be recognized by some sequence of boolean circuits (having depth 2 and exponential size)

A more interesting question: Which languages can be recognized by a sequence of circuits of polynomial size?