The Probabilistic Rabin Tree Theorem

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Highlights 2023 / LICS 2023

Theorem

We can compute the probability that a random infinite tree belongs to a given regular language L.

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given by, e.g.
- an MSO formula
- a nondeterministic parity automaton

full binary tree,
each label chosen
independently in random

- the result is an algebraic number
- can be computed in 3-EXPTIME
- can be compared with a given rational q in 2-EXPSPACE

Context

Decidable

- some results for ω-words (probability always rational)
- infinite trees: the probability exists (not clear because regular languages of infinite trees need not to be Borel) *[Gogacz, Michalewski, Mio, Skrzypczak 2017]*
- determ. top-down parity autom. *[Chen, Dräger, Kiefer 2012]*
- game automata *[Michalewski, Mio 2015]*
- weak MSO *[Niwiński, Przybyłko, Skrzypczak 2020]*

Undecidable

- nonemptiness for probabilistic automata (exists a finite word accepted with probability >0.5)
- value-1 for probabilistic automata (exists a sequence of finite words where acceptance probability tends to 1)
- exists a ω-word accepted by a probabilistic Büchi automaton with probability >0.

Open

- Satisfiability of PCTL*

Two worlds

Languages $\longrightarrow$ Probabilities

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Key difficulty:

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & A \cup B & \end{tikzcd}
\qquad\qquad
\begin{tikzcd} \mathbb{P}(A) \arrow[dr] & & \mathbb{P}(B) \arrow[dl] \\ & \mathbb{P}(A \cup B) & \end{tikzcd}$$

Two worlds

$$\text{Languages} \longrightarrow \text{Probabilities}$$

Key difficulty:

$$\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl] \\
& A \cup B &
\end{tikzcd}
\qquad\qquad
\cancel{\begin{tikzcd}
\mathbb{P}(A) \arrow[dr] & & \mathbb{P}(B) \arrow[dl] \\
& \mathbb{P}(A \cup B) &
\end{tikzcd}}$$

Another aspect:

(random variables)

$$\begin{tikzcd}
X \arrow[dr] & & Y \arrow[dl] \\
& F(X,Y) &
\end{tikzcd}$$

$$\cancel{\begin{tikzcd}
\text{distribution of } X, \arrow[dr] & \text{distribution of } Y \arrow[d] \\
& \text{distribution of } F(X,Y)
\end{tikzcd}}$$

$$\begin{tikzcd}
\text{distribution of } X \times Y \arrow[d] \\
\text{distribution of } F(X,Y)
\end{tikzcd}$$

✔

Step 1

~~Nondeterministic automata~~ → μ-calculus / powersets

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~~Nondeterministic automata~~ $\longrightarrow$ μ-calculus / powersets

$$\mu x_1.\nu x_2.\mu x_3.\nu x_4 \ldots \mu x_{d-1}.\nu x_d.\delta(x_1,x_2,\ldots,x_d)$$

Still not good – a function of many variables!

Step 2

$\cancel{\delta(\tau_1,\tau_2,\ldots,\tau_d)}$ $\longrightarrow$ $\bar{\tau}=(\tau_1,\tau_2,\ldots,\tau_d)$

$\Delta(\bar{\tau})$

Not so nice – we need $\vee,\wedge$ to interact between coordinates!

e.g. $\mu x.F(x\vee y)$

Step 3

$$\cancel{\mu x.F(x \vee y)} \longrightarrow F{\uparrow}(y)$$

Intuition behind $\mu x.F(x \vee y)$ (but not precise meaning):

least fixed point of F above y

We define: $F{\uparrow}(y)$ = least fixed point of F above y

Unary μ-calculus

Syntax: H, $F_1;F_2$, $F\uparrow$, $F\downarrow$ (defines a one-argument function $V \to V$)

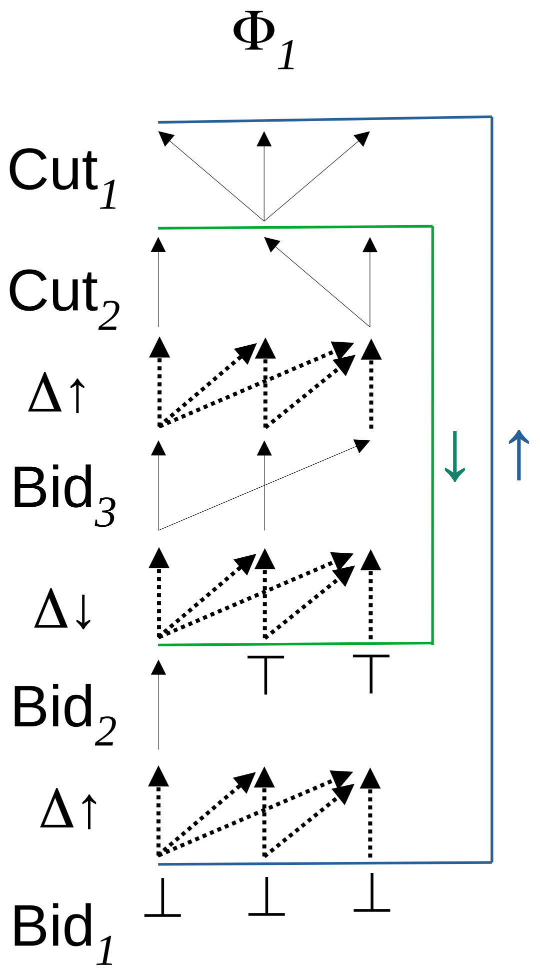
composition

fixed base functions

$F\downarrow(y)$ = greatest fixed point of F below y

$F\uparrow(y)$ = least fixed point of F above y

Unary μ-calculus – the formula



What has to be shown?

1) all fixpoints in Φ_1 exist

2) Φ_1 computes $\mu x_1.\nu x_2.\dots\mu x_{d-1}.\nu x_d.\delta(x_1,,\dots,x_d)$

3) all intermediate sets used while computing Φ_1 are measurable

4) the same computation can be done on distributions

Last step

Sets $\longrightarrow$ Distributions

τ : trees $\to$ P($Q\times\{1,\ldots,d\}$)

$\hat{\tau}$: $\mathbb{D}$(P($Q\times\{1,\ldots,d\}$))

$\hat{\tau}(R)=\mathbb{P}(\{t \mid \tau(t)=R\})$

Φ_1 can be expressed in first-order logic over reals – decidable by Tarski

(the formula is of exponential size)

Conclusions

- We shown how to compute the probability that a random infinite tree belongs to a given regular language.
- We introduced unary μ-calculus, which works well for orders without $\vee$ and $\wedge$ (e.g. probability distributions)

Thank you