Linear Algebra

Lecture 7 - Application of Determinants

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Determinant and Linear Dependence

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iii) *columns of matrix A form a linearly independent set.*

Recall that n linearly independent vectors in $\mathbb{R}^n$ form a basis.

Example

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Take matrix A and use elementary row operations to get an upper-triangular matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

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Then $\det A = \det B = 0$. The rows are linearly dependent

$$(1,-1,1) - (2,0,3) + (1,1,2) = (0,0,0).$$

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Then $\det A = \det B = 0$. The rows are linearly dependent

$$(1, -1, 1) - (2, 0, 3) + (1, 1, 2) = (0, 0, 0).$$

The columns are linearly dependent

$$-3(1, 2, 1) - (-1, 0, 1) + 2(1, 3, 2) = (0, 0, 0).$$

Identity Matrix

Definition

The **identity matrix** $I_n \in M(n \times n; \mathbb{R})$ is defined by

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

That is, it has $1'$s on the diagonal and 0's elsewhere.

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Note that for any $A \in M(n \times n; \mathbb{R})$ the following holds

$$I_n A = A I_n = A,$$

that is I_n is a neutral element with respect to matrix multiplication.

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Note that for any $A \in M(n \times n; \mathbb{R})$ the following holds

$$I_n A = A I_n = A,$$

that is I_n is a neutral element with respect to matrix multiplication. This follows also from the fact that $M(id_{\mathbb{R}^n})^{\mathcal{A}}_{\mathcal{A}} = I_n$ for any basis $\mathcal{A}$ of $\mathbb{R}^n$.

Invertible Matrix

Definition

A matrix $A \in M(n \times n; \mathbb{R})$ is called **invertible** if there exists matrix $B \in M(n \times n; \mathbb{R})$ such that $AB = I_n$. Such matrix B is unique and it satisfies the equality $BA = I_n$. The matrix B is called the inverse of A and is denoted A^{-1}, that is

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$$AA^{-1} = A^{-1}A = I_n.$$

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If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

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If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

Proposition

Let $\mathcal{A} = (v_1, \ldots, v_n)$ and $\mathcal{B} = (w_1, \ldots, w_n)$ be ordered bases of vector space V. Let M be the change-of-coordinate matrix from the basis $\mathcal{A}$ to the basis $\mathcal{B}$, that is $M = M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}}$. Let N be the change-of-coordinate matrix from the basis $\mathcal{B}$ to the basis $\mathcal{A}$, that is $N = M(\mathrm{id})_{\mathcal{B}}^{\mathcal{A}}$. Then $N = M^{-1}$.

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Proof.

It is enough to use the formula relating composition of linear transformations with matrix multiplication and the uniqueness of the inverse.

$$MN = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{A}} M(\mathrm{id})^{\mathcal{A}}_{\mathcal{B}} = M(\mathrm{id})^{\mathcal{B}}_{\mathcal{B}} = I_n.$$

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Example

Let $V = \mathbb{R}^2$, $\mathcal{A} = ((2,1),(5,3)), \mathcal{B} = st = ((1,0),(0,1))$. Then $M = M(\mathrm{id})^{st}_{\mathcal{A}} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $N = M(\mathrm{id})^{\mathcal{A}}_{st} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Example (continued)

Let $V = \mathbb{R}^2$, $\mathcal{A} = ((2,1),(5,3)), \mathcal{B} = st = ((1,0),(0,1))$. Then $M = M(\mathrm{id})_{\mathcal{A}}^{st} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $N = M(\mathrm{id})_{st}^{\mathcal{A}} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

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For example, take vector $v = (3,1)$. It's coordinates relative to the standard basis are $3, 1$ that is $(3,1) = 3(1,0) + 1(0,1)$.

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For example, take vector $v = (3,1)$. It's coordinates relative to the standard basis are $3, 1$ that is $(3,1) = 3(1,0) + 1(0,1)$. To compute coordinates of v relative to the basis $\mathcal{A}$ we use the change-of-coordinate matrix $N = M(\mathrm{id})^{\mathcal{A}}_{st}$.

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Example (continued)

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The coordinates of v relative to the basis $\mathcal{A}$ are $4, -1$ that is

$$(3,1) = 4(2,1) - 1(5,3).$$

Determinants and Invertible Matrices

Theorem

Let $A \in M(n \times n; \mathbb{R})$. Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let $\mathcal{A}, \mathcal{B}$ be bases of $\mathbb{R}^n$ such that $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = A$. The following conditions are equivalent:

i) *the matrix A is invertible,*

ii) $\det A \neq 0$,

iii) *rows of A form a linearly independent set,*

iv) *columns of A form a linearly independent set,*

v) *for any* $K = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$ *if* $AK = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ *then* $K = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$,

vi) *the linear transformation φ is injective,*

vii) *the linear transformation φ is surjective,*

viii) *the linear transformation φ is bijective (invertible).*

Computing the Inverse

For any $A = [a_{ij}], B = [b_{ij}] \in M(n \times n; \mathbb{R})$ denote by $[A|B]$ the matrix

$$\left[\begin{array}{ccc|ccc} a_{11} & \dots & a_{1n} & b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} & b_{n1} & \dots & b_{nn} \end{array}\right] \in M(n \times 2n; \mathbb{R}).$$

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Matrix A is invertible if and only if matrix $[A|I_n]$ can be transformed by elementary row operations to the matrix $[I_n|B]$. Then $B = A^{-1}$.

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Proof.

Use multiplication by elementary matrices (cf. Lecture 5). $\square$

Example

Let $A = \left[\begin{array}{ccc} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$. Then $\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 - r_2}$
$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_2}$
$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right].$

Therefore

$$A^{-1} = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 1 & -2 & 1 \\ -1 & 2 & 0 \end{array}\right].$$

Minors

Definition

Let $A = [a_{ij}] \in M(m \times n; \mathbb{R})$ be a matrix. **Minor (determinant) of matrix** A **of order** k, where $1 \leqslant k \leqslant \min\{m, n\}$, is the determinant of any k–by–k submatrix of A. In particular, for any

$$1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant m,$$

$$1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n,$$

and

$$A_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \begin{bmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_k} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k j_1} & a_{i_k j_2} & \cdots & a_{i_k j_k} \end{bmatrix},$$

the number $\det A_{i_1,\ldots,i_k;j_1,\ldots,j_k}$ is a minor of A of order k.

Rank of Matrix

Recall

Definition

Let $A \in M(m \times n; \mathbb{R})$. The rank of A is the dimension of the space $\text{lin}(r_1, \ldots, r_m)$ where $r_1, \ldots, r_m \in \mathbb{R}^n$ are rows of A. The rank of A is denoted $r(A)$.

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Example

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in an echelon form with two non-zero rows therefore $r(A) = \dim \text{lin}((1,2,1,1),(3,7,3,4),(1,3,1,2)) =$
$= \dim \text{lin}((1,2,1,1),(0,1,0,1)) = 2.$

Rank of Matrix

Remark

In the previous example

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{\textit{elementary row} \\ \textit{operations}}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A',$$

and

$$\mathrm{colsp}(A) = \mathrm{lin}((1,3,1),(1,4,2)),$$

$$\mathrm{colsp}(A') = \mathrm{lin}((1,0,0),(0,1,0)).$$

It follows that

$$\mathrm{colsp}(A) \neq \mathrm{colsp}(A').$$

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It follows that

$$\operatorname{colsp}(A) \neq \operatorname{colsp}(A').$$

However

$$\dim \operatorname{colsp}(A) = \dim \operatorname{colsp}(A'),$$

which is also equal to the dimension of $\operatorname{rowsp}(A) = \operatorname{rowsp}(A')$*.*
This is a general phenomenon.

Rank of Matrix

Theorem

For any matrix $A \in M(m \times n; \mathbb{R})$ the following numbers are equal:

i) $\dim \operatorname{lin}(r_1, \ldots, r_m)$ *where* $r_1, \ldots, r_m$ *are rows of* A*,*

ii) $\dim \operatorname{lin}(c_1, \ldots, c_n)$ *where* $c_1, \ldots, c_n$ *are columns of* A*,*

iii) *the highest order of a non–zero minor of matrix* A*.*

Proof

Matrix A can be put into a reduced echelon form by elementary row operations, and then, by elementary operations on columns, it can be put into the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}.$$

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Elementary row and column operations do not change those three numbers. Therefore the rank of A is equal to the number of pivots in an echelon form.

Example

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Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{bmatrix}$. It can be checked that

$$\det \begin{bmatrix} 2 & 1 & 1 \\ 7 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 4 \\ 1 & 3 & 2 \end{bmatrix} =$$

$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 1 & 3 & 1 \end{bmatrix} = 0.$$

On the other hand

$$\det A_{1,2;1,2} = \det \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = 1 \neq 0,$$

hence $r(A) = 2$.

Kronecker-Capelli Theorem

Consider a system of linear equations and two associated matrices

$$U : \left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right], \ B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

Kronecker-Capelli Theorem (continued)

Theorem (Kronecker-Capelli)

i) *the system U has a solution if and only if $r(A) = r(B)$,*

ii) *if the system U has a solution then exactly $n - r(A)$ variables are free variables,*

iii) *if $(s_1, \dots, s_n) \in \mathbb{R}^n$ is any solution of U and W is the subspace of all solutions of a homogeneous system of linear equations given by the matrix A then solutions of U are of the form $(s_1, \dots, s_n) + W = \{(s_1, \dots, s_n) + w \mid w \in W\}$.*

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Proof.

Adding one column to a matrix can only increase its rank by at most 1. If $r(B) = r(A) + 1$ then in the echelon form of B there is a pivot in the column of constant terms. The pivots correspond to dependent variables and the number of pivots is equal to the rank of the matrix. The difference of any two solutions of U is a solution of the homogeneous system of linear equations associated to the matrix A. $\square$

Kronecker-Capelli Theorem (continued)

Remark

Alternatively,

$$\begin{array}{c}\textit{system } U\\ \textit{has a solution}\end{array} \iff \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \operatorname{colsp}(A) \iff r(A) = r(B).$$

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Fix $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ *a solution of the system* U *and let* $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ *be any solution of the system* U*. Then*

$$r - s \in W$$

because

$$\begin{cases} a_{11}(r_1 - s_1) & + & a_{12}(r_2 - s_2) & + & \ldots & + & a_{1n}(r_n - s_n) & = & 0 \\ a_{21}(r_1 - s_1) & + & a_{22}(r_2 - s_2) & + & \ldots & + & a_{2n}(r_n - s_n) & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}(r_1 - s_1) & + & a_{m2}(r_2 - s_2) & + & \ldots & + & a_{mn}(r_n - s_n) & = & 0 \end{cases}$$

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Theorem

Let $A \in M(n \times n; \mathbb{R})$ be an invertible matrix. Then

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Proof.

The equality $A \frac{1}{\det A} \operatorname{adj}(A) = I_n$ can be checked directly using the Laplace expansion. □

Example

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For example $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Matrix Inverse Formula (continued)

Remark

For any matrices $A, B \in M(n \times n; \mathbb{R})$ *such that* $\det A, \det B \neq 0$ *and* $k \geqslant 0$

$$\operatorname{adj}(I_n) = I_n,$$

$$\det \operatorname{adj}(A) = (\det A)^{n-1},$$

$$(\operatorname{adj}(A))^{-1} = \operatorname{adj}(A^{-1}) = \frac{1}{\det A} A,$$

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A),$$

$$\operatorname{adj}(A^k) = (\operatorname{adj}(A))^k,$$

$$\operatorname{adj}(\operatorname{adj}(A)) = (\det A)^{n-2} A.$$

Cramer's Rule

Let U be a system of linear equations with n unknowns and n equations:

$$U : \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{cases}$$

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Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be the associated matrix of coefficients and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be the matrix of constant terms.

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The system U can be written as $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Cramer's Rule (continued)

Therefore, if $\det A \neq 0$ the system U has exactly one solution given by
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

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Theorem (Cramer's Rule)

If $\det A \neq 0$ *then the unique solution of the system* U *is given by* $x_i = \frac{\det A_i}{\det A}$ *for* $i = 1, \ldots, n$*, where* A_i *is the matrix* A *with* i*-th column replaced by* B*.*

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Theorem (Cramer's Rule)

If $\det A \neq 0$ then the unique solution of the system U is given by $x_i = \frac{\det A_i}{\det A}$ for $i = 1, \ldots, n$, where A_i is the matrix A with i-th column replaced by B.

Proof.

Use the Laplace expansion and the inverse matrix formula. $\square$

Cramer's Rule (continued)

Proof.

Alternatively, let $A(i \to v)$ denote matrix A with the i-th column replaced by vector v. It is easy to see that

$$\det I(i \to x) = x_i,$$

and that the equation $Ax = b$ is equivalent to

$$A\left(I(i \to x)\right) = A(i \to b) = A_i.$$

Taking determinants of both sides gives

$$(\det A)x_i = \det A_i.$$

□

Example

Let

$$U : \begin{cases} 2x_1 + 3x_2 = -1 \\ 3x_1 + 4x_2 = -3 \end{cases}$$

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Then

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & 3 \\ -3 & 4 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix}.$$

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Therefore, $x_1 = \frac{\det A_1}{\det A} = \frac{5}{-1} = -5, \; x_2 = \frac{\det A_2}{\det A} = \frac{-3}{-1} = 3.$

Matrix Algebra

Remarks

i) *if* $A, B \in M(n \times n; \mathbb{R})$ *and* $\det A \neq 0, \det B \neq 0$ *then the matrix* AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$,

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iv) *for* $n > 0$ *define*

$$A^n = A \cdots A \ (n - times),$$

if $\det A \neq 0$ *for* $n < 0$ *define*

$$A^n = (A^{-1})^{-n}$$

and $A^0 = I$.

Matrix Algebra (continued)

Remarks

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$$A^n A^m = A^{n+m},$$
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v) *note that unless* $AB = BA$, *in general,* $(AB)^n = (AB)(AB)\cdots(AB) \neq A^n B^n$.

Matrix Inverse Formula

Proposition

For any square matrix $A \in M(n \times n; \mathbb{R})$ the following formula holds

$$\sum_{j=1}^{n} (-1)^{i+j} a_{jk} \det A_{ji} = \begin{cases} \det A & k = i \\ 0 & k \neq i \end{cases}$$

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Proof.

Follows by the use of Laplace formula along columns of A and the fact that

$$\det \left(c_1, \ldots, \overbrace{c_k}^{k}, \ldots, \overbrace{c_k}^{i}, \ldots, c_n \right) = 0,$$

where $c_1, \ldots, c_n$ denote columns of matrix A.

Matrix Inverse Formula (continued)

Theorem

Let $A \in M(n \times n; \mathbb{R})$ be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

Proof.

Recall that

$$\operatorname{adj}(A) = \begin{bmatrix} (-1)^{1+1} \det A_{11} & (-1)^{2+1} \det A_{21} & \cdots & (-1)^{n+1} \det A_{n1} \\ (-1)^{1+2} \det A_{12} & (-1)^{2+2} \det A_{22} & \cdots & (-1)^{n+1} \det A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{1+n} \det A_{1n} & (-1)^{2+n} \det A_{2n} & \cdots & (-1)^{n+n} \det A_{nn} \end{bmatrix}.$$

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Then

$$A\operatorname{adj}(A) = \operatorname{adj}(A)A = B = [b_{ik}] = (\det A)I_n,$$

by the previous formula because

$$b_{ik} = \sum_{j=1}^{n}(-1)^{i+j}a_{jk}\det A_{ji} = \begin{cases} \det A & k = i \\ 0 & k \neq i \end{cases}$$

Cramer's Rule – Proof

Theorem (Cramer's Rule)

If $\det A \neq 0$ *then the unique solution of the system* $AX = B$ *is given by* $x_i = \frac{\det A_i}{\det A}$ *for* $i = 1, \dots, n$, *where* A_i *is the matrix* A *with* i*-th column replaced by* B.

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Since $\det A \neq 0$ matrix A is invertible therefore

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$$X = \frac{1}{\det A}\begin{bmatrix} (-1)^{1+1}\det A_{11} & (-1)^{2+1}\det A_{21} & \cdots & (-1)^{n+1}\det A_{n1} \\ (-1)^{1+2}\det A_{12} & (-1)^{2+2}\det A_{22} & \cdots & (-1)^{n+1}\det A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{1+n}\det A_{1n} & (-1)^{2+n}\det A_{2n} & \cdots & (-1)^{n+n}\det A_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

By the Laplace formula along the i-th column

$$x_i = \frac{\det\left(c_1, \ldots, \overbrace{B}^{i}, \ldots, c_n \right)}{\det A},$$

for $i = 1, \ldots, n$, where $c_1, \ldots, c_n$ are columns of matrix A. $\square$

Adjugate of a Symmetric Matrix

Proposition

Let $A \in M(n \times n; \mathbb{R})$ be a symmetric matrix of rank at most $n-1$ such that

$$A\mathbb{1} = 0,$$

where $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. Then there exists $c \in \mathbb{R}$ such that

$$\operatorname{adj}(A) = c\mathbb{1}\mathbb{1}^\intercal,$$

(i.e. all expressions $(-1)^{i+j} \det A_{ij}$ (i.e. cofactors) are equal).

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Proof.

Let $A = Q \operatorname{diag}(\lambda_1, \ldots, \lambda_n) Q^\intercal$, where Q is an orthogonal matrix (i.e. $Q^\intercal Q = I$). Since

$$\operatorname{adj}(A) = \operatorname{adj}(Q^\intercal) \operatorname{adj}(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)) \operatorname{adj}(Q) = Q \operatorname{adj}(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)) Q^\intercal,$$

$\operatorname{adj}(A) = 0$ if $\operatorname{rk}(A) \leqslant n-2$ and $\operatorname{adj}(A)$ is a rank 1 symmetric matrix if $\operatorname{rk}(A) = n-1$. In the latter case, since $A \operatorname{adj}(A) = 0$, and the kernel of A is $1-$dimensional, the columns of $\operatorname{adj}(A)$ must be equal to a multiple of $\mathbb{1}$. By symmetry all multiples must be equal, hence $\operatorname{adj}(A) = c\mathbb{1}\mathbb{1}^\intercal$.

Adjugate of a Symmetric Matrix (continued)

Remark

If $B \in M(n \times n; \mathbb{R})$ is a symmetric matrix of rank 1, then there exist $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, such that

$$\|v\| = 1,$$

$$B = cvv^{\mathsf{T}}.$$

Incidence and Adjacency Matrices

Definition

Let $G = (V, E)$ be a finite, undirected graph, where $V = \{v_1, \dots, v_n\}, E = \{e_1, \dots, e_m\}$. Then $B_G = B = [b_{ij}] \in M(n \times m; \mathbb{R})$ is the **incidence matrix** of graph G if

$$b_{ij} = 1 \text{ if and only if } v_i \in e_j,$$

and $b_{ij} = 0$ otherwise. Analogously, the matrix $A_G = A = [a_{ij}] \in M(n \times n; \mathbb{R})$ is the **adjacency matrix** of graph G if

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and $a_{ij} = 0$ otherwise.

Proposition

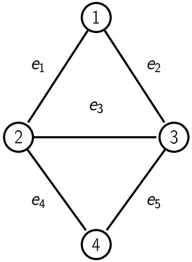
For any finite, undirected graph G

$$A^\intercal = A,$$

$$A + D = BB^\intercal,$$

where $D \in M(n \times n; \mathbb{R})$ is the **degree matrix**.

Incidence and Adjacency Matrices (continued)



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(rows of B correspond to vertices, columns of B to edges)

$$A + D = BB^{\mathsf{T}} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Kirchhoff's Theorem

Theorem

Let G be finite, connected, simple, undirected graph with n vertices. Let Q be the Laplacian matrix of G, i.e.

$$Q = D - A,$$

where D is the degree matrix of the graph G and A is the adjacency matrix of G.

Kirchhoff's Theorem

Theorem

Let G be finite, connected, simple, undirected graph with n vertices. Let Q be the Laplacian matrix of G, i.e.

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where D is the degree matrix of the graph G and A is the adjacency matrix of G. Then the number of spanning trees of G is equal to any principal minor of order $n-1$ (or to any cofactor) of matrix Q.

Incidence and Adjacency Matrices (continued)

Proof.

(sketch) Let B' be a modified incidence matrix B, where in each column the topmost 1 is exchanged to -1. Then (exercise)

$$Q = D - A = B' B'^{\mathsf{T}}.$$

Since $Q\mathbb{1} = 0$, all cofactors of Q are equal. Choose $(-1)^{1+1} \det Q_{11} = \det Q_{11}$. By the generalized Cauchy–Binet formula

$$\det Q_{11} = \det A_{S,T} = \sum_{\substack{P=\{k_1,\dots,k_{n-1}\} \\ 1\leqslant k_1<\dots<k_{n-1}\leqslant m}} \det B'_{S,P} \det B'^{\mathsf{T}}_{P,T} =$$

$$= \sum_{\substack{P=\{k_1,\dots,k_{n-1}\} \\ 1\leqslant k_1<\dots<k_{n-1}\leqslant m}} \det B'^2_{S,P},$$

where m is the number of edges and $S = T = \{2, \dots, n\}$.

Incidence and Adjacency Matrices (continued)

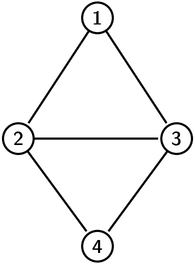
Proof.

It can be checked (by induction, exercise) that a subgraph of G spanned by $n-1$ edges contained in the set $P = \{k_1, \ldots, k_{n-1}\}$ is a spanning tree if and only if

$$\left|\det B'_{S,P}\right| = 1.$$

$\square$

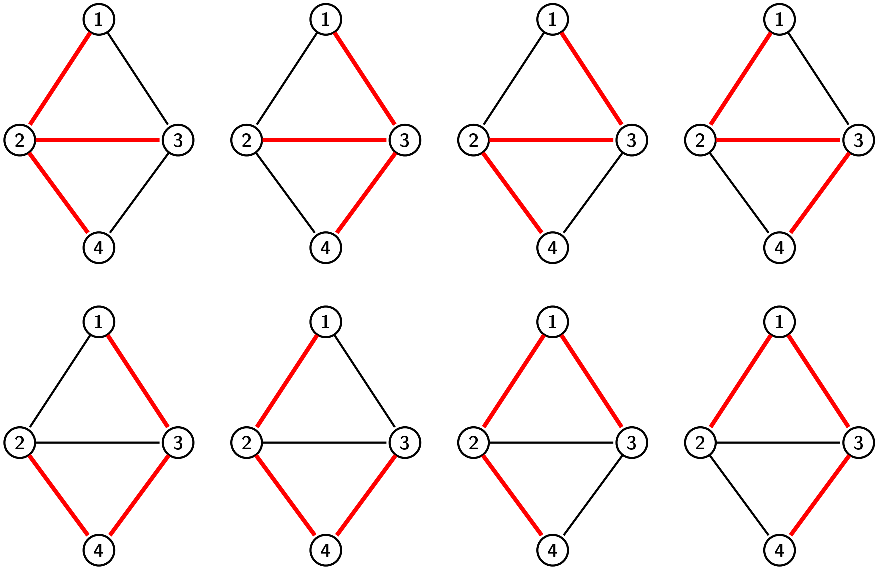
Example



$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad \mathrm{adj}(Q) = \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix}.$$

Example – 8 Spanning Trees



Cayley's formula

Theorem

The number of spanning trees of a complete n-graph is equal to n^{n-2}.

[0] for proof, see for example J. Harris, J. L. Hirst, M. Mossinghoff

Cayley's formula

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Corollary

$$\det \underbrace{\begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}}_{n-1} = n^{n-2}.$$

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$$\det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 4^2 = 16.$$

[0] for proof, see for example J. Harris, J. L. Hirst, M. Mossinghoff

Totally Unimodular Matrices

Definition

Matrix $A \in M(m \times n; \mathbb{R})$ is **totally unimodular** if any minor of A is equal to $-1, 0, 1$.

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Example

Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

is totally unimodular.

Totally Unimodular Matrices (continued)

Proposition

If matrix $A \in M(m \times n; \mathbb{R})$ *is totally unimodular then matrices*

$$-A, A^{\mathsf{T}}, \left[\begin{array}{c|c} A & I \end{array}\right], \left[\begin{array}{c|c} A & A \end{array}\right], \left[\begin{array}{c|c} A & -A \end{array}\right],$$

are totatlly unimodular.

Proof.

First two are obvious. For $\left[\begin{array}{c|c} A & I \end{array}\right]$ if the submatrix contains a column of I use Laplace's formula and the induction on the number of columns. In the two last cases, if a square submatrix contains columns i and $n+i$ then determinant is 0. Otherwise, it is equal to ± 1 to determinant of a square submatrix of A. $\square$

Bipartite Graphs

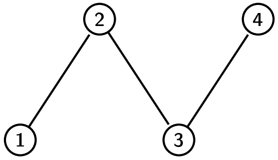
Definition

A simple undirected graph $G = (V, E)$ is a **bipartite graph** if there exists a partition of the vertex set V into to non–empty, disjoint parts V_1, V_2, i.e.

$$V = V_1 \sqcup V_2,$$

where $V_1, V_2 \neq \varnothing$ and any edge $e \in E$ join a vertex from V_1 with a vertex from V_2 (so no edge joins two vertices from V_1 or two vertices from V_2).

Example



$$V_1 = \{1, 3\}, \quad V_2 = \{2, 4\}.$$

Incidence Matrix of a Bipartite Graph is Totally Unimodular

Proposition

Let $G = (V, E)$ be a bipartite graph. Let $B_G \in M(|V| \times |E|, \mathbb{Z})$ be the incidence matrix of the graph G. Then B_G is a totally unimodular matrix.

Proof.

Let $K \in M(n \times n; \mathbb{R})$ be a square submatrix of B. Induction on n. If $n = 1$ then $\det K \in \{0, 1\}$ as entries of B are equal either to 0 or to 1. Assume $n > 1$ and all minors of order $n - 1$ are equal to $-1, 0, 1$. Any column of K contains at most two ones. If K has a zero column, then $\det K = 0$. If K has a column which contains exactly one 1 then by the Laplace formula it is equal to ± 1 times a minor of B of order $n - 1$.

Incidence Matrix of a Bipartite Graph is Totally Unimodular (continued)

Proof.

If none of the above holds every column of A contains exactly two $1's$. This means that sum of rows of K corresponding to vertices in V_1 is equal to the sum of rows of K corresponding to vertices in V_2, i.e. $\det K = 0$ (rows of K are linearly dependent). $\square$

Incidence Matrix of a Bipartite Graph is Totally Unimodular (continued)

Obviously, not every totally unimodular matrix is an incidence matrix of some bipartite graph (it can contain $-1's$). However,

Proposition

Let M be a unimodular incidence matrix of a graph G. Then G is bipartite.

Proof.

If G is not bipartite then it contains an odd cycle. Let N be a $(2k+1) \times (2k+1)$ submatrix of M corresponding to that cycle. Then (up to a permutation of rows and columns)

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

and $\det M = 1 + (-1)^{1+(2k+1)} = 2$ (Laplace's formula for the first row).

Incidence Matrix of a Bipartite Graph is Totally Unimodular (continued)

Remark

Adjacency matrix of a tree is totally unimodular (exercise). There are several characterizations of unimodular matrices (see Camion's Theorem and Ghouila-Houri's Theorem).

Lattice Points

Proposition

Let $v_1, \ldots, v_n \in \mathbb{Z}^n$ *be linearly independent over* $\mathbb{Z}$. *Let*

$$P = \{\lambda_1 v_1 + \ldots + \lambda_n v_n \in \mathbb{R}^n \mid 0 \leqslant \lambda_i < 1 \text{ for } i = 1, \ldots, n\}.$$

Then

$$|P \cap \mathbb{Z}^n| = |\det(v_1, \ldots, v_n)|.$$

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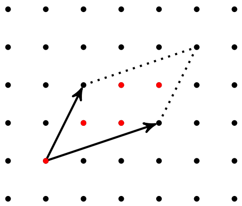

$$\left|\det \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\right| = 5 \text{ points}$$

Lattice Points (continued)

Proof.

(sketch) Let $A \in M(n \times n; \mathbb{Z})$ be a matrix with columns equal to $v_1, \ldots, v_n$. The proof follows from the Smith normal form, i.e. there exists matrices $P, Q \in M(n \times n; \mathbb{Z})$ with $\det P, \det Q = \pm 1$, such that

$$PAQ = \mathrm{diag}(a_1, \ldots, a_n).$$

Matrix of determinant ± 1, with integral coefficients induces a bijection on the lattice points and in the hyperrectangle spanned by vectors $(a_1, 0, \ldots, 0), (0, a_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n) \in \mathbb{Z}^n$ there are $|a_1| \cdot \ldots |a_n| = \det(v_1, \ldots, v_n)$ lattice points., i.e.

$$|\mathbb{Z}^n \cap \{\lambda_1 a_1 \varepsilon_1 + \ldots + \lambda_n a_n \varepsilon_n \in \mathbb{R}^n \mid 0 \leqslant \lambda_i < 1 \text{ for } i = 1, \ldots, n\}| =$$

$$= |a_1| \cdot \ldots |a_n|.$$

□

Lattice Points

Theorem

Let $P = \mathrm{conv}(p_1, \dots, p_k) \subset \mathbb{R}^n$ be a convex $n-$dimensional lattice polyhedron, i.e. $p_i \in \mathbb{Z}^n \subset \mathbb{R}^n$ for $i = 1, \dots, k$. Then

i) *there exists a degree n* **Erhart polynomial** *of P*

$$L_P(m) = c_n m^n + c_{n-1} m^{n-1} + \ldots + c_1 m + c_0,$$

such that

$$|mP \cap \mathbb{Z}^n| = L_P(m),$$

(i.e. polynomial P counts the number of lattice points in the dilated polyhedron mP),

Lattice Points

Theorem

ii) *there exists a rational function* $\mathrm{Erh}_P(x)$ *of the form*

$$\mathrm{Erh}_P(x) = \frac{h_n^* x^n + h_{n-1}^* x^{n-1} + h_1^* x + h_0^*}{(1-x)^{n+1}},$$

which Taylor–Maclaurin series at $x_0 = 0$ *is equal to the* **Erhart series**, *i.e.*

$$\mathrm{Erh}_P(x) = L_P(0) + L_P(1)x + L_P(2)x^2 + \dots,$$

that is

$$\frac{\mathrm{Erh}_P^{(m)}(0)}{m!} = L_P(m),$$

Lattice Points

Theorem

iii)

$$L_P(m) = h_n^* \binom{m}{n} + h_{n-1}^* \binom{m+1}{n} + \ldots + h_1^* \binom{m+n-1}{n} + h_0^* \binom{m+n}{n},$$

where $\binom{m}{n} = 0$ *if* $m < n$,

iv) *for* $m \geqslant 1$

$$L_P(-m) = (-1)^n L_{P^\circ}(m),$$

counts the lattice points in the **interior** *of polyhedron* P,

v) $c_n = \mathrm{vol}_n P, c_0 = h_0^* = 1.$

---

[0]See M. Beck, S.Robins *Computing the Continuous Discretely*, Springer.

Lattice Points

Theorem

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Proof.

Omitted. □

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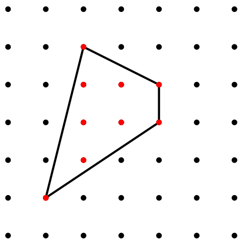
Example

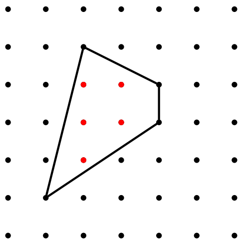


$$L_P(m) = c_2 m^2 + c_1 m + c_0,$$

$$L_P(1) = c_2 + c_1 + c_0 = 1 + 4 + 2 + 2 = 9,$$

(the lattice points are counted in columns).

Example



$$L_P(m) = c_2 m^2 + c_1 m + c_0,$$

$$L_{P^\circ}(-1) = c_2 - c_1 + c_0 = 3 + 2 = 5.$$

(the lattice points are counted in columns).

Example

That is (note $c_0 = 1$)

$$\begin{cases} c_2 & + & c_1 & = & 8 \\ c_2 & - & c_1 & = & 4 \end{cases}$$

which gives

$$\begin{cases} c_1 & = & 2 \\ c_2 & = & 6 \end{cases},$$

i.e.,

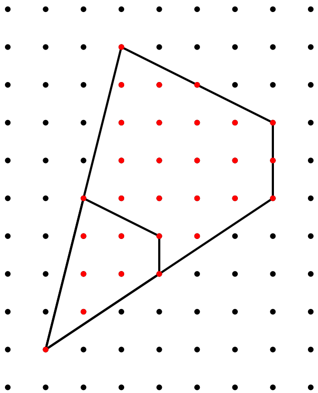
$$L_P(m) = 6m^2 + 2m + 1.$$

Moreover,

$$\mathrm{vol}_2\, P = A = I + \frac{B}{2} - 1 = 6,$$

where $I = 5,\ B = 4,$ from the Pick's formula.

Example

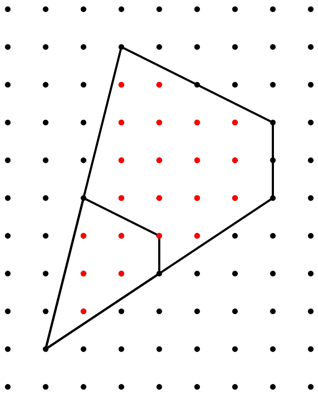


$$L_P(m) = 6m^2 + 2m + 1,$$

$$L_P(2) = 6 \cdot 2^2 + 2 \cdot 2 + 1 = 29 =$$

$$= 1 + 4 + 7 + 6 + 5 + 3 + 3.$$

Example

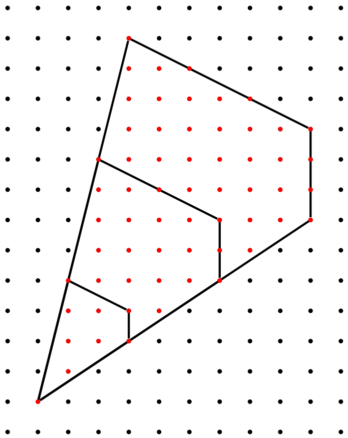


$$L_P(m) = 6m^2 + 2m + 1,$$

$$L_P(-2) = 6 \cdot (-2)^2 + 2 \cdot (-2) + 1 = 21 =$$

$$= 3 + 6 + 5 + 4 + 3 = 29 - 8.$$

Example

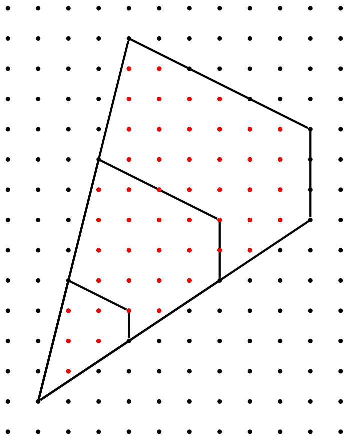


$$L_P(m) = 6m^2 + 2m + 1,$$

$$L_P(3) = 6 \cdot 3^2 + 2 \cdot 3 + 1 = 61 =$$

$$= 1 + 4 + 7 + 11 + 9 + 8 + 7 + 6 + 4 + 4.$$

Example



$$L_P(m) = 6m^2 + 2m + 1,$$

$$L_P(-3) = 6 \cdot (-3)^2 + 2 \cdot (-3) + 1 = 49 =$$

$$= 3 + 6 + 9 + 9 + 7 + 6 + 5 + 4 = 61 - 12.$$

Example

From

$$L_P(m) = h_n^* \binom{m}{n} + h_{n-1}^* \binom{m+1}{n} + \ldots + h_0^* \binom{m+n}{n},$$

for $n = 2, m = 1, 2$ we have

$$h_2^* \binom{1}{2} + h_1^* \binom{2}{2} + h_0^* \binom{3}{2} = L + P(1) = 9,$$

$$h_2^* \binom{2}{2} + h_1^* \binom{3}{2} + h_0^* \binom{4}{2} = L + P(2) = 29,$$

that is

$$\begin{cases} h_2^* & + & 3h_1^* & = & 23 \\ & & h_1^* & = & 6 \end{cases}.$$

Therefore $h_2^* = 5,\ h_2^* = 6$, that is

$$\mathrm{Erh}_P(x) = \frac{5x^2 + 6x + 1}{(1-x)^3}.$$

Example

It can be checked that the Erhart series of P is equal to

$$\mathrm{Erh}_P(x) = \frac{5x^2 + 6x + 1}{(1-x)^3} =$$

$$= L_P(0) + L_P(1)x + L_P(2)x^2 + L_P(3)x^3 + \ldots =$$

$$= 1 + 9z + 29x^2 + 61x^3 + 105x^4 + 161x^5 + 229x^6 + 309x^7 + \ldots$$