Linear Algebra

Lecture 6 - Determinants

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13 November 2023

Notation

Definition

A matrix $A \in M(n \times n; \mathbb{R})$ is called a square matrix. For any square matrix A let $A_{ij} \in M((n-1) \times (n-1); \mathbb{R})$ denote the submatrix of A formed by deleting the i-th row and j-th column of A.

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Example

$$A = \begin{bmatrix} -1 & 5 & 0 \\ 4 & -2 & 3 \\ 2 & -1 & 0 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -1 & 5 \\ 2 & -1 \end{bmatrix}.$$

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ii) if $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ and $n > 1$ then

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.$$

Examples

In particular, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\det A = (-1)^{1+1} a_{11} a_{22} + (-1)^{1+2} a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$$

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For example, $\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1 \cdot 4 - 3 \cdot 2 = -2.$

Examples (continued)

In particular, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

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$(-1)^{1+2} a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + (-1)^{1+3} a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} =$

$a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}.$

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For example, $\det \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} = 1 \cdot 3 \cdot 2 + 2 \cdot 1 \cdot 2 = 10.$

Rule of Sarrus

$$
\begin{array}{c}
\begin{array}{ccccc} + & + & + & \phantom{a_{11}} & \phantom{a_{12}} \end{array} \\
\left[\begin{array}{ccc|cc}
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
\end{array}\right] \\
\begin{array}{ccccc} - & - & - & \phantom{a_{11}} & \phantom{a_{12}} \end{array}
\end{array}
$$

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Note this DOES NOT work for n-by-n matrices for $n \geqslant 4$.

Properties of Determinants

Let $A, B, C \in M(n \times n; \mathbb{R})$

Theorem

i) *Let* $1 \leqslant k \leqslant n$. *If matrices* A, B, C *have all rows the same (resp. columns) except the* k*-th row (resp.column) and* k*-th row of* C *is the sum of* k*-th rows (resp. columns) of matrices* A *and* B *then* $\det C = \det A + \det B$*,*

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Proof.

Use induction on the matrix size. □

Examples

i)

$$\det\begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & 2 \\ 2 & -5 & 3 \\ 0 & 2 & 2 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

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ii)

$$\det\begin{bmatrix}1&3&0\\1&0&2\\0&2&2\end{bmatrix}=-\det\begin{bmatrix}1&0&2\\1&3&0\\0&2&2\end{bmatrix}$$

iii)

$$\det\begin{bmatrix}1&0&2\\3&9&0\\0&2&2\end{bmatrix}=3\det\begin{bmatrix}1&0&2\\1&3&0\\0&2&2\end{bmatrix}$$

Transposition

Definition

Let $A = [a_{ij}] \in M(m \times n; \mathbb{R})$. The matrix $B = [b_{ij}] \in M(n \times m; \mathbb{R})$ where $b_{ij} = a_{ji}$ is called **the transpose** of matrix A. We write $B = A^{\mathsf{T}}$.

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Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}, \quad A^\intercal = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 0 & 5 \end{bmatrix}.$$

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Example

$$\det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}^{\mathsf{T}} = \det \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

Laplace expansion

Theorem

Let $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ *and let* $n > 1$. *Then for any* $1 \leqslant i \leqslant n$ *(fixed* i*-th row and fixed* j*-th column, respectively)*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

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Example

$$\det \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 9 & 2 & 0 \\ 3 & 8 & 4 & 3 \\ 2 & 6 & 5 & 0 \end{bmatrix} = (-1)^{3+4} 3 \det \begin{bmatrix} 0 & 2 & 0 \\ 1 & 9 & 2 \\ 2 & 6 & 5 \end{bmatrix} =$$

$$= -3(-1)^{1+2} 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 6.$$

Determinants and Matrix Multiplication

Theorem (Special case of Cauchy-Binet formula)

Let $A, B \in M(n \times n; \mathbb{R})$. *Then* $\det AB = \det A \det B$.

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v) *an elementary operation of multiplying a row (resp. a column) of matrix A by a constant $c \in \mathbb{R}$ multiplies the determinant by constant c,*

vi) *if rows (resp. columns) of matrix A form are linearly dependent then* $\det A = 0$*.*

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vi) a row (resp. a column) is a linear combination of the other, use elementary row (resp. column) operations to get a zero row (resp. a zero column). Then use i).

Computing Determinants

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Example

Matrix $\begin{bmatrix} 1 & 0 & 1 & -1 & 7 \\ 0 & 3 & 0 & 2 & 3 \\ 0 & 0 & 5 & 0 & -2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is upper-triangular.

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Proof.

Use induction and the Laplace expansion formula along the first column of A. $\square$

Computing Determinants (continued)

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$(\Leftarrow)$ matrix A can be transformed by elementary row operations to an echelon form with a zero row. $\square$

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Use elementary operations on rows and columns in order to get as many zeroes as possible in a row or a column and use the Laplace expansion.

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How to compute determinant of matrix?

Use elementary operations on rows and columns in order to get as many zeroes as possible in a row or a column and use the Laplace expansion.

or

Put matrix in an upper-triangular form using elementary operations and take product of the diagonal entries.

Example

$$
\det \begin{bmatrix} 1 & 2 & 2 & 6 \\ 1 & 2 & 2 & 5 \\ 1 & 1 & 2 & 8 \\ 2 & 5 & 6 & 2 \end{bmatrix} \overset{r_1 - r_2}{=} \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 5 \\ 1 & 1 & 2 & 8 \\ 2 & 5 & 6 & 2 \end{bmatrix} =
$$

$$
(-1)^{1+4} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 5 & 6 \end{bmatrix} = -2 \det \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} \overset{c_3 - c_1}{=}
$$

$$
-2 \det \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} = -2(-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 2.
$$

Block Matrices

Theorem

Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ *where* A, C *are square matrices and* 0 *is a zero matrix. Then* $\det M = \det A \det C$.

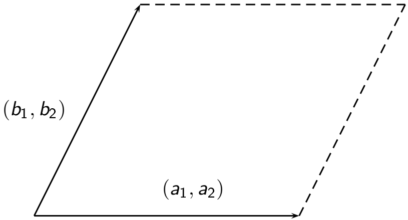
Example

Let

$$\det \begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 3 & 0 & 2 & 10 & 22 \\ 4 & 5 & 0 & 7 & 9 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \det \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} =$$

$$(2 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 5 - 1 \cdot 2 \cdot 5)(2 \cdot 2 - 1 \cdot 5) = -21.$$

Area (2−dimensional volume)



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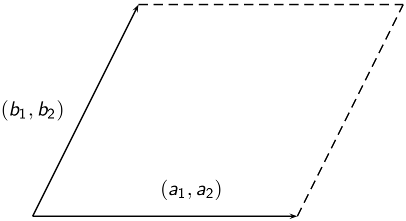
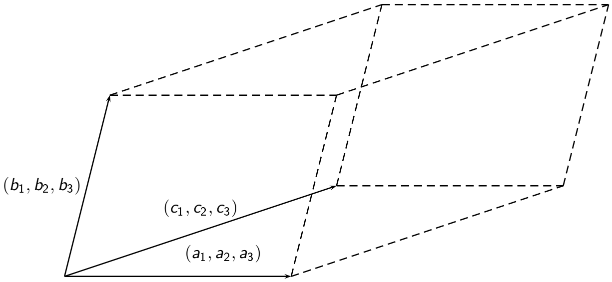


The area of a parallelogram spanned by vectors $(a_1, a_2), (b_1, b_2)$ is equal to the absolute value of $\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

Volume (3−dimensional volume)



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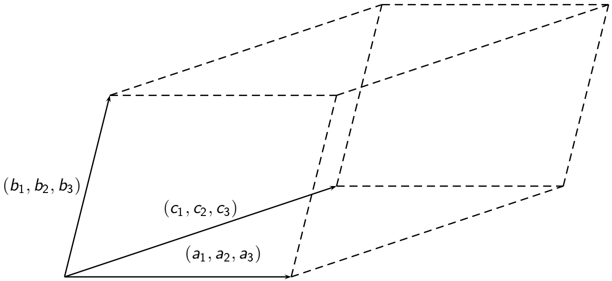


The volume of a parallelepiped spanned by vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ is equal to the absolute value of

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Volume – Motivation

Let $I_n = [0,1] \times [0,1] \times \ldots \times [0,1] = [0,1]^n \subset \mathbb{R}^n$ be an $n-$dimensional unit hypercube. The result relating volume to the determinant can be understood by checking how the elementary matrices change the $n-$dimensional volume of I (they multiply the volume by the absolute value of the determinant of the elementary matrix), i.e.

$$\mathrm{vol}_n\, D_{1,\alpha}(I_n) = \mathrm{vol}_n [0,\alpha] \times [0,1] \times \cdots \times [0,1] = \alpha\, \mathrm{vol}_n\, I_n,$$

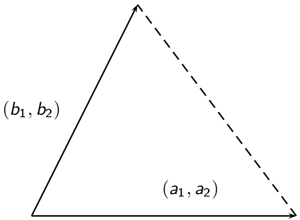
$$\mathrm{vol}_n\, L_{1,2}(I_n) = \mathrm{vol}_n\, \mathrm{conv}((0,0),(1,0),(2,1),(1,1)) \times [0,1] \times \cdots \times [0,1] =$$

$$= \mathrm{vol}_n\, I_n,$$

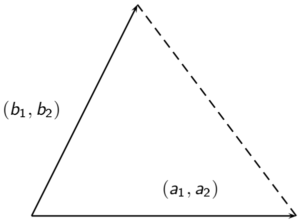
$$\mathrm{vol}_n\, T_{i,j}(I_n) = \mathrm{vol}_n\, I_n.$$

The same happens for small hypercubes and volume approximately is a sum volumes of small hypercubes (this is not a formal proof – just a loose explanation!).

Area of a 2−dimensional Simplex

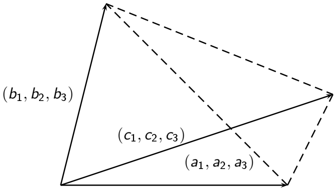


Area of a 2−dimensional Simplex

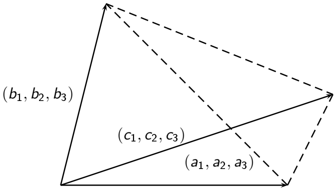


The area of a 2−dimensional simplex with vertices at $0, (a_1, a_2), (b_1, b_2)$ is equal to the absolute value of $\frac{1}{2!} \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

Volume of a 3−dimensional Simplex



Volume of a 3−dimensional Simplex



The volume of a 3−dimensional simplex with vertices in $0, (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ is equal to the absolute value of

$$\frac{1}{3!}\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Volume of a Simplex

Definition

A **simplex** with vertices in $0, v_1, \dots, v_n \in \mathbb{R}^n$ is equal to the set

$$s(v_1, \dots, v_n) = \operatorname{conv}(0, v_1, \dots, v_n) =$$

$$= \left\{ \sum_{i=1}^{n} \lambda_i v_i \in \mathbb{R}^n \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geqslant 0 \;\; i = 0, \dots, n \right\} =$$

$$= \left\{ \sum_{i=1}^{n} \lambda_i v_i \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i \leqslant 1, \lambda_i \geqslant 0 \;\; i = 1, \dots, n \right\}.$$

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Proposition

$$\operatorname{vol}_n s(v_1, \ldots, v_n) = \frac{1}{n!} |\det(v_1, \ldots, v_n)|.$$

Volume of a Simplex (continued)

Proof.

Let $V_n = \mathrm{vol}_n(\varepsilon_1, \ldots, \varepsilon_n)$. Obviously $V_1 = 1, V_2 = \frac{1}{2!}$. Assume $V_{n-1} = \frac{1}{(n-1)!}$. By the Cavalieri's principle or Fubini's theorem

$$V_n = \int_0^1 (1 - x_n)^{n-1} V_{n-1} \, \mathrm{d}x_n = \begin{vmatrix} 1 - x_n = t \\ -\mathrm{d}x_n = \mathrm{d}t \end{vmatrix} = \frac{1}{(n-1)!} \frac{t^n}{n} \bigg|_0^1 = \frac{1}{n!}.$$

By the mathematical induction $V_n = \frac{1}{n!}$ for any $n \geqslant 1$. Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear diffeomorphism given by the conditions

$$\varphi(\varepsilon_i) = v_i,$$

for $i = 1, \ldots, n$.

Volume of a Simplex (continued)

Proof.

Then (φ preserves linear combinations)

$$\varphi(s(\varepsilon_1, \ldots, \varepsilon_n)) = s(v_1, \ldots, v_n),$$

$$\det D\varphi = \det(v_1, \ldots, v_n),$$

where $D\varphi = M(\varphi)^{st}_{st}$ is the determinant of the Jacobi matrix (the derivative) of φ. Let $X = s(\varepsilon_1, \ldots, \varepsilon_n)$. By the change–of–coordinates formula

$$\operatorname{vol}_n s(v_1, \ldots, v_n) = \int_{\varphi(X)} \mathrm{d}x_1 \ldots \mathrm{d}x_n =$$

$$= \int_X |\det D\varphi| \, \mathrm{d}x_1 \ldots \mathrm{d}x_n = V_n |\det(v_1, \ldots, v_n)| =$$

$$= \frac{1}{n!} |\det(v_1, \ldots, v_n)|.$$

$\square$

Determinant of Block Matrix

Proposition

If $M \in M(n \times n; \mathbb{R})$ *is a block matrix and*

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where A *and* D *are square matrices then*

$$\det M = \begin{cases} \det A \det\left(D - CA^{-1}B\right) & \text{if} \quad \det A \neq 0 \\ \det D \det\left(A - BD^{-1}C\right) & \text{if} \quad \det D \neq 0 \\ \det A \det D & \text{if} \quad B = 0 \text{ or } C = 0 \end{cases}$$

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Proof.

$$\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline -CA^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1}B \\ \hline 0 & D - CA^{-1}B \end{array}\right]$$

$$\left[\begin{array}{c|c} I & -BD^{-1} \\ \hline 0 & D^{-1} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A - BD^{-1}C & 0 \\ \hline D^{-1}C & I \end{array}\right]$$

Example

$$M = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 6 \\ 1 & 2 & 2 & 5 \\ \hline 1 & 1 & 2 & 8 \\ 2 & 5 & 6 & 2 \end{array}\right]$$

$$D = \begin{bmatrix} 2 & 8 \\ 6 & 2 \end{bmatrix}, \quad D^{-1} = -\frac{1}{44}\begin{bmatrix} 2 & -8 \\ -6 & 2 \end{bmatrix},$$

$$\det M = \det D \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 2 & 5 \end{bmatrix}(-1)\frac{1}{44}\begin{bmatrix} 2 & -8 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}\right) =$$

$$= 44 \cdot \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - \frac{1}{44}\begin{bmatrix} 40 & 52 \\ 38 & 56 \end{bmatrix}\right) = 44 \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -\frac{10}{11} & -\frac{13}{11} \\ -\frac{19}{22} & -\frac{14}{11} \end{bmatrix}\right) =$$

$$= 44 \begin{bmatrix} \frac{1}{11} & \frac{9}{11} \\ \frac{3}{22} & \frac{8}{11} \end{bmatrix} = 44 \cdot \frac{16 - 27}{22 \cdot 11} = -2.$$

Sylvester's Determinant Theorem/Weinstein–Aronszajn Identity

Corollary

Let $A \in M(m \times n; \mathbb{R})$, $B \in M(n \times m; \mathbb{R})$ *be two matrices. Then*

$$\det(AB + I_m) = \det(BA + I_n).$$

Sylvester's Determinant Theorem/Weinstein–Aronszajn Identity

Corollary

Let $A \in M(m \times n; \mathbb{R})$, $B \in M(n \times m; \mathbb{R})$ *be two matrices. Then*

$$\det(AB + I_m) = \det(BA + I_n).$$

Proof.

Let

$$M = \left[\begin{array}{c|c} I_n & -B \\ \hline A & I_m \end{array}\right].$$

Since $\det I_n = \det I_m = 1 \neq 0$ from both formulas for the determinant of a block matrix one gets

$$\det M = \det I_n \det\big(I_m - AI_n^{-1}(-B)\big) = \det(AB + I_m),$$

$$\det M = \det I_m \det\big(I_n - (-B)I_m^{-1}A\big) = \det(BA + I_n).$$

Determinant as a Function of Matrix Rows

For matrix $A = [a_{ij}]$ let

$$r_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ldots, r_n = (a_{n1}, a_{n2}, \ldots, a_{nn}),$$

be the rows of A. Set

$$\det(r_1, \ldots, r_n) = \det A.$$

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Proposition

$$\det(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_n) = 0 \text{ for } i = 1, \dots, n.$$

Determinant as a Function of Matrix Rows

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Proposition

$$\det(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_n) = 0 \textit{ for } i = 1, \dots, n.$$

Proof.

For $i = 1$ it follows from the definition, for $i > 1$ it follows by induction (the $(i-1)$-th row in matrices A_{1j} is zero). $\square$

Determinant as a Function of Matrix Rows (continued)

Definition

Let V, W be a vector spaces. Function $\varphi \colon \underbrace{V \times \ldots \times V}_{n-\text{times}} \to W$ is called

i) **multilinear** if for any $i = 1, \ldots, n$ and $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in V$ the function

$$\varphi(v_1, \ldots, v_{i-1}, \cdot, v_{i+1}, \ldots, v_n) \colon V \to W$$

is linear,

Determinant as a Function of Matrix Rows (continued)

Definition

Let V, W be a vector spaces. Function $\varphi\colon \underbrace{V \times \ldots \times V}_{n-\text{times}} \to W$ is called

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$$\varphi(v_1, \ldots, v_{i-1}, \cdot, v_{i+1}, \ldots, v_n)\colon V \to W$$

is linear,

ii) **antisymmetric** (or skew-symmetric) if for any $1 \leqslant i < j \leqslant n$ and $v_1, \ldots, v_n \in V$

$$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -\varphi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n),$$

Determinant as a Function of Matrix Rows (continued)

Definition

iii) **alternating** if

$$\varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_n) = 0,$$

for any $v_1, \dots, v_n \in V$.

Determinant as a Function of Matrix Rows (continued)

Proposition

Over the real numbers a multilinear function f is antisymmetric if and only if it is alternating.

Proof.

Assume φ is alternating. Then

$$0 = \varphi(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_n) =$$

$$= \varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_n) + \varphi(v_1, \dots, v_j, \dots, v_i, \dots, v_n) +$$

$$+ \varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_n) + \varphi(v_1, \dots, v_j, \dots, v_j, \dots, v_n).$$

Determinant as a Function of Matrix Rows (continued)

Proof.

Assume φ is antisymmetric. Then

$$\varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_n) = -\varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_n),$$

(after replacing first v_i with the second v_i), therefore

$$2\varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_n) = 0.$$

$\square$

Determinant as a Function of Matrix Rows (continued)

Proposition

For $i = 1, \ldots, n$ *and any* $\alpha \in \mathbb{R}, r_1, \ldots, r_i, r_i', \ldots, r_n \in \mathbb{R}^n$

i) $\det(r_1, \ldots, r_{i-1}, \alpha r_i, r_{i+1}, \ldots, r_n) =$
$\alpha \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n)$,

Determinant as a Function of Matrix Rows (continued)

Proposition

For $i = 1, \ldots, n$ and any $\alpha \in \mathbb{R}, r_1, \ldots, r_i, r_i', \ldots, r_n \in \mathbb{R}^n$

i) $\det(r_1, \ldots, r_{i-1}, \alpha r_i, r_{i+1}, \ldots, r_n) =$
$\alpha \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n)$,

ii) $\det(r_1, \ldots, r_{i-1}, r_i + r_i', r_{i+1}, \ldots, r_n) =$
$\det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n) +$
$\det(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n)$.

Determinant as a Function of Matrix Rows (continued)

Proposition

For $i = 1, \ldots, n$ and any $\alpha \in \mathbb{R}, r_1, \ldots, r_i, r_i', \ldots, r_n \in \mathbb{R}^n$

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that is, determinant is a multilinear functions of matrix rows.

Determinant as a Function of Matrix Rows (continued)

Proposition

For $i = 1, \dots, n$ and any $\alpha \in \mathbb{R}, r_1, \dots, r_i, r_i', \dots, r_n \in \mathbb{R}^n$

i) $\det(r_1, \dots, r_{i-1}, \alpha r_i, r_{i+1}, \dots, r_n) = \alpha \det(r_1, \dots, r_{i-1}, r_i, r_{i+1}, \dots, r_n)$,

ii) $\det(r_1, \dots, r_{i-1}, r_i + r_i', r_{i+1}, \dots, r_n) = \det(r_1, \dots, r_{i-1}, r_i, r_{i+1}, \dots, r_n) + \det(r_1, \dots, r_{i-1}, r_i', r_{i+1}, \dots, r_n)$.

that is, determinant is a multilinear functions of matrix rows.

Proof.

For $i = 1$ it follows from the definition, for $i > 1$ it follows by induction (in matrices A_{1j} the $(i-1)$-th rows is multiplied by α or is a sum of rows r_i and r_i' with j-th coordinate removed). $\square$

Determinant as a Function of Matrix Rows (continued)

Proposition

For any $1 \leqslant i < j \leqslant n$ *and* $r_1, \ldots, r_n \in \mathbb{R}^n$

$$\det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_{j-1}, r_i, r_{j+1}, \ldots, r_n) = 0,$$

that is, determinant is alternating (hence antisymmetric) multilinear map.

Proof.

For $n = 2$ and $i = 1, j = 2$ the claim follows from the definition.

Determinant as a Function of Matrix Rows (continued)

Proposition

For any $1 \leqslant i < j \leqslant n$ *and* $r_1, \ldots, r_n \in \mathbb{R}^n$

$$\det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_{j-1}, r_i, r_{j+1}, \ldots, r_n) = 0,$$

that is, determinant is alternating (hence antisymmetric) multilinear map.

Proof.

For $n = 2$ and $i = 1, j = 2$ the claim follows from the definition.
For $n \geqslant 3$ and $i, j \neq 1$ the claim follows by induction (in matrices A_{1j} two rows are the same).

Determinant as a Function of Matrix Rows (continued)

Proposition

For any $1 \leqslant i < j \leqslant n$ *and* $r_1, \ldots, r_n \in \mathbb{R}^n$

$$\det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_{j-1}, r_i, r_{j+1}, \ldots, r_n) = 0,$$

that is, determinant is alternating (hence antisymmetric) multilinear map.

Proof.

For $n = 2$ and $i = 1, j = 2$ the claim follows from the definition.
For $n \geqslant 3$ and $i, j \neq 1$ the claim follows by induction (in matrices A_{1j} two rows are the same). It is enough to prove the case $n \geqslant 3, i = 1, j > 1$. Let $r_i^{(p)} \in \mathbb{R}^{n-1}, r_i^{(pq)} \in \mathbb{R}^{n-2}$ denote respectively, i-th row with p-th coordinate removed and i-th row with p-th and q-th coordinates removed.

Determinant as a Function of Matrix Rows (continued)

Proof.

Then

$$\det(r_1, \ldots, r_{j-1}, r_1, r_{j+1}, \ldots, r_n) =,$$

$$= \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det\left(r_2^{(k)}, \ldots, r_{j-1}^{(k)}, r_1^{(k)}, r_{j+1}^{(k)}, \ldots, r_n^{(k)}\right) =$$

(by definition)

$$= \sum_{k=1}^{n} (-1)^{(1+k)+(j-2)} a_{1k} \det\left(r_1^{(k)}, r_2^{(k)}, \ldots, r_{j-1}^{(k)}, r_{j+1}^{(k)}, \ldots, r_n^{(k)}\right) =$$

(by the inductive assumption and antisymmetry)

Determinant as a Function of Matrix Rows (continued)

Proof.

$$= \sum_{k=1}^{n} (-1)^{k+j-1} a_{1k} \left(\sum_{l=1}^{k-1} (-1)^{1+l} a_{1l} \det\left(r_2^{(kl)}, \ldots, r_{j-1}^{(kl)}, r_{j+1}^{(kl)}, \ldots, r_n^{(kl)}\right) + \right.$$

$$\left. + \sum_{l=k+1}^{n} (-1)^{l} a_{1l} \det\left(r_2^{(kl)}, \ldots, r_{j-1}^{(kl)}, r_{j+1}^{(kl)}, \ldots, r_n^{(kl)}\right) \right) = 0,$$

since the term $a_{1l} a_{1k} \det\left(r_2^{(kl)}, \ldots, r_{j-1}^{(kl)}, r_{j+1}^{(kl)}, \ldots, r_n^{(kl)}\right)$ appears in the sum exactly twice but with different signs. □

Determinant as a Function of Matrix Rows (continued)

Corollary

Adding a row multiplied by a constant to another of a matrix does not change its determinant.

Determinant as a Function of Matrix Rows (continued)

Corollary

Adding a row multiplied by a constant to another of a matrix does not change its determinant.

Proof.

$$\det(r_1 + \alpha r_2, r_2, \ldots, r_n) = \det(r_1, r_2, \ldots, r_n) +$$
$$+\alpha \det(r_2, r_2, \ldots, r_n) = \det(r_1, r_2, \ldots, r_n).$$

$\square$

Determinant as a Function of Matrix Rows (continued)

Corollary

The Laplace formula for rows holds.

Proof.

$$\det(r_1, \ldots, r_i, \ldots, r_n) =$$

$$= (-1)^{i-1} \det(r_i, r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n) =$$

$$= (-1)^{i-1} \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \det\left(r_1^{(j)}, \ldots, r_{i-1}^{(j)}, r_{i+1}^{(j)}, r_n^{(j)}\right).$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(r_1^{(j)}, \ldots, r_{i-1}^{(j)}, r_{i+1}^{(j)}, r_n^{(j)}\right).$$

$\square$

Determinant of Matrix Product

Proposition

If A is one of the elementary matrices $D_{i,\alpha}, L_{ij}, T_{ij}$ or matrix A is in the reduced echelon form then

$$\det AB = \det A \det B.$$

Proof.

Multiplying matrix B on the left by matrix $D_{i,\alpha}$ corresponds to an elementary operation of multiplying i-th row of matrix B by the constant $\alpha \in \mathbb{R}$, multiplying matrix B on the left by matrix D_{ij} corresponds an elementary operation of adding j-th row of matrix B to the i-th one, multiplying matrix B on the left by matrix T_{ij} corresponds an elementary operation of swapping i-th and j-th row of matrix B. Determinant is an antisymmetric multilinear maps, therefore

Determinant of Matrix Product (continued)

Proof.

$$\det D_i = \alpha, \qquad \det L_{ij} = 1, \qquad \det T_{ij} = -1,$$

$$\det D_i B = \alpha \det B, \qquad \det L_{ij} B = 1 \cdot \det B, \qquad \det T_{ij} B = (-1) \cdot \det B.$$

Determinant of Matrix Product (continued)

Proof.

$$\det D_i = \alpha, \qquad \det L_{ij} = 1, \qquad \det T_{ij} = -1,$$

$$\det D_i B = \alpha \det B, \qquad \det L_{ij} B = 1 \cdot \det B, \qquad \det T_{ij} B = (-1) \cdot \det B.$$

If A if in the reduced echelon form the either A has a zero row or $A = I$. Then, respectively

$$\det A = 0, \qquad \det A = 1,$$

$$\det AB = 0 \cdot \det B, \qquad \det AB = 1 \cdot \det B,$$

since matrix AB has a zero row too.

□

Determinant of Matrix Product (continued)

Corollary

For any matrices $A, B \in M(n \times n; \mathbb{R})$

$$\det AB = \det A \det B.$$

Determinant of Matrix Product (continued)

Corollary

For any matrices $A, B \in M(n \times n; \mathbb{R})$

$$\det AB = \det A \det B.$$

Proof.

Matrix A can be brought by elementary row operations to the reduced echelon form. Therefore there exist elementary matrices $E_1, \ldots, E_k$ and matrix S in the reduced echelon form such that

$$A = E_1 E_2 \ldots E_k S.$$

Therefore

$$\det AB = (\det E_1 \det E_2 \cdots \det E_k \det S) \det B = \det A \det B.$$

□

Determinant of a Transposed Matrix

Proposition

For any matrix $A \in M(n \times n; \mathbb{R})$

$$\det A^{\mathsf{T}} = \det A.$$

Proof.

Matrix A can be brought by elementary row operations to the reduced echelon form. Therefore there exist elementary matrices $E_1, \ldots, E_k$ and matrix S in the reduced echelon form such that

$$A = E_1 E_2 \ldots E_k S.$$

Determinant of a Transposed Matrix (continued)

Proof.

Matrix S has either a zero row or $S = I$. Then, respectively $\det S^{\intercal} = 0$ (since matrix $S^{\intercal}$ has a zero column, therefore its reduced echelon form has a zero row) or $\det S^{\intercal} = \det I = 1$. Moreover

$$\det D_{i,\alpha}^{\intercal} = \det D_{i,\alpha} = \alpha, \quad \det L_{ij}^{\intercal} = \det L_{ji} = 1, \quad \det T_{ij}^{\intercal} = \det T_{ij} = -1,$$

therefore

$$\det A^{\intercal} = \det S^{\intercal} \det E_k^{\intercal} \det E_{k-1}^{\intercal} \cdots \det E_1^{\intercal} = \det A.$$

$\square$

Corollary

The Laplace formula for columns hold. Determinant is antisymmetric mulitilinear map of its columns.

Inverse of an Elementary Matrix

Proposition

It can be directly checked that

$$D_{i,\alpha}D_{i,\alpha^{-1}} = D_{i,\alpha^{-1}}D_{i,\alpha} = I,$$

$$L_{ij}(2I - L_{ij}) = (2I - L_{ij})L_{ij} = I,$$

$$T_{ij}T_{ij} = I,$$

i.e., elementary matrices have left- and right-hand side inverse matrices which are equal to each other.

Inverse of an Elementary Matrix (continued)

Remark

$$L_{ij}^{-1} = 2I - L_{ij} = \begin{array}{cc} & \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{array}{c} \\ i \\ \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array} = D_{j,-1} L_{ij} D_{j,-1},$$

and

$$(2I - L_{ij})A = \textit{matrix } A, \ \textit{with } j-th \textit{ row subtracted from the } i-th,$$

$$A(I-2L_{ij}) = \textit{matrix } A, \ \textit{with } i-th \textit{ column subtracted from the } j-thone,$$

in particular, the inverse of matrix L_{ij} is a product of elementary matrices.

Inverse Matrix

Corollary

Let $A \in M(n \times n; \mathbb{R})$ *be any matrix. Then* $\det A = 0$ *if and only if the reduced echelon form of* A *has a zero row and* $\det A \neq 0$ *if and only if the reduced echelon form of* A *is equal to* I *(the unit matrix).*

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Corollary

If $AB = A'B = I$ *then* $A = A'$*, i.e the left–hand side inverse, if it exists, is unique.*

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If $AB = A'B = I$ *then* $A = A'$*, i.e the left–hand side inverse, if it exists, is unique. Analogously, the right–hand side inverse is unique.*

Proof.

If $AB = I$ then $\det B \neq 0$ so $B = E_1 E_2 \cdots E_k I$, where $E_1, \ldots, E_k$ are elementary matrices, which have inverses.

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Corollary

Let $A \in M(n \times n; \mathbb{R})$ be any matrix. Then $\det A = 0$ if and only if the reduced echelon form of A has a zero row and $\det A \neq 0$ if and only if the reduced echelon form of A is equal to I (the unit matrix).

Corollary

If $AB = A'B = I$ then $A = A'$, i.e the left–hand side inverse, if it exists, is unique. Analogously, the right–hand side inverse is unique.

Proof.

If $AB = I$ then $\det B \neq 0$ so $B = E_1 E_2 \cdots E_k I$, where $E_1, \ldots, E_k$ are elementary matrices, which have inverses. Multiplying respectively by $E_k^{-1}, \ldots, E_1^{-1}$ the equation on the right $AB = A'B$ we get $A = A'$. $\square$

Inverse Matrix (continued)

Corollary

If $AB = I$ then $BA = I$ (i.e. the right-hand side inverse of A is also its left-hand side inverse).

Proof.

If $AB = I$ then $\det B \neq 0$ so $B = E_1 E_2 \cdots E_k I$, where $E_1, \dots, E_k$ are elementary matrices, which have inverses. Therefore

$$A = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1},$$

and

$$BA = E_1 \cdots E_k E_k^{-1} \cdots E_1^{-1} = I.$$

$\square$

Inverse Matrix (continued)

Corollary

Matrix A is invertible if and only if $\det A \neq 0$. *Moreover*

$$[A \mid I] \xrightarrow{\substack{\textit{elt. row}\\ \textit{operations}}} \left[I \mid A^{-1}\right].$$

Proof.

If $AB = I$ then $\det A \det B = 1$, therefore $\det A \neq 0$. If $\det A \neq 0$ then $A = E_1 E_2 \cdots E_k I$, where $E_1, \ldots, E_k$ are elementary matrices. Then

$$B = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} I,$$

and $AB = I$. $\square$

Gram Determinant

Definition

For any vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ let $A \in M(n \times k; \mathbb{R})$ be a matrix with columns equal to $v_1, \ldots, v_k$. The **Gram determinant** is

$$G(v_1, \ldots, v_k) = \det \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_k \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_k \\ \vdots & \vdots & \ddots & \vdots \\ v_k \cdot v_1 & v_k \cdot v_2 & \cdots & v_k \cdot v_k \end{bmatrix} = \det A^\intercal A.$$

k-dimensional Volume of k-dimensional Parallelotope

Theorem

The $k-$dimensional volume of a parallelotope spanned by vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is equal to $\sqrt{G(v_1, \ldots, v_k)}$.

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Proof.

The proof follows by induction on k. For $k = 1$

$$\sqrt{G(v_1)} = \sqrt{\det \left[\begin{array}{c} v_1 \cdot v_1 \end{array} \right]} = ||v_1||.$$

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Proof.

The proof follows by induction on k. For $k = 1$

$$\sqrt{G(v_1)} = \sqrt{\det \left[\; v_1 \cdot v_1 \; \right]} = ||v_1||.$$

If $k \geqslant 2$ let $V(v_1, \dots, v_k)$ denote the k-dimensional volume of a parallelotope spanned by $v_1, \dots, v_k$. Assume that

$$V(v_1, \dots, v_k) = V(v_1, \dots, v_{k-1})h,$$

where h is the distance of the vector v_k from the subspace $V = \mathrm{lin}(v_1, \dots, v_{k-1})$ (k-dimensional volume is equal to the $(k-1)$-dimensional volume of the base times the height).

k-dimensional Volume of k-dimensional Parallelotope

Theorem

The $k-$dimensional volume of a parallelotope spanned by vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is equal to $\sqrt{G(v_1, \ldots, v_k)}$.

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where h is the distance of the vector v_k from the subspace $V = \operatorname{lin}(v_1, \ldots, v_{k-1})$ (k-dimensional volume is equal to the $(k-1)$-dimensional volume of the base times the height). Let $w = \sum_{i=1}^{k-1} \alpha_i v_i$ be such a vector in V that $h = ||v_k - w||$.

k-dimensional Volume of k-dimensional Parallelotope (continued)

Proof.

That is

$$(v_k - w) \perp v_j \iff \sum_{i=1}^{k-1} \alpha_i v_j \cdot v_i = v_j \cdot v_k \quad \text{for} \quad j = 1, \ldots, k-1.$$

Moreover

$$h^2 = (v_k - w) \cdot (v_k - w) = (v_k - w) \cdot v_k \iff$$
$$\iff \sum_{i=1}^{k-1} \alpha_i (v_k \cdot v_i) + h^2 = v_k \cdot v_k.$$

k-dimensional Volume of k-dimensional Parallelotope (continued)

Proof.

The system of k linear equations with variables $\alpha_1, \dots, \alpha_{k-1}$ and h^2

$$\left\{\begin{array}{ccccccccccc} (v_1 \cdot v_1)\alpha_1 & + & (v_1 \cdot v_2)\alpha_2 & + & \dots & + & (v_1 \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & v_1 \cdot v_k \\ (v_2 \cdot v_1)\alpha_1 & + & (v_2 \cdot v_2)\alpha_2 & + & \dots & + & (v_2 \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & v_2 \cdot v_k \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \vdots \\ (v_{k-1} \cdot v_1)\alpha_1 & + & (v_{k-1} \cdot v_2)\alpha_2 & + & \dots & + & (v_{k-1} \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & (v_{k-1} \cdot v_k) \\ (v_k \cdot v_1)\alpha_1 & + & (v_k \cdot v_2)\alpha_2 & + & \dots & + & (v_k \cdot v_{k-1})\alpha_{k-1} & + & h^2 & = & v_k \cdot v_k \end{array}\right.$$

k-dimensional Volume of k-dimensional Parallelotope (continued)

Proof.

The system of k linear equations with variables $\alpha_1, \ldots, \alpha_{k-1}$ and h^2

$$\begin{cases} (v_1 \cdot v_1)\alpha_1 & + & (v_1 \cdot v_2)\alpha_2 & + & \ldots & + & (v_1 \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & v_1 \cdot v_k \\ (v_2 \cdot v_1)\alpha_1 & + & (v_2 \cdot v_2)\alpha_2 & + & \ldots & + & (v_2 \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & v_2 \cdot v_k \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \vdots \\ (v_{k-1} \cdot v_1)\alpha_1 & + & (v_{k-1} \cdot v_2)\alpha_2 & + & \ldots & + & (v_{k-1} \cdot v_{k-1})\alpha_{k-1} & + & 0h^2 & = & (v_{k-1} \cdot v_k) \\ (v_k \cdot v_1)\alpha_1 & + & (v_k \cdot v_2)\alpha_2 & + & \ldots & + & (v_k \cdot v_{k-1})\alpha_{k-1} & + & h^2 & = & v_k \cdot v_k \end{cases}$$

can be solved by Cramer's rule, i.e.

$$h^2 = \frac{G(v_1, \ldots, v_k)}{G(v_1, \ldots, v_{k-1})}.$$

$\square$

k-dimensional Volume of k-dimensional Parallelotope (continued)

Proposition

Let $v_1, \dots, v_n \in \mathbb{R}^n$ be any vectors. Let $A \in M(n \times n; \mathbb{R})$ be a matrix which columns are equal to $v_1, \dots, v_n$. Then

$$V(v_1, \dots, v_n) = |\det A|,$$

where $V(v_1, \dots, v_n)$ is the n-dimensional volume of a parallelotope spanned by $v_1, \dots, v_n$.

k-dimensional Volume of k-dimensional Parallelotope (continued)

Proposition

Let $v_1, \dots, v_n \in \mathbb{R}^n$ be any vectors. Let $A \in M(n \times n; \mathbb{R})$ be a matrix which columns are equal to $v_1, \dots, v_n$. Then

$$V(v_1, \dots, v_n) = |\det A|,$$

where $V(v_1, \dots, v_n)$ is the n-dimensional volume of a parallelotope spanned by $v_1, \dots, v_n$.

Proof.

$$V(v_1, \dots, v_n) = \sqrt{\det A^{\intercal} A} = \sqrt{(\det A)^2} = |\det A|.$$

$\square$

Cauchy–Binet Formula

Theorem (Cauchy–Binet)

Let $A \in M(m \times n; \mathbb{R})$, $B \in M(n \times m; \mathbb{R})$ *be matrices such that* $m \leqslant n$. *For any subset* $S \subset \{1, \dots, n\}$ *of* m *elements let* $A_{m,S} \in M(m \times m; \mathbb{R})$ *denote the square submatrix of matrix* A *consisting of columns indexed by* S. *Let* $B_{S,m} \in M(m \times m; \mathbb{R})$ *denote the square submatrix of matrix* B *consisting of rows indexed by* S. *Then*

$$\det AB = \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} \det A_{m,S} \det B_{S,m}.$$

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$$\det AB = \sum_{\substack{S \subset \{1,\ldots,n\} \\ \#S = m}} \det A_{m,S} \det B_{S,m}.$$

If $m > n$ *then* $\det(AB) = 0$

Cauchy–Binet Formula (continued)

Proof.

If $A = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ then the claim holds because

$$\det AB = \det A_{m,\{1,\ldots,m\}} B_{\{1,\ldots,m\},m}.$$

Cauchy–Binet Formula (continued)

Proof.

If $A = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ then the claim holds because

$$\det AB = \det A_{m,\{1,\ldots,m\}} B_{\{1,\ldots,m\},m}.$$

In particular, if $m > n$ then $k \leqslant n < m$ therefore the matrix $AB \in M(m \times m; \mathbb{R})$ has a zero row hence $\det AB = 0$ (columns of AB are linear combinations of n vectors in $\mathbb{R}^m$).

Cauchy–Binet Formula (continued)

Proof.

If the claim holds for some matrices A, B then it holds for matrices EA, BF where $E, F \in M(m \times m; \mathbb{R})$ are any elementary matrices because

$$\det(EA)_{m,S} = \det E \det A_{m,S}, \quad \det(BF)_{S,m} = \det B_{S,m} \det F,$$

and

$$\det(EA)(BF) = \det E \det AB \det F =$$

$$= \det E \left(\sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} \det A_{m,S} \det B_{S,m} \right) \det F =$$

$$= \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} \det(EA)_{m,S} \det(BF)_{S,m}.$$

Cauchy–Binet Formula (continued)

Proof.

If the claim holds for some matrices A, B then it holds for matrices $AE, E^{-1}B$ when

i) $E = D_{i,\alpha}$ because if $i \in S$ then

$$\det(AD_{i,\alpha})_{m,S} = \alpha \det A_{m,S}, \quad \det\left(D_{i,\alpha}^{-1}B\right)_{S,m} = \alpha^{-1} \det B_{S,m},$$

and if $i \notin S$ then

$$\det(AD_{i,\alpha})_{m,S} = \det A_{m,S}, \quad \det\left(D_{i,\alpha}^{-1}B\right)_{S,m} = \det B_{S,m},$$

$$\det(AD_{i,\alpha})(D_{i,\alpha}^{-1}B) = \det AB = \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} \det A_{m,S} \det B_{S,m} =$$

$$= \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} \det(AD_{i,\alpha})_{m,S} \det\left(D_{i,\alpha}^{-1}B\right)_{S,m}.$$

Cauchy–Binet Formula (continued)

Proof.

ii) $E = L_{ij}$ because for any $S \subset \{1, \dots, n\}$

$$\det(AL_{ij}) = \det A_{m,S}, \quad \det\left(L_{ij}^{-1}B\right)_{S,m} = \det B_{S,m}.$$

The claim holds for matrices AL_{ij} and $L_{ij}^{-1}B$ by the similar formula as above.

Cauchy–Binet Formula (continued)

Proof.

For any $S \subset \{1, \ldots, n\}$ and $1 \leqslant i, j \leqslant n$ define the map $f\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $f(i) = j, f(j) = i, f(k) = k, k \neq i, j$ and let $S_{ij} = f(S)$.

iii) $E = T_{ij}$ because

$$\det (AT_{ij})_{m,S} = \varepsilon_S \det A_{m,S_{ij}}, \quad \det \left(T_{ij}^{-1} B\right)_{m,S} = \varepsilon_S \det B_{S_{ij},m},$$

where $\varepsilon_S \in \{-1, 1\}$ (for example $\varepsilon_S = 1$ if $i, j \notin S$ and $\varepsilon_S = -1$ if $i, j \in S$). Therefore

$$\det(AT_{ij})(T_{ij}^{-1} B) = \det AB = \sum_{\substack{S \subset \{1,\ldots,n\} \\ \# S = m}} (\varepsilon_S \det A_{m,S}) (\varepsilon_S \det B_{S,m}) =$$

$$= \sum_{\substack{S \subset \{1,\ldots,n\} \\ \# S = m}} \det(AT_{ij})_{m,S} \det\left(T_{ij}^{-1} B\right)_{S,m}.$$

Cauchy–Binet Formula (continued)

Proof.

By elementary row operation (i.e. by multiplying by elementary matrices on the left) matrix A can be put into the reduced echelon form and then the reduced echelon form of A can be put by elementary column operations (i.e. by multiplying the reduced echelon form by elementary matrices on the right) into the form

$$\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right].$$

Therefore the Cauchy–Binet formula holds for any matrices $A \in M(m \times n; \mathbb{R}), B \in M(n \times m; \mathbb{R})$.

$\square$

Cauchy–Binet Formula (continued)

Corollary

For any $A \in M(n \times m; \mathbb{R})$

$$\det(A^{\mathsf{T}}A) = \begin{cases} 0 & m > n \\ (\det A)^2 & m = n \\ \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} (\det A_{S,m})^2 & m < n \end{cases}$$

Cauchy–Binet Formula (continued)

Corollary

For any $A \in M(n \times m; \mathbb{R})$

$$\det(A^\intercal A) = \begin{cases} 0 & m > n \\ (\det A)^2 & m = n \\ \sum_{\substack{S \subset \{1,\dots,n\} \\ \#S = m}} (\det A_{S,m})^2 & m < n \end{cases}$$

Corollary (Generalized Pythagorean Theorem)

The square of m-dimensional volume of a parallelotope spanned by m vectors in $\mathbb{R}^n$ is equal to the sum of squares of the m-dimensional volumes of its projections on all m-dimensional coordinate subspaces for any $m \leqslant n$.

Generalized Cauchy–Binet Formula

Let $A_{S,T}$ denote the submatrix of matrix $A \in M(m \times n; \mathbb{R})$ consisting of rows $S \subset \{1, \ldots, m\}$ and columns $T \subset \{1, \ldots, n\}$.

Generalized Cauchy–Binet Formula

Let $A_{S,T}$ denote the submatrix of matrix $A \in M(m \times n; \mathbb{R})$ consisting of rows $S \subset \{1, \ldots, m\}$ and columns $T \subset \{1, \ldots, n\}$.

Proposition

For any matrices $A \in M(m \times n; \mathbb{R}), B \in M(n \times k; \mathbb{R})$, *any* $q \leqslant \max\{m, n, k\}$ *and any* $S = \{i_1, \ldots, i_q\}, \; T = \{j_1, \ldots, j_q\}$

$$\det(AB)_{S,T} = \sum_{\substack{Q=\{k_1,\ldots,k_q\} \\ 1 \leqslant k_1 < \ldots < k_q \leqslant n}} \det A_{S,Q} \det B_{Q,T}.$$

Generalized Cauchy–Binet Formula

Let $A_{S,T}$ denote the submatrix of matrix $A \in M(m \times n; \mathbb{R})$ consisting of rows $S \subset \{1, \ldots, m\}$ and columns $T \subset \{1, \ldots, n\}$.

Proposition

For any matrices $A \in M(m \times n; \mathbb{R}), B \in M(n \times k; \mathbb{R})$*, any* $q \leqslant \max\{m, n, k\}$ *and any* $S = \{i_1, \ldots, i_q\}, \ T = \{j_1, \ldots, j_q\}$

$$\det(AB)_{S,T} = \sum_{\substack{Q=\{k_1,\ldots,k_q\} \\ 1\leqslant k_1<\ldots<k_q\leqslant n}} \det A_{S,Q} \det B_{Q,T}.$$

Proof.

(sketch)

$$\bigwedge^q AB = \bigwedge^q A \bigwedge^q B,$$

and the entries of $\bigwedge^q A$ (resp. $\bigwedge^q B$) are all order q minors of the matrix A (resp. the matrix B). □

Sylvester's Theorem

Proposition

Let $A \in M(n \times n; \mathbb{R})$ be a square matrix. Let $1 \leqslant p < q \leqslant n$ and $1 \leqslant s < t \leqslant n$. Then

$$\det A \det A_{pq,st} = \det \begin{bmatrix} \det A_{ps} & \det A_{pt} \\ \det A_{qs} & \det A_{qt} \end{bmatrix},$$

where $A_{pq,ij} \in M((n-2) \times (n-2); \mathbb{R})$ denotes matrix A with rows p and q and columns s and t removed.

Proof.

It is enough to prove the theorem for $p = s = n-1$ and $q = t = n$ (exercise). The proof is taken from G. A. Baker, P. Graves–Morris, Padé Approximants, Cambridge University Press.

Sylvester's Theorem

Proof.

Let $A = [a_{ij}] \in M(n \times n; \mathbb{R})$ and let $R_{n-1}, R_n, C_{n-1}, C_n$ denote the corresponding rows and columns of matrix A but without the last two entries. Let $B = A_{(n-1)n,(n-1)n} \in M((n-2) \times (n-2); \mathbb{R})$ be the remaining matrix, i.e.

$$A = \left[\begin{array}{c|c|c} B & C_{n-1} & C_n \\ \hline R_{n-1} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ \hline R_n & a_{n(n-1)} & a_{nn} \end{array}\right].$$

Let

$$C = \left[\begin{array}{c|c|c|c} B & C_{n-1} & C_n & 0 \\ \hline R_{n-1} & a_{(n-1)(n-1)} & a_{(n-1)n} & 0 \\ \hline R_n & a_{n(n-1)} & a_{nn} & R_n \\ \hline 0 & 0 & 0 & B \end{array}\right].$$

Sylvester's Theorem

Proof.

$$\det C = \det A \det A_{pq,ij} \overset{r_4 + r_1}{=}$$

$$= \det \left[\begin{array}{c|c|c|c} B & C_{n-1} & C_n & 0 \\ \hline R_{n-1} & a_{(n-1)(n-1)} & a_{(n-1)n} & 0 \\ \hline R_n & a_{n(n-1)} & a_{nn} & R_n \\ \hline B & C_{n-1} & C_n & B \end{array}\right] \overset{c_1 - c_4}{=}$$

$$= \det \left[\begin{array}{c|c|c|c} B & C_{n-1} & C_n & 0 \\ \hline R_{n-1} & a_{(n-1)(n-1)} & a_{(n-1)n} & 0 \\ \hline 0 & a_{n(n-1)} & a_{nn} & R_n \\ \hline 0 & C_{n-1} & C_n & B \end{array}\right] =$$

(sum in the second column)

Sylvester's Theorem

Proof.

$$= \det \left[\begin{array}{c|c|c|c} B & C_{n-1} & C_n & 0 \\ \hline R_{n-1} & a_{(n-1)(n-1)} & a_{(n-1)n} & 0 \\ \hline 0 & 0 & a_{nn} & R_n \\ \hline 0 & 0 & C_n & B \end{array} \right] +$$

$$+ \det \left[\begin{array}{c|c|c|c} B & 0 & C_n & 0 \\ \hline R_{n-1} & 0 & a_{(n-1)n} & 0 \\ \hline 0 & a_{n(n-1)} & a_{nn} & R_n \\ \hline 0 & C_{n-1} & C_n & B \end{array} \right] =$$

Sylvester's Theorem

Proof.

$$= \det \left[\begin{array}{c|c} B & C_{n-1} \\ \hline R_{n-1} & a_{(n-1)(n-1)} \end{array}\right] \det \left[\begin{array}{c|c} a_{nn} & R_n \\ \hline C_n & B \end{array}\right] -$$

$$- \det \left[\begin{array}{c|c} B & C_n \\ \hline R_{n-1} & a_{(n-1)n} \end{array}\right] \det \left[\begin{array}{c|c} a_{n(n-1)} & R_n \\ \hline C_{n-1} & B \end{array}\right] =$$

$$= \det A_{(n-1)(n-1)} \det A_{nn} - \det A_{(n-1)n} \det A_{n(n-1)}.$$

Exchanging two appropriate rows and columns does not change signs in the above equation.

Permutation Matrix

Definition

For any permutation $\sigma \in S_n$ let $P_\sigma = [p_{ij}] \in M(n \times n; \mathbb{R})$ be its **permutation matrix** given by

$$p_{ij} = \begin{cases} 0 & i \neq \sigma(j) \\ 1 & i = \sigma(j) \end{cases}$$

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iv) $\mathrm{sgn}(\sigma) = \det P_\sigma$.

Permutation Matrix (continued)

Example

Let $\sigma = (1,2,3) \in S_3$. Then $P_\sigma^3 = P_{\sigma^3} = I$

$$P_\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{\sigma^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

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Remark

Some sources define $P_\sigma^\top$ as the permutation matrix of σ.

Permutation Matrix (continued)

Proposition

For any permutation $\sigma \in S_n$ *and matrix* $A \in M(n \times m; \mathbb{R})$ *with rows* $r_1, \ldots, r_n$ *and matrix* $B \in M(m \times n; \mathbb{R})$ *with columns* $c_1, \ldots, c_n$

v) $P_\sigma A$ *has rows* $r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)}$,

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$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

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Triangular and Unitriangular Matrices

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General Linear Group

Definition

The (real) general linear group $\mathrm{GL}(n, \mathbb{R})$ is the group of all (real) invertible n-by-n matrices, i.e.,

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M(n \times n; \mathbb{R}) \mid \det A \neq 0\}.$$

Weyl Subgroup and Borel Subgroup

Definition

The **Weyl subgroup** $W = W_n$ is the subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R})$ consisting of all permutation matrices, i.e.,

$$W_n = \{P_\sigma \in GL(n, \mathbb{R}) \mid \sigma \in S_n\}.$$

The **standard Borel subgroup** $B = B_n$ is the subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R})$ consisting of all invertible upper triangular matrices, i.e.,

$$B_n = \{A \in GL(n, \mathbb{R}) \mid A \text{ is upper triangular}\}.$$

Borel subgroup of $\mathrm{GL}(n, \mathbb{R})$ is any subgroup conjugated with the standard Borel subgroup, i.e. is of the form hBh^{-1} for some matrix $h \in \mathrm{GL}(n, \mathbb{R})$.

Transvections

Definition

For any $\alpha \in \mathbb{R}$ and $i \neq j$ where $1 \leqslant i, j \leqslant n$ a **transvection** is a matrix $X_{ij}(\alpha) \in M(n \times n; \mathbb{R})$ given by the condition

$$X_{ij}(\alpha) = I_n + \alpha E_{ij},$$

where $E_{ij} = [e_{ij}] \in M(n \times n; \mathbb{R})$ and

$$e_{kl} = \begin{cases} 1 & k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$$

[0]I am following J. L. Alperin, R. B. Bell *Groups and Representations*, Springer 1995

Transvections (continued)

Proposition

Let $\alpha, \beta \in \mathbb{R}$ and let i, j, k be any pairwise distinct numbers. Then

i) $\det X_{ij}(\alpha) = 1$ *hence* $X_{ij}(\alpha) \in GL(n, \mathbb{R})$,

ii) *if* $\alpha \neq 0$ *then*
$$X_{ij}(\alpha) \in B \;\Leftrightarrow\; i < j,$$

iii) $X_{ij}(\alpha)X_{ij}(\beta) = X_{ij}(\alpha + \beta)$,

iv) $X_{ij}(\alpha)^{-1} = X_{ij}(-\alpha)$,

v) $[X_{ij}(\alpha), X_{jk}(\beta)] = X_{ik}(\alpha\beta)$, *where* $[A, B] = AB - BA$,

vi) $P_\sigma X_{ij}(\alpha) P_\sigma^\intercal = X_{\sigma(i)\sigma(j)}(\alpha)$ *for any* $P_\sigma \in W$.

Transvections (continued)

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vi) $P_\sigma X_{ij}(\alpha) P_\sigma^\top = X_{\sigma(i)\sigma(j)}(\alpha)$ *for any* $P_\sigma \in W$.

Proof.

Exercise. $\square$

Transvections (continued)

Proposition

Let $\alpha \in \mathbb{R}$ *and let* $i \neq j$. *Assume* $A \in M(n \times n; \mathbb{R})$ *has rows* $r_1, \ldots, r_n$ *and columns* $c_1, \ldots, c_n$. *Then*

i) $X_{ij}(\alpha)A$ *is equal to matrix* A *whose* i*-th row is equal to* $r_i + \alpha r_j$,

ii) $AX_{ij}(\alpha)$ *is equal to matrix* A *whose* j*-column row is equal to* $c_j + \alpha c_i$.

Transvections (continued)

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Bruhat Decomposition of $\mathrm{GL}(n, \mathbb{R})$

The following result is a simple particular case of a more general result valid for any algebraic group G. This particular case is closely related to the reduced echelon form.

Proposition

For any matrix $A \in \mathrm{GL}(n, \mathbb{R})$ there exists a matrix $P_\sigma \in W_n$ and matrices $b, b' \in B$ such that

$$A = bP_\sigma b'.$$

Bruhat Decomposition of $\mathrm{GL}(n, \mathbb{R})$ (continued)

Proof.

There exist pairwise different numbers $k_1, \ldots, k_n \in \{1, \ldots, n\}$ and a matrix $b \in B$ such that for any $j = 1, \ldots, n$ the only non–zero entry in the j-th column of matrix bA, excluding rows $k_1, \ldots, k_{j-1}$ is in the k_j-th row (for $j = 1$ this condition is empty).
Let k_1 be the biggest number such that $a_{k_1 1} \neq 0$ (there exists such k_1 as matrix A is invertible). Multiplying A by a product of transvections $X_{ik_1}(\alpha)$ with $i < k_1$, equal to $b_1 \in B$ one can make the entry $(k_1, 1)$ the only non–zero entry in the 1st column of $b_1 A$.

Bruhat Decomposition of $\mathrm{GL}(n, \mathbb{R})$ (continued)

Proof.

Analogously, let k_2 be the biggest number, different from k_1 such that $a_{k_1 2} \neq 0$ (there exists such k_2 as matrix A is invertible). Multiplying $b_1 A$ by a product of transvections $X_{ik_2}(\alpha)$ with $i < k_2$, equal to $b_2 \in B$, one can make the entry $(k_2, 2)$ the only non–zero entry in the 2nd row of $b_2 b_1 A$, excluding row k_1. And so on, finally let $b = (b_n \cdots b_2 b_1)^{-1} \in B$. Let $\sigma(j) = k_j$ for $j = 1, \ldots, n$. Multiplying bA on the right by the appropriate product of transvections with $X_{ik_j}(\alpha)$ where $i < k_j$ one can get

$$A = bP_\sigma b'.$$

$\square$

Bruhat Decomposition

Theorem

$$GL(n;\mathbb{R}) = BWB,$$

that is, the general linear group is a disjoint union of $n!$ the double cosets.

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that is, the general linear group is a disjoint union of $n!$ the double cosets.

Proof.

It is enough to prove that the cosets are disjoint. Assume that

$$b_1 P_\sigma b_1' = b_2 P_\tau b_2',$$

then

$$bP_\sigma = P_\tau b',$$

for some $b_i, b_i', b, b' \in B$.

Bruhat Decomposition (continued)

Proof.

Let k be the smallest number such that $\sigma(k) \neq \tau(k)$. Then the largest index of a non–zero entry of the k-th column of $bP\sigma$ is $(\sigma(k), k)$. The only (possibly) non–zero entries in $P_\tau b'$ in the k-th column are $(\sigma(1), k), (\sigma(2), k), \ldots, (\sigma(k-1), k), (\tau(k), k)$. Since $\sigma(k) \neq \sigma(j)$ for $j = 1, \ldots, k-1$ this leads to a contradiction.

Root Subgroups and Matrix Exponential

Definition

For any $i \neq j$ the root subgroup $X_{ij} \subset \mathrm{GL}(n; \mathbb{R})$ is given by

$$X_{ij} = \{X_{ij}(\alpha) \in \mathrm{GL}(n; \mathbb{R}) \mid \alpha \in \mathbb{R}\}.$$

Definition

For any matrix $A \in M(n \times n; \mathbb{R})$ there is well defined matrix $exp(A) \in \mathrm{GL}(n, \mathbb{R})$ given by

$$exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Root Subgroups and Matrix Exponential (continued)

Remark

Observe that for any $i \neq j$

$$exp(tE_{i,j}) = X_{ij}(t),$$

which gives the group homomorphism

$$\exp\colon (\mathbb{R}, +) \ni t \mapsto exp(tE_{i,j}) \in X_{ij} \subset \mathrm{GL}(n, \mathbb{R}).$$

Complete Flag

Definition

A complete flag in $\mathbb{R}^n$ is a sequence of subspaces $V_i \subset \mathbb{R}^n$ such that $\dim V_i = i$ for $i = 0, \dots, n$ and

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{R}^n.$$

The standard complete flag is given by the condition

$$V_i = \operatorname{lin}(\varepsilon_1, \dots, \varepsilon_i),$$

i.e., the $i-$th subspace is spanned by the first i vectors of the standard basis of $\mathbb{R}^n$.

(Complete) Flag Variety

Definition

Flag variety $F = F(1, 2, \ldots, n)$ is the set of all complete flags in $\mathbb{R}^n$.

Proposition

The general linear group $\mathrm{GL}(n; \mathbb{R})$ *acts transitively on the flag variety with the stabilizer (at the standard complete flag) equal to the standard Borel subgroup* B*.*

Corollary

The (complete) flag variety is a homogenous variety, i.e.,

$$F = \mathrm{GL}(n; \mathbb{R})/B = \bigsqcup_{w \in W} BwB/B.$$

Schubert/Bruhat Cell

Definition

For any $\sigma \in S_n$ the set

$$C_w = BwB/B,$$

where $w = P_\sigma$ is called Schubert/Bruhat cell.
The closure X_w of C_w is called Schubert variety, i.e.

$$X_w = \overline{C}_w.$$

Definition

For any permutation $\sigma \in S_n$ a pair (i, j) such that $\sigma(i) > \sigma(j)$ and $1 \leqslant i < j \leqslant n$ is called an inversion. The number of all inversions of permutation σ is called the length od σ and is denoted $l(\sigma)$.

Schubert/Bruhat Cell (continued)

Proposition

Each Schubert/Bruhat C_w cell is isomorphic to $\mathbb{R}^{l(w)}$. The dimension of F is the maximal number of inversions that is

$$\dim F(1,2,\ldots,n) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

The cohomology classes $[X_w]$ form a basis of the (integral) cohomology of the complete flag variety.

[0] see V Lakshmibai, J. Brown *Flag Varieties*, Springer 2018.

Bruhat Order

Definition

The transitive closure of the relation

$$\sigma \leqslant \tau \ \Leftrightarrow \ \begin{cases} \sigma = t\tau \text{ for some transposition } t \\ l(\sigma) < l(\tau) \end{cases},$$

induces a (ranked) partial order on all permutations in S_n.

Proposition

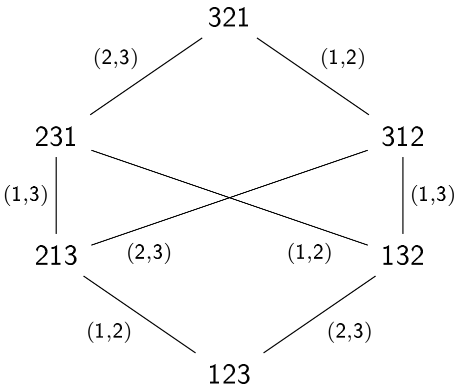
The Schubert/Bruhat cells form a CW complex and

$$\overline{C}_v \subset \overline{C}_w \ \Leftrightarrow v \leqslant w,$$

where $\overline{C}_v$ denotes the closure of cell C_v and v, w are identified with corresponding permutations.

Bruhat Order – Example

For $n = 3$ identifty the permutation $\sigma \in S_3$ with the sequence $\sigma(1)\sigma(2)\sigma(3)$.



Antisymmetric Matrices

Definition

Matrix $A \in M(n \times n; \mathbb{R})$ is antisymmetric if

$$A^{\mathsf{T}} = -A.$$

Proposition

If $A \in M((2k+1) \times (2k+1); \mathbb{R})$ *is antisymmetric then* $\det A = 0$.

Proof.

Exercise. $\square$

Pfaffian

Proposition

Let $A = [x_{ij}] \in M(2k \times 2k; \mathbb{R})$ *an antisymmetric matrix with entries equal to degree one monomials* x_{ij}. *Then there exists*[1] *a unique (up to a sign) polynomial* $P \in \mathbb{Z}[x_{ij}]$ *(i.e. with integral coefficients) such that*

$$\det A = [P(x_{ij})]^2.$$

[1] S. Lang, *Algebra*, Springer

Pfaffian (continued)

Definition

For any antisymmetric matrix $A = [a_{ij}] \in M(2k \times 2k; \mathbb{R})$ **the Pfaffian** of matrix A is a scalar determined by the above polynomial with a sign chosen such that

i)
$$[\mathrm{Pf}(A)]^2 = \det A,$$

ii)
$$\mathrm{Pf}\left(\begin{bmatrix} J & 0 & \dots & 0 \\ 0 & J & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J \end{bmatrix}\right) = 1,$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Pfaffian (continued)

Remark

For any antisymmetric matrix $A^\intercal = -A$ *and* $A = [a_{ij}] \in M(2k \times 2k; \mathbb{R})$ *then* $\mathrm{Pf}(A)$ *is a scalar such that if*

$$\omega = \sum_{1 \leqslant i < j \leqslant n} a_{ij} \varepsilon_i \wedge \varepsilon_j,$$

then

$$\frac{\omega^k}{k!} = \mathrm{Pf}(A) \varepsilon_1 \wedge \ldots \wedge \varepsilon_{2k}.$$

Pfaffian (continued)

Proposition

For any antisymmetric matrix $A^\intercal = -A$ *and* $A = [a_{ij}] \in M(2k \times 2k; \mathbb{R})$ *and any matrix* $B \in M(2k \times 2x; \mathbb{R})$

$$\mathrm{Pf}(B^\intercal AB) = \det B \,\mathrm{Pf}(A).$$

Proof.

Since $\det(B^\intercal AB) = \det B^2 [\mathrm{Pf}(A)]^2$ it is obvious that

$$\mathrm{Pf}(B^\intercal AB) = \pm \det B \,\mathrm{Pf}(A),$$

where the sign does not depend on matrix B (consider matrices with entries in a polynomial ring). Substituting $B = I_{2k}$ gives the result. $\square$

Pfaffian - Equivalence of Definitions

Remark

For any real skew–symmetric matrix A there exists an orhogonal matrix Q such that

$$Q^\mathsf{T} A Q = \begin{bmatrix} a_1 J & 0 & \dots & 0 \\ 0 & a_2 J & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_k J \end{bmatrix}.$$

Moreover, by replacing Q with TQ where T is the transposition matrix (say of rows 1 and 2), one can assume that $\det Q = 1$*. Therefore*

$$\mathrm{Pf}(A) = a_1 \dots a_k,$$

which shows that the two definitions are equivalent.

Pfaffian (continued)

Proposition

Let $A = [a_{ij}] \in M(k \times k; \mathbb{R})$ *be any matrix. Then*

$$\mathrm{Pf}\left(\left[\begin{smallmatrix} 0 & A \\ -A^{\intercal} & 0 \end{smallmatrix}\right]\right) = (-1)^{\frac{k(k-1)}{2}} \det A.$$

Proof.

Again, it is clear that $\mathrm{Pf}\left(\left[\begin{smallmatrix} 0 & A \\ -A^{\intercal} & 0 \end{smallmatrix}\right]\right) = \pm \det A$. Take $A = I$. Then for

$$\omega = \varepsilon_1 \wedge \varepsilon_{k+1} + \varepsilon_2 \wedge \varepsilon_{k+2} + \ldots + \varepsilon_k \wedge \varepsilon_{2k},$$

we have

$$\frac{\omega^k}{k!} = \varepsilon_1 \wedge \varepsilon_{k+1} \wedge \varepsilon_2 \wedge \varepsilon_{k+2} \wedge \ldots \wedge \varepsilon_k \wedge \varepsilon_{2k} =$$

$$= (-1)^{1+2+\ldots+(k-1)} \varepsilon_1 \wedge \ldots \wedge \varepsilon_{2k}..$$

$\square$

Pfaffians – Examples

$$\mathrm{Pf}\left(\begin{bmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{bmatrix}\right) = x_{12},$$

$$\mathrm{Pf}\left(\begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}\right) = x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14}.$$

$$\mathrm{Pf}\left(\begin{bmatrix} a_1 J & 0 & \dots & 0 \\ 0 & a_2 J & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_k J \end{bmatrix}\right) = a_1 a_2 \cdots a_k.$$

Laplace–type Formula for Pfaffians

Proposition

Let $A = [a_{ij}] \in M(2k \times 2k; \mathbb{R})$ *be an antisymmetric matrix. Then*

$$\mathrm{Pf}(A) = \sum_{j=2}^{2k} (-1)^j a_{1j}\, \mathrm{Pf}(A_{1j,1j}).$$

Proof.

By the Sylverster's Theorem

$$\det A \det A_{1j,1j} = \det \begin{bmatrix} \det A_{11} & \det A_{1j} \\ \det A_{j1} & \det A_{jj} \end{bmatrix} =$$

$$= \det \begin{bmatrix} 0 & \det A_{1j} \\ \det\left(-A_{1j}^{\mathsf{T}}\right) & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & \det A_{1j} \\ (-1)^{2k-1} \det A_{1j} & 0 \end{bmatrix} = \det(A_{1j})^2.$$

Laplace–type Formula for Pfaffians

Proof.

It turns out that

$$\mathrm{Pf}(A)\,\mathrm{Pf}(A_{1j,1j}) = -\det A_{1j}.$$

To see this consider the form (and the corresponding matrix)

$$\omega = \varepsilon_1 \wedge \varepsilon_j + \varepsilon_2 \wedge \varepsilon_3 + \ldots + \varepsilon_{j-2} \wedge \varepsilon_{j-1} + \varepsilon_{j+1} \wedge \varepsilon_{j+2} + \ldots,$$

for even j and

$$\omega = \varepsilon_1 \wedge \varepsilon_j + \varepsilon_2 \wedge \varepsilon_3 + \ldots + \varepsilon_{j-1} \wedge \varepsilon_{j+1} + \varepsilon_{j+2} \wedge \varepsilon_{j+3} + \ldots,$$

for odd j. In both cases

$$A_{1j,1j} = \begin{bmatrix} J & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & J \end{bmatrix},$$

hence $\mathrm{Pf}(A_{1j,1j}) = 1$. In $\frac{\omega^k}{k!}$ in both cases on have to exchange ε_j with $\varepsilon_2, \ldots, \varepsilon_{j-1}$ hence $\mathrm{Pf}(A) = (-1)^{j-1}$.

Laplace–type Formula for Pfaffians

Proof.

Finally, the matrix A_{1j} has a unique -1 in the first column and the $(j-1)$–th row, by the Laplace formula for the first column

$$\det A_{1j} = (-1)^{(j-1)+1}(-1)\det \begin{bmatrix} J & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J \end{bmatrix} = (-1)^{j-1}.$$

By the Laplace formula in the first row for matrix A

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j},$$

$$[\mathrm{Pf}(A)]^2 = \sum_{j=2}^{n} (-1)^{j} a_{1j} \,\mathrm{Pf}(A)\,\mathrm{Pf}(A_{1j,1j}).$$

Note that $\det A_{11} = 0$ and to divide by $\mathrm{Pf}(A)$ one should switch to matrices with entries in a polynomial ring. $\square$

Example

Let

$$A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}.$$

Then

$$A_{12,12} = \begin{bmatrix} 0 & x_{34} \\ -x_{34} & 0 \end{bmatrix}, \quad A_{13,13} = \begin{bmatrix} 0 & x_{24} \\ -x_{24} & 0 \end{bmatrix},$$

$$A_{14,14} = \begin{bmatrix} 0 & x_{23} \\ -x_{23} & 0 \end{bmatrix}.$$

$$\mathrm{Pf}(A) = (-1)^2 x_{12}x_{34} + (-1)^3 x_{13}x_{24} + (-1)^4 x_{14}x_{23} =$$

$$= x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}.$$

Exercise

$$A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} & x_{36} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} & x_{46} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 & x_{56} \\ -x_{16} & -x_{26} & -x_{36} & -x_{46} & -x_{56} & 0 \end{bmatrix}.$$

$$\mathrm{Pf}(A) = x_{12}x_{34}x_{56} - x_{13}x_{24}x_{56} + x_{14}x_{23}x_{56} - x_{12}x_{35}x_{46} + x_{13}x_{25}x_{46} -$$

$$-x_{15}x_{23}x_{46} + x_{12}x_{36}x_{45} - x_{13}x_{26}x_{45} + x_{16}x_{23}x_{45} - x_{14}x_{25}x_{36} +$$

$$+x_{15}x_{24}x_{36} + x_{14}x_{26}x_{35} - x_{16}x_{24}x_{35} - x_{15}x_{26}x_{34} + x_{16}x_{25}x_{34}.$$