# Linear Algebra <br> Lecture 5 - Operations on Linear Transformations 

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## Sum and Scalar Multiplication

## Proposition

Let $V, W$ be vector spaces. Let $\varphi, \psi: V \longrightarrow W$ be linear transformations and let $\alpha \in \mathbb{R}$. The transformation
$\varphi+\psi: V \longrightarrow W$, defined by $(\varphi+\psi)(v)=\varphi(v)+\psi(v)$ for $v \in V$, and the transformation $\alpha \varphi$ defined by $(\alpha \varphi)(v)=\alpha \varphi(v)$ are linear. The transformation $\varphi+\psi$ is called the sum of $\varphi$ and $\psi$ and $\alpha \varphi$ is called the product of the transformation $\varphi$ with scalar $\alpha$.

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## Example

Let $\varphi, \psi: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ be given by
$\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}+2 x_{2}-x_{3}, x_{1}+2 x_{2}+x_{3}\right)$ and
$\psi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(-x_{1}+x_{2}+x_{3}, 3 x_{1}-2 x_{2}+x_{3}\right)$. Then
$(\varphi+\psi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(3 x_{2}, 4 x_{1}+2 x_{3}\right)$ and
$(2 \varphi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(2 x_{1}+4 x_{2}-2 x_{3}, 2 x_{1}+4 x_{2}+2 x_{3}\right)($ for $\alpha=2)$.

## Composition

## Proposition

Let $U, V, W$ be vectors spaces and let $\varphi: U \longrightarrow V, \psi: V \longrightarrow W$ be linear transformations. The transformation

$$
\psi \circ \varphi: U \longrightarrow W
$$

given by

$$
(\psi \circ \varphi)(v)=\psi(\varphi(v)))
$$

for $v \in U$, is linear. It is called the composition of $\psi$ with $\varphi$.

## Example

Let $\varphi: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ and $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be linear transformations given by $\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}-x_{2}+2 x_{3},-x_{1}+3 x_{2}-x_{3}\right)$ and $\left.\psi\left(\left(y_{1}, y_{2}\right)\right)=\left(y_{1}-y_{2}, y_{1}+2 y_{2}\right)\right)$. Then
$(\psi \circ \varphi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\psi\left(\left(x_{1}-x_{2}+2 x_{3},-x_{1}+3 x_{2}-x_{3}\right)\right)=$ $\left(\left(x_{1}-x_{2}+2 x_{3}\right)-\left(-x_{1}+3 x_{2}-x_{3}\right),\left(x_{1}-x_{2}+2 x_{3}\right)+2\left(-x_{1}+3 x_{2}-x_{3}\right)\right)=$ $\left(2 x_{1}-4 x_{2}+3 x_{3},-x_{1}+5 x_{2}\right)$.

## Operations on Matrices

Definition
Let $A, B \in M(m \times n ; \mathbb{R}), \alpha \in \mathbb{R}, A=\left[a_{i j}\right], B=\left[b_{i j}\right]$. The sum of matrices $A$ and $B$ is the matrix $A+B=\left[a_{i j}+b_{i j}\right]$. The product of matrix $A$ by scalar $\alpha$ is the matrix $\alpha A=\left[\alpha a_{i j}\right]$.

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Example
Let $\alpha=2$ and let $A, B \in M(2 \times 3 ; \mathbb{R})$ be given by

$$
A=\left[\begin{array}{rrr}
1 & 2 & -1 \\
0 & 1 & 0
\end{array}\right], B=\left[\begin{array}{rrr}
-1 & 3 & 2 \\
1 & 0 & 1
\end{array}\right] .
$$

Then

$$
A+B=\left[\begin{array}{lll}
0 & 5 & 1 \\
1 & 1 & 1
\end{array}\right], \alpha A=\left[\begin{array}{rrr}
2 & 4 & -2 \\
0 & 2 & 0
\end{array}\right] .
$$

## Matrix Multiplication

## Definition

Let $A \in M(m \times n ; \mathbb{R})$ and let $B \in M(n \times l ; \mathbb{R})$. The matrix product of $A$ by $B$ is the matrix $A B=\left[c_{i j}\right] \in M(m \times l ; \mathbb{R})$ where $c_{i j}=\sum_{s=1}^{n} a_{i s} b_{s j}=a_{i 1} b_{1 j}+a_{i 2} b_{2 j}+\ldots+a_{i n} b_{n j}$ for $i=1, \ldots, m$ and $j=1, \ldots, l$.

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In particular, if $R_{i}=\left[\begin{array}{llll}a_{i 1} & a_{i 2} & \ldots & a_{i n}\end{array}\right] \in M(1 \times n ; \mathbb{R})$ is the
$i$-th row of matrix $A$ and $C_{j}=\left[\begin{array}{c}b_{1 j} \\ b_{2 j} \\ \vdots \\ b_{n j}\end{array}\right] \in M(n \times 1 ; \mathbb{R})$ is the $j$-th
column of matrix $B$ then $R_{i} C_{j}=\left[a_{i 1} b_{1 j}+\ldots+a_{i n} b_{n j}\right]$ is a $1 \times 1$ matrix which can be identified with a real number.

## Matrix Multiplication (continued)

Using this identification we can write

$$
A B=\left[\begin{array}{cccc}
R_{1} C_{1} & R_{1} C_{2} & \ldots & R_{1} C_{l} \\
R_{2} C_{1} & R_{2} C_{2} & \ldots & R_{2} C_{l} \\
\vdots & \vdots & \ddots & \vdots \\
R_{m} C_{1} & R_{m} C_{2} & \ldots & R_{m} C_{l}
\end{array}\right]
$$

## Matrix Multiplication (continued)

Using this identification we can write

$$
A B=\left[\begin{array}{cccc}
R_{1} C_{1} & R_{1} C_{2} & \ldots & R_{1} C_{l} \\
R_{2} C_{1} & R_{2} C_{2} & \ldots & R_{2} C_{l} \\
\vdots & \vdots & \ddots & \vdots \\
R_{m} C_{1} & R_{m} C_{2} & \ldots & R_{m} C_{l}
\end{array}\right]
$$

For example

$$
\begin{gathered}
{\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m 1} & a_{m 2} & \ldots & a_{m n}
\end{array}\right]\left[\begin{array}{c}
1 \\
0 \\
\vdots \\
0
\end{array}\right]=\left[\begin{array}{c}
a_{11} \\
a_{21} \\
\vdots \\
a_{m 1}
\end{array}\right]=\text { the first column of } A} \\
{\left[\begin{array}{llll}
1 & 0 & \cdots & 0
\end{array}\right]\left[\begin{array}{cccc}
b_{11} & b_{12} & \ldots & b_{1 k} \\
b_{21} & b_{22} & \ldots & b_{2 k} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n 1} & b_{n 2} & \ldots & b_{n k}
\end{array}\right]=} \\
=\left[\begin{array}{llll}
b_{11} & b_{12} & \cdots & b_{1 k}
\end{array}\right]=\text { the first row of } B
\end{gathered}
$$

and so on.

## Example

Let $A \in M(3 \times 2 ; \mathbb{R})$ and $B \in M(2 \times 2 ; \mathbb{R})$ be given by

$$
A=\left[\begin{array}{rr}
1 & 2 \\
2 & 3 \\
-1 & 1
\end{array}\right], B=\left[\begin{array}{rr}
1 & 1 \\
1 & -2
\end{array}\right]
$$

Then

$$
A B=\left[\begin{array}{l}
R_{1} \\
R_{2} \\
R_{3}
\end{array}\right]\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right]=\left[\begin{array}{ll}
R_{1} C_{1} & R_{1} C_{2} \\
R_{2} C_{1} & R_{2} C_{2} \\
R_{3} C_{1} & R_{3} C_{3}
\end{array}\right]=\left[\begin{array}{ll}
3 & -3 \\
5 & -4 \\
0 & -3
\end{array}\right] .
$$

The first column of $A B$ is the sum of columns of $A$ and the second one is the first column of $A$ minus twice the second column of $A$.

## Warning

The matrix multiplication is, in general, not commutative. For example

$$
\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]
$$

but

$$
\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]
$$

## Operations on Linear Transformations and Matrices

Theorem (Addition)
Let $V, W$ be vector spaces and let $\varphi, \psi: V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}$ be bases of $V$ and $W$ respectively. Then $M(\varphi+\psi)_{\mathcal{A}}^{\mathcal{B}}=M(\varphi)_{\mathcal{A}}^{\mathcal{B}}+M(\psi)_{\mathcal{A}}^{\mathcal{B}}$.
Theorem (Composition and multiplication)
Let $U, V, W$ be vectors spaces and let $\varphi: U \longrightarrow V, \psi: V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the bases of $U, V$ and $W$, respectively. Then $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.

## Example (continued)

Let $\varphi: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ and $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be linear transformations given by $\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}-x_{2}+2 x_{3},-x_{1}+3 x_{2}-x_{3}\right)$ and $\left.\psi\left(\left(y_{1}, y_{2}\right)\right)=\left(y_{1}-y_{2}, y_{1}+2 y_{2}\right)\right)$. Recall that $(\psi \circ \varphi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(2 x_{1}-4 x_{2}+3 x_{3},-x_{1}+5 x_{2}\right)$.

## Example (continued)

Let $\varphi: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ and $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be linear transformations given by $\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}-x_{2}+2 x_{3},-x_{1}+3 x_{2}-x_{3}\right)$ and $\left.\psi\left(\left(y_{1}, y_{2}\right)\right)=\left(y_{1}-y_{2}, y_{1}+2 y_{2}\right)\right)$. Recall that $(\psi \circ \varphi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(2 x_{1}-4 x_{2}+3 x_{3},-x_{1}+5 x_{2}\right)$. We will compute this again, using matrix multiplication.

## Example (continued)

Let $\varphi: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ and $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be linear transformations given by $\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}-x_{2}+2 x_{3},-x_{1}+3 x_{2}-x_{3}\right)$ and $\left.\psi\left(\left(y_{1}, y_{2}\right)\right)=\left(y_{1}-y_{2}, y_{1}+2 y_{2}\right)\right)$. Recall that $(\psi \circ \varphi)\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(2 x_{1}-4 x_{2}+3 x_{3},-x_{1}+5 x_{2}\right)$. We will compute this again, using matrix multiplication. Let $\mathcal{A}$ be the standard basis in $\mathbb{R}^{3}$ and let $\mathcal{B}=\mathcal{C}$ be the standard basis in $\mathbb{R}^{2}$.
Then

$$
\begin{gathered}
M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right]\left[\begin{array}{rrr}
1 & -1 & 2 \\
-1 & 3 & -1
\end{array}\right]= \\
=\left[\begin{array}{rrr}
2 & -4 & 3 \\
-1 & 5 & 0
\end{array}\right]
\end{gathered}
$$

This agrees with the formula of $\psi \circ \varphi$.

## Applications

## Proposition

Let $V, W$ be vector spaces and let $\varphi: V \longrightarrow W$ be a linear transformation. Let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ be an ordered basis of $V$ and let $\mathcal{B}=\left(w_{1}, \ldots, w_{m}\right)$ be an ordered basis of $W$. For any vector $v \in V$ let $\alpha_{1}, \ldots, \alpha_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{A}$ and let $\beta_{1}, \ldots, \beta_{m}$ be the coordinates of $\varphi(v)$ relative to the basis $\mathcal{B}$, that is $v=\alpha_{1} v_{1}+\ldots+\alpha_{n} v_{n}$ and $\varphi(v)=\beta_{1} w_{1}+\ldots+\beta_{m} w_{m}$. Then

$$
M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{c}
\beta_{1} \\
\beta_{2} \\
\vdots \\
\beta_{m}
\end{array}\right]
$$

## Example

Let $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be a linear transformations given by $\psi\left(\left(x_{1}, x_{2}\right)\right)=\left(x_{1}-x_{2}, x_{1}+2 x_{2}\right)$. Let $s t=((1,0),(0,1))$ be the standard basis in $\mathbb{R}^{2}$ and let $\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$ be other two bases of $\mathbb{R}^{2}$.

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$$
\begin{aligned}
& \psi(1,2)=(-1,5)=4(1,0)-5(1,-1), \\
& \psi(0,1)=(-1,2)=1(1,0)-2(1,-1) .
\end{aligned}
$$

Therefore

$$
M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right] .
$$

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$\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$ be other two bases of $\mathbb{R}^{2}$.
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Therefore

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M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]
$$

Pick, say, $v=(1,1)$. Since $v=1(1,2)-1(0,1)$, the coordinates of $v$ relative to $\mathcal{A}$ are $1,-1$. Since $\psi(v)=(0,3)=3(1,0)-3(1,-1)$, the coordinates of $\psi(v)$ relative to $\mathcal{B}$ are $3,-3$.

## Example (continued)

$$
M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]
$$

## Example (continued)

$$
M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]
$$

the coordinates of $v=(1,1)$ relative to the basis $\mathcal{A}$ are $1,-1$
the coordinates of $\psi(v)=(0,3)$ relative to the basis $\mathcal{B}$ are $3,-3$

## Example (continued)

$$
M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right] .
$$

the coordinates of $v=(1,1)$ relative to the basis $\mathcal{A}$ are $1,-1$
the coordinates of $\psi(v)=(0,3)$ relative to the basis $\mathcal{B}$ are $3,-3$

$$
\begin{gathered}
M(\psi)_{s t}^{s t}\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
0 \\
3
\end{array}\right] \\
M(\psi)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{r}
1 \\
-1
\end{array}\right]=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]\left[\begin{array}{r}
1 \\
-1
\end{array}\right]=\left[\begin{array}{r}
3 \\
-3
\end{array}\right] .
\end{gathered}
$$

## Applications (continued)

Let $V$ be a vector space. The function $\mathrm{id}_{V}: V \longrightarrow V$ given by $i d_{V}(v)=v$ for any $v \in V$ is a linear transformation called the identity.

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Let $V$ be a vector space. The function $\mathrm{id}_{V}: V \longrightarrow V$ given by $\operatorname{id}_{V}(v)=v$ for any $v \in V$ is a linear transformation called the identity.

Corollary
Let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ and $\mathcal{B}=\left(w_{1}, \ldots, w_{n}\right)$ be two ordered bases of $V$. For any $v \in V$ let $\alpha_{1}, \ldots, \alpha_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{A}$ and let $\beta_{1}, \ldots, \beta_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{B}$. Then

$$
M\left(\operatorname{id}_{V}\right)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{c}
\beta_{1} \\
\beta_{2} \\
\vdots \\
\beta_{n}
\end{array}\right] .
$$

## Applications (continued)

Let $V$ be a vector space. The function $\mathrm{id}_{V}: V \longrightarrow V$ given by $i^{\prime} V(v)=v$ for any $v \in V$ is a linear transformation called the identity.

## Corollary

Let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ and $\mathcal{B}=\left(w_{1}, \ldots, w_{n}\right)$ be two ordered bases of $V$. For any $v \in V$ let $\alpha_{1}, \ldots, \alpha_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{A}$ and let $\beta_{1}, \ldots, \beta_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{B}$. Then

$$
M\left(\operatorname{id}_{V}\right)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{c}
\beta_{1} \\
\beta_{2} \\
\vdots \\
\beta_{n}
\end{array}\right] .
$$

The matrix $M\left(\mathrm{id}_{V}\right){ }_{\mathcal{A}}^{\mathcal{B}}$ is called a change-of-coordinates matrix.

## Applications (continued)

## Proposition

Let $V, W$ be vector spaces and let $\varphi: V \longrightarrow W$ be a linear transformation. Let $\mathcal{A}, \mathcal{A}^{\prime}$ be (ordered) bases of $V$ and let $\mathcal{B}, \mathcal{B}^{\prime}$ be (ordered) bases of W. Then

$$
M(\varphi)_{\mathcal{A}^{\prime}}^{\mathcal{B}^{\prime}}=M\left(\mathrm{id}_{W}\right)_{\mathcal{B}}^{\mathcal{B}^{\prime}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}} M\left(\mathrm{id}_{V}\right)_{\mathcal{A}^{\prime}}^{\mathcal{A}}
$$

## Proof.

This follows directly from the fact that $\mathrm{id}_{w} \circ \varphi \circ \mathrm{id}_{V}=\varphi$ and the formula relating composition of linear transformations with matrix multiplication.

## Example (continued)

Let $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be a linear transformation given by the formula $\psi\left(\left(x_{1}, x_{2}\right)\right)=\left(x_{1}-x_{2}, x_{1}+2 x_{2}\right)$. Let $s t=((1,0),(0,1))$ be the standard basis of $\mathbb{R}^{2}$ and let $\mathcal{A}=((1,2),(0,1))$,
$\mathcal{B}=((1,0),(1,-1))$ be other two bases of $\mathbb{R}^{2}$.

## Example (continued)

Let $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be a linear transformation given by the formula $\psi\left(\left(x_{1}, x_{2}\right)\right)=\left(x_{1}-x_{2}, x_{1}+2 x_{2}\right)$. Let $s t=((1,0),(0,1))$ be the standard basis of $\mathbb{R}^{2}$ and let $\mathcal{A}=((1,2),(0,1))$,
$\mathcal{B}=((1,0),(1,-1))$ be other two bases of $\mathbb{R}^{2}$. We have already checked that

$$
M(\psi)_{s t}^{s t}=\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]
$$

## Example (continued)

Let $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be a linear transformation given by the formula $\psi\left(\left(x_{1}, x_{2}\right)\right)=\left(x_{1}-x_{2}, x_{1}+2 x_{2}\right)$. Let $s t=((1,0),(0,1))$ be the standard basis of $\mathbb{R}^{2}$ and let $\mathcal{A}=((1,2),(0,1))$,
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1 & -1 \\
1 & 2
\end{array}\right], M(\psi)_{\mathcal{A}}^{\mathcal{B}}=\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]
$$

Let check this again using the previous Proposition. It says that

$$
M(\psi)_{\mathcal{A}}^{\mathcal{B}}=M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}} M(\psi)_{s t}^{s t} M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}
$$

## Example (continued)

Let $\psi: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{2}$ be a linear transformation given by the formula $\psi\left(\left(x_{1}, x_{2}\right)\right)=\left(x_{1}-x_{2}, x_{1}+2 x_{2}\right)$. Let $s t=((1,0),(0,1))$ be the standard basis of $\mathbb{R}^{2}$ and let $\mathcal{A}=((1,2),(0,1))$,
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Let check this again using the previous Proposition. It says that

$$
M(\psi)_{\mathcal{A}}^{\mathcal{B}}=M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}} M(\psi)_{s t}^{s t} M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}
$$

We need to compute $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}$ and $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$.

## Example (continued)

We need to compute $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}$ and $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$. Recall that $\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$.

## Example (continued)

We need to compute $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}$ and $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$. Recall that $\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$. Since

$$
\begin{aligned}
& \mathrm{id}((1,2))=1(1,0)+2(0,1), \\
& \operatorname{id}(0,1)=0(1,0)+1(0,1),
\end{aligned}
$$

we have $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}=\left[\begin{array}{ll}1 & 0 \\ 2 & 1\end{array}\right]$.

## Example (continued)

We need to compute $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}$ and $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$. Recall that $\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$. Since

$$
\begin{gathered}
\operatorname{id}((1,2))=1(1,0)+2(0,1) \\
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\end{gathered}
$$

we have $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}=\left[\begin{array}{ll}1 & 0 \\ 2 & 1\end{array}\right]$. Since

$$
\begin{aligned}
& \mathrm{id}((1,0))=1(1,0)+0(1,-1) \\
& \mathrm{id}((0,1))=1(1,0)-1(1,-1)
\end{aligned}
$$

we have $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}=\left[\begin{array}{rr}1 & 1 \\ 0 & -1\end{array}\right]$.

## Example (continued)

We need to compute $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}$ and $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$. Recall that $\mathcal{A}=((1,2),(0,1)), \mathcal{B}=((1,0),(1,-1))$. Since

$$
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\operatorname{id}((1,2))=1(1,0)+2(0,1) \\
i d(0,1)=0(1,0)+1(0,1)
\end{gathered}
$$

we have $M\left(\operatorname{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}=\left[\begin{array}{ll}1 & 0 \\ 2 & 1\end{array}\right]$. Since

$$
\begin{aligned}
& \operatorname{id}((1,0))=1(1,0)+0(1,-1) \\
& \operatorname{id}((0,1))=1(1,0)-1(1,-1)
\end{aligned}
$$

we have $M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}}=\left[\begin{array}{rr}1 & 1 \\ 0 & -1\end{array}\right]$. Using
$M(\psi))_{\mathcal{A}}^{\mathcal{B}}=M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{s t}^{\mathcal{B}} M(\psi)_{s t}^{s t} M\left(\mathrm{id}_{\mathbb{R}^{2}}\right)_{\mathcal{A}}^{s t}$ one can check that

$$
\left[\begin{array}{rr}
4 & 1 \\
-5 & -2
\end{array}\right]=\left[\begin{array}{rr}
1 & 1 \\
0 & -1
\end{array}\right]\left[\begin{array}{rr}
1 & -1 \\
1 & 2
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
2 & 1
\end{array}\right]
$$

## Elementary Matrices

Fix $\alpha \in \mathbb{R}, n>0$ and define the following matrices
$D_{i, \alpha}=\left[d_{k l}\right], L_{i j}=\left[\ell_{k l}\right], T_{i j}=\left[t_{k l}\right] \in M(n \times n ; \mathbb{R})$ as follows

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i) $d_{k k}=1$ for $k \neq i, d_{i i}=\alpha, d_{k l}=0$ elsewhere,
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Fix $\alpha \in \mathbb{R}, n>0$ and define the following matrices
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i) $d_{k k}=1$ for $k \neq i, d_{i i}=\alpha, d_{k l}=0$ elsewhere,
ii) $\ell_{k k}=1$ for $k=1, \ldots, n, \ell_{i j}=1, \ell_{k l}=0$ elsewhere,
iii) $t_{k k}=1$ for $k \notin\{i, j\}, t_{i j}=t_{j i}=1, t_{k l}=0$ elsewhere.

## Elementary Matrices (continued)

$$
\begin{gathered}
D_{i, \alpha}=\left[\begin{array}{ccccccc}
i_{1} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \alpha & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right], L_{i j}=\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right], \\
\\
\quad i\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right]
\end{gathered}
$$

## Elementary Matrices (continued)

Proposition
Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,

## Elementary Matrices (continued)

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Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
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## Elementary Matrices (continued)

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
ii) $L_{i j} A=$ matrix $A$ with the $j$-th row added to the $i$-th row,
iii) $T_{i j} A=$ matrix $A$ with the $i$-th and $j$-th rows swapped,

## Elementary Matrices (continued)

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
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iii) $T_{i j} A=$ matrix $A$ with the $i$-th and $j$-th rows swapped,
that is, elementary row operations correspond to multiplication by elementary matrices from the left.

## Elementary Matrices (continued)

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Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
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iii) $T_{i j} A=$ matrix $A$ with the $i$-th and $j$-th rows swapped,
that is, elementary row operations correspond to multiplication by elementary matrices from the left.

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $A D_{i, \alpha}=$ matrix $A$ with the $i$-th column multiplied by $\alpha$,

## Elementary Matrices (continued)

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
ii) $L_{i j} A=$ matrix $A$ with the $j$-th row added to the $i$-th row,
iii) $T_{i j} A=$ matrix $A$ with the $i$-th and $j$-th rows swapped,
that is, elementary row operations correspond to multiplication by elementary matrices from the left.

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $A D_{i, \alpha}=$ matrix $A$ with the $i$-th column multiplied by $\alpha$,
ii) $A L_{i j}=$ matrix $A$ with the $i$-th column added to the $j$-th one,

## Elementary Matrices (continued)

## Proposition

Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $D_{i, \alpha} A=$ matrix $A$ with the $i$-th row multiplied by $\alpha$,
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Let $A \in M(n \times m ; \mathbb{R})$. Then
i) $A D_{i, \alpha}=$ matrix $A$ with the $i$-th column multiplied by $\alpha$,
ii) $A L_{i j}=$ matrix $A$ with the $i$-th column added to the $j$-th one,
iii) $A T_{i j}=$ matrix $A$ with the $i-$ th and $j$-th columns swapped,
that is, elementary column operations correspond to multiplication by elementary matrices from the right.

## Matrix Multiplication is Associative

Proposition
For any matrices
$A \in M(m \times n ; \mathbb{R}), B \in M(n \times I ; \mathbb{R}), C \in M(I \times k ; \mathbb{R})$
$(A B) C=A(B C)$.

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$$
(A B) C=A(B C)
$$

Proof.
Let $A B=\left[f_{i j}\right] \in M(m \times l ; \mathbb{R}), B C=\left[g_{i j}\right] \in M(n \times k ; \mathbb{R})$. Then

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$$
(A B) C=A(B C)
$$

Proof.
Let $A B=\left[f_{i j}\right] \in M(m \times l ; \mathbb{R}), B C=\left[g_{i j}\right] \in M(n \times k ; \mathbb{R})$. Then

$$
\begin{aligned}
& f_{i r}=\sum_{s=1}^{n} a_{i s} b_{s r}, \\
& g_{s j}=\sum_{r=1}^{l} b_{s r} c_{r j} .
\end{aligned}
$$

## Matrix Multiplication is Associative (continued)

## Proof.

The entry in the $i$-th row and the $j$-th column of the matrix $(A B) C$ is equal to

$$
\sum_{r=1}^{1} f_{i r} c_{r j}=\sum_{r=1}^{1}\left(\sum_{s=1}^{n} a_{i s} b_{s r}\right) c_{r j}=\sum_{r=1}^{l} \sum_{s=1}^{n} a_{i s} b_{s r} c_{r j}
$$

The entry in the $i$-th row and the $j$-th column of the matrix $A(B C)$ is equal to

$$
\begin{gathered}
\sum_{s=1}^{n} a_{i s} g_{s j}=\sum_{s=1}^{n} a_{i s}\left(\sum_{r=1}^{l} b_{s r} c_{r j}\right)=\sum_{s=1}^{n} \sum_{r=1}^{l} a_{i s} b_{s r} c_{r j}= \\
=\sum_{r=1}^{l} \sum_{s=1}^{n} a_{i s} b_{s r} c_{r j}
\end{gathered}
$$

## Coordinate Vector

## Definition

Let $V$ be a vector space and let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ be its ordered basis. For any $v \in V$ by $[v]_{\mathcal{A}}$ we denote the coordinate vector of $v$ relative to $\mathcal{A}$, i.e. a $n$-by-1 matrix with coordinates of $v$ relative to $\mathcal{A}$. In particular, if $v=\alpha_{1} v_{1}+\alpha_{2} v_{2}+\ldots+\alpha_{n} v_{n}$ then

$$
[v]_{\mathcal{A}}=\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]
$$

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$$
[v]_{\mathcal{A}}=\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]
$$

Example
If $\mathcal{A}=((1,1),(1,2)), \quad v=(1,3)$ then

$$
[v]_{s t}=\left[\begin{array}{l}
1 \\
3
\end{array}\right] \text { and }[v]_{\mathcal{A}}=\left[\begin{array}{r}
-1 \\
2
\end{array}\right] .
$$

## Image by the Matrix of Linear Transformation

## Proposition

Let $V, W$ be vector spaces and let $\varphi: V \longrightarrow W$ be a linear transformation. Let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ be an ordered basis of $V$ and let $\mathcal{B}=\left(w_{1}, \ldots, w_{m}\right)$ be an ordered basis of $W$. For any vector $v \in V$ let $\alpha_{1}, \ldots, \alpha_{n}$ be the coordinates of $v$ relative to the basis $\mathcal{A}$ and let $\beta_{1}, \ldots, \beta_{m}$ be the coordinates of $\varphi(v)$ relative to the basis $\mathcal{B}$, that is $v=\alpha_{1} v_{1}+\ldots+\alpha_{n} v_{n}$ and $\varphi(v)=\beta_{1} w_{1}+\ldots+\beta_{m} w_{m}$. Then

$$
M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{c}
\beta_{1} \\
\beta_{2} \\
\vdots \\
\beta_{m}
\end{array}\right]
$$

or equivalently

$$
M(\varphi)_{\mathcal{A}}^{\mathcal{B}}[v]_{\mathcal{A}}=[\varphi(v)]_{\mathcal{B}}
$$

## Image by the Matrix of Linear Transformation - Proof

$$
\begin{aligned}
& M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m 1} & a_{m 2} & \ldots & a_{m n}
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]= \\
& =\alpha_{1}\left[\begin{array}{c}
a_{11} \\
a_{21} \\
\vdots \\
a_{m 1}
\end{array}\right]+\alpha_{2}\left[\begin{array}{c}
a_{12} \\
a_{22} \\
\vdots \\
a_{m 2}
\end{array}\right]+\ldots+\alpha_{n}\left[\begin{array}{c}
a_{1 n} \\
a_{2 n} \\
\vdots \\
a_{m n}
\end{array}\right] .
\end{aligned}
$$

## Image by the Matrix of Linear Transformation - Proof

$$
\begin{aligned}
& M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m 1} & a_{m 2} & \ldots & a_{m n}
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{n}
\end{array}\right]= \\
& =\alpha_{1}\left[\begin{array}{c}
a_{11} \\
a_{21} \\
\vdots \\
a_{m 1}
\end{array}\right]+\alpha_{2}\left[\begin{array}{c}
a_{12} \\
a_{22} \\
\vdots \\
a_{m 2}
\end{array}\right]+\ldots+\alpha_{n}\left[\begin{array}{c}
a_{1 n} \\
a_{2 n} \\
\vdots \\
a_{m n}
\end{array}\right] .
\end{aligned}
$$

On the other hand

$$
\begin{gathered}
\alpha_{1} \varphi\left(v_{1}\right)=\alpha_{1}\left(a_{11} w_{1}+a_{21} w_{2}+\ldots+a_{m 1} w_{m}\right), \\
\alpha_{2} \varphi\left(v_{2}\right)=\alpha_{2}\left(a_{12} w_{1}+a_{22} w_{2}+\ldots+a_{m 2} w_{m}\right), \\
\vdots \\
\alpha_{n} \varphi\left(v_{n}\right)=\alpha_{n}\left(a_{1 n} w_{1}+a_{2 n} w_{2}+\ldots+a_{m n} w_{m}\right), \\
\text { and }
\end{gathered}
$$

$$
\varphi(v)=\varphi\left(\alpha_{1} v_{1}+\alpha_{2} v_{2}+\ldots+\alpha_{n} v_{n}\right)=\alpha_{1} \varphi\left(v_{1}\right)+_{\alpha} \varphi\left(\underline{\alpha}_{2}\right)+\ldots+\alpha_{\underline{\underline{\underline{1}}}} \alpha_{n} \varphi\left(v_{n}\right) .
$$

## Matrix of the Sum of Linear Transformations

## Proposition

Let $V, W$ be vector spaces and let $\varphi, \psi: V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}$ be bases of $V$ and $W$ respectively and $\alpha \in \mathbb{R}$ any scalar. Then $M(\varphi+\psi)_{\mathcal{A}}^{\mathcal{B}}=M(\varphi)_{\mathcal{A}}^{\mathcal{B}}+M(\psi)_{\mathcal{A}}^{\mathcal{B}}$ and $M(\alpha \varphi)_{\mathcal{A}}^{\mathcal{B}}=\alpha M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.
Proof.
Let $\mathcal{A}=\left(v_{1}, \ldots, v_{n}\right)$ and $\mathcal{B}=\left(w_{1}, \ldots, w_{m}\right)$ be the ordered bases of $V$ and $W$, respectively. If

$$
\begin{aligned}
& \varphi\left(v_{i}\right)=a_{1 i} w_{1}+a_{2 i} w_{2}+\ldots+a_{m i} w_{m} \\
& \psi\left(v_{i}\right)=b_{1 i} w_{1}+b_{2 i} w_{2}+\ldots+b_{m i} w_{m}
\end{aligned}
$$

then

$$
\begin{aligned}
(\varphi+\psi)\left(v_{i}\right)=\varphi\left(v_{i}\right)+ & \psi\left(v_{i}\right)=\left(a_{1 i}+b_{1 i}\right) w_{1}+\left(a_{2 i}+b_{2 i}\right) w_{2}+\ldots \\
& \ldots+\left(a_{m i}+b_{m i}\right) w_{m} \\
(\alpha \varphi)\left(v_{i}\right)=\alpha \varphi\left(v_{i}\right)= & \left(\alpha a_{1 i}\right) w_{1}+\left(\alpha a_{2 i}\right) w_{2}+\ldots+\left(\alpha a_{m i}\right) w_{m}
\end{aligned}
$$

## Matrix of the Composition of Linear Transformations

## Proposition

Let $U, V, W$ be vectors spaces and let $\varphi: U \longrightarrow V, \psi: V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the bases of $U, V$ and $W$, respectively. Then $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.

## Matrix of the Composition of Linear Transformations

## Proposition

Let $U, V, W$ be vectors spaces and let $\varphi: U \longrightarrow V, \psi: V \longrightarrow W$ be linear transformations. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the bases of $U, V$ and $W$, respectively. Then $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$.

Proof.
By the Proposition on the image by matrix of a linear transformation

$$
\begin{gathered}
M(\varphi)_{\mathcal{A}}^{\mathcal{B}}[v]_{\mathcal{A}}=[\varphi(v)]_{\mathcal{B}} \\
M(\psi)_{\mathcal{B}}^{\mathcal{C}}[\varphi(v)]_{\mathcal{B}}=[\psi(\varphi(v))]_{\mathcal{C}}=[(\psi \circ \varphi)(v)]_{\mathcal{C}}
\end{gathered}
$$

that is, by associativity of matrix product

$$
[(\psi \circ \varphi)(v)]_{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}}\left(M(\varphi)_{\mathcal{A}}^{\mathcal{B}}[v]_{\mathcal{A}}\right)=\left(M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\right)[v]_{\mathcal{A}}
$$

## Matrix of the Composition of Linear Transformations (continued)

Proof.
Therefore, for any $v \in V$

$$
M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}[v]_{\mathcal{A}}=[(\varphi \circ \psi)(v)]_{\mathcal{C}}=\left(M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\right)[v]_{\mathcal{A}} .
$$

Substituting $v=v_{i}$ for $i=1, \ldots, n$ we see that matrices $M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}, M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ have the same columns, in particular

$$
M(\psi \circ \varphi)_{\mathcal{A}}^{\mathcal{C}}=M(\psi)_{\mathcal{B}}^{\mathcal{C}} M(\varphi)_{\mathcal{A}}^{\mathcal{B}}
$$

## Block Matrix

## Definition

Matrix $A \in M(m \times n ; \mathbb{R})$ is a block matrix if

$$
A=\left[\begin{array}{c|c|c|c}
A_{11} & A_{12} & \cdots & A_{1 q} \\
\hline A_{21} & A_{22} & & A_{2 q} \\
\hline \vdots & & \ddots & \vdots \\
\hline A_{p 1} & A_{p 2} & \cdots & A_{p q}
\end{array}\right],
$$

where $A_{i, j} \in M\left(m_{i} \times n_{j} ; \mathbb{R}\right)$ and

$$
\begin{aligned}
m & =m_{1}+m_{2}+\ldots+m_{p} \\
n & =n_{1}+n_{2}+\ldots+n_{q}
\end{aligned}
$$

## Example

$$
A=\left[\begin{array}{l|lll}
1 & 2 & 2 & 6 \\
1 & 2 & 2 & 5 \\
1 & 1 & 2 & 8 \\
\hline 2 & 5 & 6 & 2
\end{array}\right]
$$

## Multiplication of Block Matrices

Proposition
If $A \in M(m \times n ; \mathbb{R}), B \in M(n \times I ; \mathbb{R})$ are block matrices, where

$$
m=m_{1}+\ldots+m_{p}, \quad n=n_{1}+\ldots+n_{q}, \quad l=I_{1}+\ldots+I_{r}
$$

$$
A=\left[A_{i j}\right], \quad B=\left[B_{j k}\right],
$$

for $i=1, \ldots, p, j=1, \ldots, q, k=1, \ldots, r$, then $C=A B$ is a block matrix such that $C=\left[C_{i k}\right]$, where

$$
C_{i k}=\sum_{j=1}^{q} A_{i j} B_{j k},
$$

that is

$$
C_{i k}=A_{i 1} B_{1 k}+A_{i 2} B_{2 k}+\ldots+A_{i q} B_{q k} .
$$

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$$
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$$

that is

$$
C_{i k}=A_{i 1} B_{1 k}+A_{i 2} B_{2 k}+\ldots+A_{i q} B_{q k} .
$$

Proof.
Exercise.

## Multiplication of Block Matrices (continued)

## Remark

The claim follows by the mathematical induction of $\max (p, q)$. The cases $p=1, q=2, p=2, q=1$ and $p=q=2$ may be checked directly. Then, by separating a single block it is possible to prove the inductive step, i.e.,

$$
\begin{aligned}
A B & =\left[\begin{array}{c|c|c|c}
A_{11} & A_{12} & \cdots & A_{1 q} \\
\hline A_{21} & A_{22} & & A_{2 q} \\
\hline \vdots & & \ddots & \vdots \\
\hline A_{p 1} & A_{p 2} & \cdots & A_{p q}
\end{array}\right]\left[\begin{array}{c|c|c|c}
B_{11} & B_{12} & \cdots & B_{1 r} \\
\hline B_{21} & B_{22} & & B_{2 r} \\
\hline \vdots & & \ddots & \vdots \\
\hline B_{q 1} & B_{q 2} & \cdots & B_{q r}
\end{array}\right]= \\
& =\left[\begin{array}{c|c|c}
A_{1,1} & A_{1,2}^{\prime} \\
\hline A^{\prime}{ }_{2,1} & A^{\prime}{ }_{2,2}
\end{array}\right]\left[\begin{array}{c|c|c|}
B_{1,1} & B_{1,2}^{\prime} \\
\hline B_{2,1}^{\prime} & B^{\prime}{ }_{2,2}
\end{array}\right]= \\
& =\left[\begin{array}{cc}
A_{1,1} B_{1,1}+A_{1,2}^{\prime} B_{2,1}^{\prime} & A_{1,1} B_{1,2}^{\prime}+A_{1,2}^{\prime} B^{\prime}{ }_{2,2} \\
\hline A_{1,2}^{\prime} B_{1,1}^{\prime}+A_{2,2}^{\prime} B_{2,1}^{\prime} & A_{2,1}^{\prime} B^{\prime}{ }_{2,2}+A^{\prime}{ }_{2,2} B_{2,2}
\end{array}\right]=C .
\end{aligned}
$$

## Multiplication of Block Matrices (continued)

Let $E_{i j}=\left[e_{k l}\right] \in M(m \times n ; \mathbb{R})$ be a matrix such that

$$
e_{k l}= \begin{cases}1 & i=k \text { and } j=l \\ 0 & \text { otherwise }\end{cases}
$$

Let $\delta_{i j}$ be the Kronecker delta, i.e.,

$$
\delta_{i j}= \begin{cases}1 & i=j \\ 0 & i \neq j\end{cases}
$$

Then, equivalently

$$
e_{k l}=\delta_{i k} \delta_{j l}
$$

and (matrices $E, E^{\prime}$ and $E^{\prime \prime}$ may have different sizes)

$$
E_{i j} E_{k l}^{\prime}=\delta j k E_{i l}^{\prime \prime}
$$

In particular, for any matrix $A=\left[a_{i} j\right] \in M(m \times n ; \mathbb{R})$

$$
A=\sum_{i, j=1}^{n, m} a_{i j} E_{i j}
$$

and the matrix multiplication can be seen as a special case of the block matrix multiplication.

## Example

$$
\begin{gathered}
A=\left[\begin{array}{l|lll}
1 & 2 & 2 & 6 \\
1 & 2 & 2 & 5 \\
1 & 1 & 2 & 8 \\
\hline 2 & 5 & 6 & 2
\end{array}\right], \quad B=\left[\begin{array}{l|ll}
1 & 0 & 2 \\
1 & 0 & 1 \\
2 & 1 & 2 \\
0 & 1 & 1
\end{array}\right] . \\
A B=\left[\begin{array}{l}
{\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right][1]+\left[\begin{array}{lll}
2 & 2 & 6 \\
2 & 2 & 5 \\
1 & 2 & 8
\end{array}\right]\left[\begin{array}{l}
1 \\
2 \\
0
\end{array}\right] \left\lvert\,\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]\left[\begin{array}{ll}
0 & 2
\end{array}\right]+\left[\begin{array}{lll}
2 & 2 & 6 \\
2 & 2 & 5 \\
1 & 2 & 8
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 2 \\
1 & 1
\end{array}\right]\right.} \\
{\left[\begin{array}{l}
2
\end{array}\right][1]+\left[\begin{array}{lll}
5 & 6 & 2
\end{array}\right]\left[\begin{array}{l}
1 \\
2 \\
0
\end{array}\right] \left\lvert\,\left[\begin{array}{ll}
0
\end{array}\right]\left[\begin{array}{ll}
0 & 2
\end{array}\right]+\left[\begin{array}{lll}
5 & 6 & 2
\end{array}\right]\left[\begin{array}{ll}
1 \\
1 & 2 \\
1 & 1
\end{array}\right]\right.}
\end{array}\right] \\
=\left[\left.\begin{array}{l}
{\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]+\left[\begin{array}{l}
6 \\
6 \\
5
\end{array}\right] \left\lvert\,\left[\begin{array}{ll}
0 & 2 \\
0 & 2 \\
0 & 2
\end{array}\right]+\left[\begin{array}{cc}
8 & 12 \\
7 & 11 \\
10 & 13
\end{array}\right]\right.} \\
2+17
\end{array} \right\rvert\, \begin{array}{ll}
0 & 4
\end{array}\right]+\left[\begin{array}{ll}
8 & 19
\end{array}\right]=\left[\begin{array}{c|cc}
7 & 8 & 14 \\
7 & 7 & 13 \\
6 & 10 & 15 \\
\hline 19 & 8 & 23
\end{array}\right] .
\end{gathered}
$$

## Markov chains - Application

Let $S=\{1, \ldots, N\}$ be the state space.

## Definition

A (discrete-time, discrete-state, time-homogenous) Markov chain) is a sequence of random variables $X_{0}, X_{1}, X_{2}, \ldots, X_{n}, \ldots$ with values in the set $S$ such that for all $i, i_{0}, \ldots, i_{n-1}, j \in S$ and all $n \in \mathbb{N}$

$$
\begin{aligned}
P\left(X_{n+1}=j \mid X_{n}\right. & \left.=i, X_{n-i}=i_{n-1}, \ldots X_{1}=i_{1}, X_{0}=i_{0}\right)= \\
& =P\left(X_{n+1}=j \mid X_{n}=i\right)
\end{aligned}
$$

and it does not depend on $n$ (i.e. the current state depends only on the previous state and this dependence is constant in time).

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\end{aligned}
$$

and it does not depend on $n$ (i.e. the current state depends only on the previous state and this dependence is constant in time).

## Remark

The number $P\left(X_{n+1}=j \mid X_{n}=i\right)$ is called the transition probability form the state $i$ to the state $j$.

## Markov chains - Application (continued)

Definition
The matrix $Q=\left[q_{i j}\right] \in M(N \times N ; \mathbb{R})$ where

$$
q_{i j}=P\left(X_{n+1}=j \mid X_{n}=i\right)
$$

is called the transition matrix.

## Markov chains - Application (continued)

Definition
The matrix $Q=\left[q_{i j}\right] \in M(N \times N ; \mathbb{R})$ where

$$
q_{i j}=P\left(X_{n+1}=j \mid X_{n}=i\right)
$$

is called the transition matrix.
Example
For $N=2$

$$
Q=\left[\begin{array}{ll}
P\left(X_{n+1}=1 \mid X_{n}=1\right) & P\left(X_{n+1}=2 \mid X_{n}=1\right) \\
P\left(X_{n+1}=1 \mid X_{n}=2\right) & P\left(X_{n+1}=2 \mid X_{n}=2\right)
\end{array}\right] .
$$

## Markov chains - Example

Each year a consumer of product $A$ switches to product $B$ with probability one-half. On the other hand, with probability two-thirds a consumer of product $B$ continues buying it and with probability one-third starts buying product $A$. If $S=\{1,2\}$ and 1 stands for product $A$ and 2 for product $B$ then

$$
Q=\left[\begin{array}{ll}
P\left(X_{n+1}=1 \mid X_{n}=1\right) & P\left(X_{n+1}=2 \mid X_{n}=1\right) \\
P\left(X_{n+1}=1 \mid X_{n}=2\right) & P\left(X_{n+1}=2 \mid X_{n}=2\right)
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right] .
$$

## Markov chains - Example (continued)



1 stands for product $A$ and 2 stands for product $B$

$$
Q=\left[\begin{array}{ll}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{array}\right]=\left[\begin{array}{ll}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right] .
$$

## Markov chains - Example (continued)

What is the probability that a consumer of product $A$ switches to product $B$ after two years?

## Markov chains - Example (continued)

What is the probability that a consumer of product $A$ switches to product $B$ after two years?

$$
\begin{gathered}
P\left(X_{2}=2 \mid X_{0}=1\right)= \\
=P\left(X_{2}=2 \mid X_{1}=1\right) P\left(X_{1}=1 \mid X_{0}=1\right)+ \\
+P\left(X_{2}=2 \mid X_{1}=2\right) P\left(X_{1}=2 \mid X_{0}=1\right)= \\
=\frac{1}{2} \cdot \frac{1}{2}+\frac{2}{3} \cdot \frac{1}{2}=\frac{1}{4}+\frac{2}{6}=\frac{7}{12} .
\end{gathered}
$$

(so a consumer switches either in the second or in the first year).

## Markov chains (continued)

## Definition

The $n$-step (conditional) probability of a Markov chain is

$$
P\left(X_{n+2}=j \mid X_{n}=i\right)=q_{i j}^{(n)}
$$

and the $n$-step condition matrix is

$$
Q^{(n)}=\left[q_{i j}^{(n)}\right] \in M(N \times N ; \mathbb{R})
$$

## Markov chains (continued)

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and the $n$-step condition matrix is

$$
Q^{(n)}=\left[q_{i j}^{(n)}\right] \in M(N \times N ; \mathbb{R})
$$

Proposition

$$
Q^{(n)}=Q^{n}=\underbrace{Q \cdot Q \cdots Q}_{n \text {-times }} .
$$

## Markov chains (continued)

## Proof.

It is enough to check the case $n=2$. By the law of total probability

$$
\begin{gathered}
q_{i j}^{(2)}=P\left(X_{n+2}=j \mid X_{n}=i\right)= \\
=\sum_{s=1}^{N} P\left(X_{n+2}=j \mid X_{n+1}=s\right) P\left(X_{n+1}=s \mid X_{n}=i\right)= \\
=\sum_{s=1}^{N} q_{s j} q_{i s}=\sum_{s=1}^{N} q_{i s} q_{s j}
\end{gathered}
$$

## Example (continued)

$$
\begin{aligned}
& \text { If } Q=\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right] \text {, then } \\
& Q^{(2)}=\left[\begin{array}{ll}
P\left(X_{n+2}=1 \mid X_{n}=1\right) & P\left(X_{n+2}=2 \mid X_{n}=1\right) \\
P\left(X_{n+2}=1 \mid X_{n}=2\right) & P\left(X_{n+2}=2 \mid X_{n}=2\right)
\end{array}\right]=Q^{2}= \\
& {\left[\begin{array}{ll}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right]\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right]=\left[\begin{array}{cc}
\frac{5}{12} & \frac{7}{12} \\
\frac{7}{18} & \frac{11}{18}
\end{array}\right] .}
\end{aligned}
$$

## Initial Conditions and Marginal Distribution

## Definition

The initial conditions is the (discrete) probability mass function of the variable $X_{0}$, i.e the vector

$$
\begin{gathered}
\mathbf{t}=\left(t_{1}, t_{2}, \ldots, t_{N}\right)= \\
=\left(P\left(X_{0}=1\right), P\left(X_{0}=2\right), \ldots, P\left(X_{0}=N\right)\right) \in \mathbb{R}^{N},
\end{gathered}
$$

and the marginal distributions are the probability mass functions of variables $X_{1}, X_{2}, \ldots$

$$
\mathbf{t}_{i}=\left(P\left(X_{i}=1\right), P\left(X_{i}=2\right), \ldots, P\left(X_{i}=N\right)\right) \in \mathbb{R}^{N}
$$

for $i=1,2, \ldots$.

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$$

and the marginal distributions are the probability mass functions of variables $X_{1}, X_{2}, \ldots$

$$
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$$

for $i=1,2, \ldots$.
Remark
The marginal distributions depend on the initial conditions.

## Initial Conditions and Marginal Distribution (continued)

Proposition
If $Q$ is the transition matrix of a Markov chain then for $k \geqslant 1$

$$
\mathbf{t}_{k}^{\top}=\mathbf{t}^{\top} Q^{k} .
$$

## Initial Conditions and Marginal Distribution (continued)

Proposition
If $Q$ is the transition matrix of a Markov chain then for $k \geqslant 1$

$$
\mathbf{t}_{k}^{\top}=\mathbf{t}^{\top} Q^{k} .
$$

Proof.
By the law of total probability

$$
\begin{gathered}
P\left(X_{k}=i\right)=\sum_{s=1}^{N} P\left(X_{k}=i \mid X_{0}=s\right) P\left(X_{0}=s\right)= \\
=\sum_{s=1}^{N} t_{s} q_{s i}^{(k)}= \\
=\text { the } i \text {-th entry of } \mathbf{t}^{\top} Q^{k} .
\end{gathered}
$$

## Example

A consumer buys product $A$ with probability $\frac{1}{5}$, product $B$ with probability $\frac{4}{5}$ and the transition matrix is equal to $Q=\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3}\end{array}\right]$, so

$$
\mathbf{t}=\left(\frac{1}{5}, \frac{4}{5}\right)
$$

and

$$
\begin{aligned}
& \mathbf{t}_{2}=\left[\begin{array}{ll}
\frac{1}{5} & \frac{4}{5}
\end{array}\right] Q^{2}=\left[\begin{array}{ll}
\frac{1}{5} & \frac{4}{5}
\end{array}\right]\left[\begin{array}{cc}
\frac{5}{12} & \frac{7}{12} \\
\frac{7}{18} & \frac{11}{18}
\end{array}\right]= \\
&=\left(\frac{71}{180}, \frac{109}{180}\right)
\end{aligned}
$$

## Example (continued)

Hence, after 2 years, a consumer buys product $A$ with probability $\frac{71}{180}$ and product $B$ with probability $\frac{109}{180}$.

## Stable State/Distribution

So, what happens after infinitely many years?

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It turns out, in a distant time a consumer buys product $A$ with probability $\frac{2}{5}$ and product $B$ with probability $\frac{3}{5}$ (and the result does not depend on the initial conditions).

## Stable State/Distribution

So, what happens after infinitely many years? Will the probability of buying product $B$ be equal to one?

It turns out, in a distant time a consumer buys product $A$ with probability $\frac{2}{5}$ and product $B$ with probability $\frac{3}{5}$ (and the result does not depend on the initial conditions).

In particular the vector $\left(\frac{2}{5}, \frac{3}{5}\right)$ is a left eigenvector of matrix $Q$, or equivalently, an eigenvector of $Q^{\top}$, i.e.,

$$
\left[\begin{array}{ll}
\frac{2}{5} & \frac{3}{5}
\end{array}\right]\left[\begin{array}{ll}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3}
\end{array}\right]=\left[\begin{array}{ll}
\frac{2}{5} & \frac{3}{5}
\end{array}\right]
$$

## Hamming Codes

$(7,4)$ Hamming code is a 2 error-detecting, 1 error-correcting linear code.

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$(7,4)$ Hamming code is a 2 error-detecting, 1 error-correcting linear code.

$$
\begin{aligned}
& 0000 \longrightarrow 0000000 \\
& 0001 \longrightarrow 1101001 \\
& 0010 \longrightarrow 0101010 \\
& 0011 \longrightarrow 1000011 \\
& 0100 \longrightarrow 1001100 \\
& 0101 \longrightarrow 0100101 \\
& 0110 \longrightarrow 1100110 \\
& 0111 \longrightarrow 0001111
\end{aligned}
$$

$$
b_{0} b_{1} b_{2} b_{3} \rightarrow p_{0} b_{0} p_{1} p_{2} b_{1} b_{2} b_{3}
$$

## Hamming Codes (continued)

$$
b_{0} b_{1} b_{2} b_{3} \rightarrow p_{0} b_{0} p_{1} p_{2} b_{1} b_{2} b_{3}
$$

where

$$
\begin{aligned}
& p_{0}=b_{0}+b_{1}+b_{3}, \\
& p_{1}=b_{0}+b_{2}+b_{3}, \\
& p_{2}=b_{0}+b_{1}+b_{3},
\end{aligned}
$$

where addition is modulo 2 , i.e $1+1=0$.

## Hamming Codes (continued)

Encoding (and decoding) can be realised by matrix multiplication.

$$
\left[\begin{array}{llll}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
b_{0} \\
b_{1} \\
b_{2} \\
b_{3}
\end{array}\right]=\left[\begin{array}{l}
p_{0} \\
b_{0} \\
p_{1} \\
p_{2} \\
b_{1} \\
b_{2} \\
b_{3}
\end{array}\right] .
$$

## Paths in Directed Graphs

Definition
A (simple, finite) directed graph $G$ is a pair $G=(V, E)$ where

$$
V=\left\{v_{1}, \ldots, v_{n}\right\}
$$

is the set of vertices and $E \subset V \times V$ is the set of edges (self-loops are allowed).

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$$

is the set of vertices and $E \subset V \times V$ is the set of edges (self-loops are allowed).

## Definition

A path from $v \in V$ to $w \in V$ of length $/$ is a sequence of edges

$$
\left(v_{i_{0}}, v_{i_{1}}\right),\left(v_{i_{1}}, v_{i_{2}}\right), \ldots,\left(v_{i_{l-1}}, v_{i_{l}}\right)
$$

such that

$$
\begin{gathered}
v_{i_{0}}=v, \quad v_{i_{l}}=w, \\
\left(v_{i_{k}}, v_{i_{k+1}}\right) \in E
\end{gathered}
$$

for $k=0, \ldots, l-1$.

## Adjacency Matrix

## Definition

For a fixed simple, finite, directed graph $G$ the adjacency matrix $A=A_{G}$ of $G$ is the matrix $A=\left[a_{i j}\right] \in M(n \times n ; \mathbb{R})$ such that

$$
a_{i j}=1 \text { if and only if }\left(v_{i}, v_{j}\right) \in E
$$

and $a_{i j}=0$ otherwise.

## Example



## Example



$$
A=\left[\begin{array}{llll}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

## Example



For example $\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right),\left(v_{3}, v_{4}\right)$ is a path from $v_{1}$ to $v_{4}$ of length 3.

## Number of Paths

## Proposition

If $A$ is the adjacency matrix of a (finite, simple) directed graph $G$, then $\left(A^{\prime}\right)_{i j}$ for any $I \geqslant 1$ is the number of paths of length I from $v_{i}$ to $v_{j}$.

## Number of Paths

## Proposition

If $A$ is the adjacency matrix of a (finite, simple) directed graph $G$, then $\left(A^{\prime}\right)_{i j}$ for any $l \geqslant 1$ is the number of paths of length I from $v_{i}$ to $v_{j}$.

Proof.
By induction.

Example (continued)


## Example (continued)



$$
A=\left[\begin{array}{llll}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right], \quad A^{2}=\left[\begin{array}{llll}
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad A^{3}=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

## Example



## Example



$$
A=\left[\begin{array}{llll}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right], \quad A^{2}=\left[\begin{array}{llll}
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{array}\right], \quad A^{3}=\left[\begin{array}{llll}
2 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 2
\end{array}\right] .
$$

## Dual Spaces

## Definition

Let $V$ be a vector space. The space dual to $V$ is a vector space

$$
V^{*}=\{\varphi: V \rightarrow \mathbb{R} \mid \varphi \text { is linear }\}=\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})
$$

with the vector space structure given by
i) $(\varphi+\psi)(v)=\varphi(v)+\psi(v)$,
ii) $(\alpha \varphi) v=\alpha \varphi(v)$,
for $\varphi, \psi \in V^{*}$ and any $v \in V, \alpha \in V$.

## Dual Basis

## Proposition

For any vector space $V$ and any basis $\mathcal{A}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ of vector space $V$, there exists the dual basis $\mathcal{A}^{*}=\left(\alpha_{1}^{*}, \ldots, \alpha_{n}^{*}\right)$ of the vector space $V^{*}$ such that

$$
\alpha_{i}^{*}\left(\alpha_{j}\right)=\left\{\begin{array}{ll}
0 & \text { for } i \neq j \\
1 & \text { for } i=j
\end{array} .\right.
$$

Proof.
There exists dual basis to the standard basis, i.e. $s t^{*}$ given by

$$
\begin{gathered}
\varepsilon_{1}^{*}\left(\left(x_{1}, \ldots, x_{n}\right)\right)=x_{1} \\
\varepsilon_{2}^{*}\left(\left(x_{1}, \ldots, x_{n}\right)\right)=x_{2} \\
\vdots \\
\varepsilon_{n}^{*}\left(\left(x_{1}, \ldots, x_{n}\right)\right)=x_{n} .
\end{gathered}
$$

## Dual Basis (continued)

## Proof.

Let $\varphi: V \rightarrow V$ be the linear transformation such that

$$
\varphi\left(\alpha_{i}\right)=\varepsilon_{i}
$$

Then

$$
\alpha_{i}^{*}=\varepsilon_{i}^{*} \circ \alpha .
$$

Assume $\alpha_{i}=\left(a_{1 j}, a_{2 j}, \ldots, a_{n j}\right)$ and let $A=\left[a_{i j}\right] \in M(n \times n ; \mathbb{R})$, then

$$
M(\varphi)_{s t}^{s t}=A^{-1}, \quad M\left(\varepsilon_{i}^{*}\right)=\left[\begin{array}{lllllll}
0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{array}\right]
$$

## Dual Basis (continued)

Proof.
Let $A^{-1}=\left[b_{i j}\right]$. Then

$$
\begin{aligned}
& M\left(\varepsilon_{i}\right)_{s t}^{s t}=M\left(\alpha_{i}^{*}\right)_{s t}^{s t} M(\varphi)_{s t}^{s t}= \\
& =\left[\begin{array}{lllll}
b_{i 1} & b_{i 2} & b_{i 3} & \cdots & b_{i n}
\end{array}\right]
\end{aligned}
$$

That is

$$
\varphi_{i}^{*}=b_{i 1} \varepsilon_{1}^{*}+b_{i 2} \varepsilon_{2}^{*}+\ldots+b_{i n} \varepsilon_{n}^{*} .
$$

## Example

Let $\alpha_{1}=(1,2), \alpha_{2}=(1,3)$ be a basis of $\mathbb{R}^{2}$. Then

$$
A=\left[\begin{array}{ll}
1 & 1 \\
2 & 3
\end{array}\right], \quad B=A^{-1}=\left[\begin{array}{rr}
3 & -1 \\
-2 & 1
\end{array}\right] .
$$

Therefore

$$
\begin{gathered}
\alpha_{1}^{*}=3 \varepsilon_{1}^{*}-\varepsilon_{2}^{*} \\
\alpha_{2}^{*}=-2 \varepsilon_{1}^{*}+\varepsilon_{2}^{*}
\end{gathered}
$$

or, in more concrete terms,

$$
\begin{gathered}
\alpha_{1}^{*}\left(\left(x_{1}, x_{2}\right)\right)=3 x_{1}-x_{2} \\
\alpha_{2}^{*}\left(\left(x_{1}, x_{2}\right)\right)=-2 x_{1}+x_{2}
\end{gathered}
$$

## Dual of a linear transformation

## Definition

For any linear transformation $\varphi: V \rightarrow W$, there exist the dual linear transformation

$$
\varphi^{*}: W^{*} \rightarrow V^{*}
$$

given by the formula

$$
\varphi^{*}(f)(v)=(f \circ \varphi)(v)
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Proposition

$$
\text { If } \begin{aligned}
& \mathcal{A}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \mathcal{B}=\left(\beta_{1}, \ldots, \beta_{m}\right) \text { then } \\
& \\
& M\left(\varphi^{*}\right)_{\mathcal{B}^{*}}^{\mathcal{A}^{*}}=\left[M(\varphi)_{\mathcal{A}}^{\mathcal{B}}\right]^{\top} .
\end{aligned}
$$

## Dual of a linear transformation (continued)

Proof.
Let $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}=\left[a_{i j}\right]$.

$$
\begin{gathered}
\varphi^{*}\left(\beta_{i}^{*}\right)\left(\alpha_{j}\right)=\left(\beta_{i} \circ \varphi\right)\left(\alpha_{j}\right)= \\
=\beta_{i}\left(\sum_{s=1}^{m} a_{s j} \beta_{j}\right)=a_{i j},
\end{gathered}
$$

that the entry in the $i-$ th column and in the $j-$ th row of $M\left(\varphi^{*}\right)_{\mathcal{B}^{*}}^{\mathcal{A}^{*}}$ is equal to $a_{i j}$.

## Dual subspaces

## Definition

Let $W \subset V$ be any subset of vector space $V$. Let

$$
W^{*}=\left\{f \in V^{*}|f| w=0\right\} .
$$

If $W \subset V$ is a subspace the $W^{*}$ called the dual subspace of $W$.

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Proof.
Omitted. Needs isomorphism Theorem.

## Bilinear Forms and Bilinear Transformations

Definition
Let $V, W, U$ be vector spaces. A function

$$
B: V \times W \rightarrow U
$$

is called a bilinear transformation if
i) $B\left(v+v^{\prime}, w\right)=B(v, w)+B\left(v^{\prime}, w\right)$ for any $v, v^{\prime} \in V, w \in W$,

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iii) $B(\alpha v, w)=\alpha B(v, w)$ for any $v \in V, w \in W, \alpha \in \mathbb{R}$,
iv) $B(v, \beta w)=\beta B(v, w)$ for any $v \in V, w \in W, \beta \in \mathbb{R}$.

Bilinear transformation $B$ is called a form if moreover $U=\mathbb{R}$.

## Equivalence Relation

Definition
An equivalence relation $R$ on the set $X$ is a relation (i.e. a subset) $R \subset X \times X$ such that
i) $\forall x \in X(x, x) \in R(R$ is reflexive $)$,

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i) $\forall_{x \in X}(x, x) \in R(R$ is reflexive $)$,
ii) $\forall_{x \in X} \forall_{y \in X}(x, y) \in R \Rightarrow(y, x) \in R$ ( $R$ is symmetric),
iii) $\forall_{x \in X} \forall_{y \in X} \forall_{z \in X}(x, y) \in R \wedge(y, z) \in R \Rightarrow(x, z) \in R(R$ is transitive).

## Partitions and Equivalence Classes

Equivalence relation $R$ on set $X$ induces a partition of $X$, given by its equivalence classes (in fact, there is a bijection between all partitions of $X$ and all equivalence relations on $X$ ).
Definition
For any $x \in X$ the set

$$
[x]_{R}=\{y \in X \mid(x, y) \in R\}
$$

is called an equivalence class of element $x \in X$. When no confusion is possible we write $[x]$.

## Quotient Vector Space

## Definition

Let $W \subset V$ be a subspace of vector space $V$. The relation $R \subset V \times V$, given by the condition

$$
\left(v, v^{\prime}\right) \in R \Leftrightarrow v-v^{\prime} \in W,
$$

is an equivalence relation on $V$, compatible with the structure of the vector space, i.e. the equivalence classes satisfy conditions
i) $[v]+\left[v^{\prime}\right]=\left[v+v^{\prime}\right]$,
ii) $\alpha[v]=[\alpha v]$,
for any $v, v^{\prime} \in V$ and $\alpha \in \mathbb{R}$.

## Tensor Product

## Definition

For any vector spaces $V, W$ let $U$ be a (infinite dimensional) vector space with basis $\left(u_{v, w}\right)_{v \in V, w \in W}$ and let $U_{0} \subset U$ be its subspace spanned by vectors
$u_{v+v^{\prime}, w}-u_{v, w}-u_{v^{\prime}, w}, u_{v, w+w^{\prime}}-u_{v, w}-u_{v, w^{\prime}}, u_{\alpha v, w}-\alpha u_{v, w}, u_{v, \beta w}-\beta u_{v, w}$,
where $v \in V, w \in V, \alpha, \beta \in \mathbb{R}$. By definition, the tensor product of vector spaces $V$ and $W$ is equal to the quotient space

$$
V \otimes W=U / U_{0}
$$

## Tensor Product (continued)

## Definition

By definition

$$
v \otimes w=\left[u_{v, w}\right] .
$$

Then, for any $v, v^{\prime} \in V, w, w^{\prime} \in W, \alpha, \beta \in \mathbb{R}$
i) $\left(v+v^{\prime}\right) \otimes w=v \otimes w+v^{\prime} \otimes w$,
ii) $v \otimes\left(w+w^{\prime}\right)=v \otimes w+v \otimes w^{\prime}$,
iii) $(\alpha v) \otimes w=\alpha(v \otimes w)$,
iv) $v \otimes(\beta w)=\beta(v \otimes w)$.

Moreover, there exists bilinear transformation

$$
\pi: V \times W \ni(v, w) \mapsto v \otimes w \in V \otimes W
$$

## Tensor Product (continued)

## Proposition

For any vector spaces $V, W$ and for any bilinear transformation

$$
B: V \times W \rightarrow U
$$

there exists a unique linear transformation

$$
\varphi_{B}: V \otimes W \rightarrow U
$$

such that for any $v \in V, w \in W$

$$
B(v, w)=\varphi_{B}(v \otimes w),
$$

i.e.


## Tensor Product (continued)

## Proof.

Let $\bar{\varphi}: \operatorname{lin}\left(\left\{u_{v, w} \mid v \in V, w \in W\right\}\right) \rightarrow U$ be a linear transformation given by

$$
\bar{\varphi}\left(u_{v, w}\right)=B(v, w)
$$

Since $B$ is bilinear, the transformation $\bar{\varphi}$ sends vectors
$u_{v+v^{\prime}, w}-u_{v, w}-u_{v^{\prime}, w}, u_{v, w+w^{\prime}}-u_{v, w}-u_{v, w^{\prime}}, u_{\alpha v, w}-\alpha u_{v, w}, u_{v, \beta w}-\beta u_{v, w}$,
to zero (in $U$ ). Since vectors $\left[u_{v, w}\right]$ span $V \otimes W, \bar{\varphi}$ descends to a linear transformation

$$
\varphi: V \otimes W \ni v \otimes w=\left[u_{v, w}\right] \mapsto B(v, w) \in U
$$

## Tensor Product (continued)

## Proposition

If $\mathcal{A}=\left(\alpha_{1}, \ldots \alpha_{n}\right)$ is a basis of $V$ and $\mathcal{B}=\left(\beta_{1}, \ldots \beta_{m}\right)$ is a basis of $W$, then

$$
\mathcal{A} \otimes \mathcal{B}=\left(\alpha_{i} \otimes \beta_{j}\right)_{i=1, \ldots, n, j=1, \ldots, m}
$$

is a basis of $V \otimes W$. It follows

$$
\operatorname{dim} V \otimes W=(\operatorname{dim} V)(\operatorname{dim} W)=n m
$$

## Tensor Product (continued)

Proof.
Consider a bilinear form $B: V \times W \rightarrow \mathbb{R}$, i.e.


If $v=\sum_{i=1}^{n} v_{i} \alpha_{i}$ and $w=\sum_{j=1}^{m} w_{j} \beta_{j}$ are bases of $V$ and $W$, respectively, then

$$
B(v, w)=\sum_{i=1}^{n} \sum_{j=1}^{m} v_{i} w_{j} B\left(\alpha_{i}, \beta_{j}\right),
$$

i.e. $B$ is uniquely determined by the values $B\left(\alpha_{i}, \beta_{j}\right)$. On the other hand if $a_{i j} \in \mathbb{R}$ are some numbers, there exists a unique bilinear form $B$ such that $B\left(\alpha_{i}, \beta_{j}\right)=a_{i j}$, given by the formula

$$
B(v, w)=\sum_{i=1}^{n} \sum_{j=1}^{m} v_{i} w_{j} a_{i j} .
$$

## Tensor Product (continued)

## Proof.

By the properties of tensor product, any linear form $\varphi$ induces a bilinear form $\pi \circ \varphi$.


By the universal property, the linear transformation

$$
(V \otimes W)^{*} \ni \varphi \mapsto \varphi \circ \pi
$$

is an isomorphism. Therefore

$$
\operatorname{dim} V \otimes W=\operatorname{dim}(V \otimes W)^{*}=\operatorname{dim} V \operatorname{dim} W
$$



## Vector Space of Linear Transformations

## Proposition

For any vector spaces $V, W$ there exists a linear isomorphism of vector spaces

$$
V^{*} \otimes W \ni f \otimes w \mapsto(V \ni v \mapsto f(v) w \in V) \in \operatorname{Hom}(V, W)
$$

where $\operatorname{Hom}(V, W)$ denotes the vector space of all linear transformations from $V$ to $W$.

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where Hom $(V, W)$ denotes the vector space of all linear transformations from $V$ to $W$.

Proof.
The transformation sends basis to basis.

## Vector Space of Linear Transformations (continued)

## Proposition

For any vector spaces $V, W$ and for any $\mathcal{A}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ a basis of $V$ and $\mathcal{B}=\left(\beta_{1}, \cdots, \beta_{m}\right)$ a basis of $W$, if

$$
M(\varphi)_{\mathcal{A}}^{\mathcal{B}}=\left[a_{i j}\right] \in M(m \times n ; \mathbb{R})
$$

then, under the above isomorphism

$$
\varphi=\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i j} \alpha_{j}^{*} \otimes \beta_{i} \in V^{*} \otimes W
$$

## Vector Space of Linear Transformations (continued)

## Proposition

For any vector spaces $V, W$ and for any $\mathcal{A}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ a basis of $V$ and $\mathcal{B}=\left(\beta_{1}, \cdots, \beta_{m}\right)$ a basis of $W$, if

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$$
\varphi=\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i j} \alpha_{j}^{*} \otimes \beta_{i} \in V^{*} \otimes W
$$

Proof.
Exercise.

## Examples

If $\varphi: \mathbb{R}^{3} \rightarrow \mathbb{R}$ is given by the formula

$$
\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=x_{1}+2 x_{2}-7 x_{3},
$$

then

$$
\varphi=\varepsilon_{1}^{*}+2 \varepsilon_{2}^{*}-7 \varepsilon_{3}^{*} .
$$

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then

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\varphi=\varepsilon_{1}^{*}+2 \varepsilon_{2}^{*}-7 \varepsilon_{3}^{*} .
$$

If $\varphi: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$, where $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$ is the standard basis of $\mathbb{R}^{3}$ and $\zeta_{1}, \zeta_{2}$ is the standard basis of $\mathbb{R}^{2}$, is given by the formula

$$
\begin{gathered}
\varphi\left(\left(x_{1}, x_{2}, x_{3}\right)\right)=\left(x_{1}+2 x_{2}-7 x_{3}, 5 x_{1}-8 x_{2}\right)= \\
x_{1}(1,5)+x_{2}(2,-8)+x_{3}(-7,0)=x_{1}\left(\zeta_{1}+5 \zeta_{2}\right)+x_{2}\left(2 \zeta_{1}-8 \zeta_{2}\right)+x_{3}\left(-7 \zeta_{1}\right),
\end{gathered}
$$ then

$$
\begin{gathered}
M(\varphi)_{s t}^{s t}=\left[\begin{array}{rrr}
1 & 2 & -7 \\
5 & -8 & 0
\end{array}\right] \\
\varphi=\varepsilon_{1}^{*} \otimes \zeta_{1}+2 \varepsilon_{2}^{*} \otimes \zeta_{1}-7 \varepsilon_{3}^{*} \otimes \zeta_{1}+5 \varepsilon_{1}^{*} \otimes \zeta_{2}-8 \varepsilon_{2}^{*} \otimes \zeta_{2} .
\end{gathered}
$$

## Trace

## Definition

Let $\varphi: V \rightarrow V$, i.e., $\varphi \in V^{*} \otimes V$ be a linear endomorphism. The trace of $\varphi$ is equal to $\operatorname{Tr}(\varphi)$, where

$$
\operatorname{Tr}: V^{*} \otimes V \ni \alpha \otimes \beta \mapsto \alpha(\beta) \in \mathbb{R}
$$

is the unique linear transformation corresponding to the bilinear transformation

$$
V^{*} \times V \ni(\alpha, \beta) \mapsto \alpha(\beta) \in \mathbb{R}
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Proposition
For any basis $\mathcal{A}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ of $V$, if $M(\varphi)_{\mathcal{A}}^{\mathcal{A}}=\left[a_{i j}\right]$, then

$$
\operatorname{Tr}(\varphi)=\sum_{i=1}^{n} a_{i i} .
$$

## Trace (continued)

Proof.
By the definition

$$
\varphi=\sum_{i, j=1}^{n} a_{i j} \alpha_{j}^{*} \otimes \alpha_{i}
$$

and

$$
\operatorname{Tr}\left(\alpha_{j}^{*} \otimes \alpha_{i}\right)=\left\{\begin{array}{lll}
1 & \text { if } & i=j \\
0 & \text { if } & i \neq j,
\end{array}\right.
$$

## Trace of a Matrix

## Definition

Let $A=\left[a_{i j}\right] \in M(n \times n ; \mathbb{R})$ be a matrix. The trace of matrix $A$ is equal to

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\operatorname{Tr}(A)=\sum_{i=1}^{n} \mathrm{a}_{i j}
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## Corollary

For any linear endomorphism $\varphi: V \rightarrow V$ and any basis $\mathcal{A}$ of vector space V

$$
\operatorname{Tr}(\varphi)=\operatorname{Tr}\left(M(\varphi)_{\mathcal{A}}^{\mathcal{A}}\right)
$$

that is for any invertible matrix $C \in M(n \times n ; \mathbb{R})$

$$
\operatorname{Tr}(A)=\operatorname{Tr}\left(C^{-1} A C\right)
$$

i.e., the trace admits the same value on similar matrices.

## Trace of a Matrix (continued)

## Proposition

For any matrices $A, B, C \in M(n \times n ; \mathbb{R})$ and scalars $\alpha, \beta \in \mathbb{R}$
i) $\operatorname{Tr}(A)=\operatorname{Tr}\left(A^{\top}\right)$,
ii) $\operatorname{Tr}(\alpha A+\beta B)=\alpha \operatorname{Tr}(A)+\beta \operatorname{Tr}(B)$,
iii) $\operatorname{Tr}(A B)=\operatorname{Tr}(B A)$,
iv) $\operatorname{Tr}(A B C)=\operatorname{Tr}(B C A)=\operatorname{Tr}(C A B)$.

Proof.
Points i) and ii) are obvious and iv) follows from iii).

## Trace of a Matrix (continued)

## Proof.

iii) let $C=\left[c_{i j}\right]=A B$ and $C^{\prime}=\left[c_{i j}^{\prime}\right]=B A$. Then

$$
\begin{gathered}
\operatorname{Tr}(C)=\sum_{i=1}^{n} c_{i i}=\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i j} b_{j i} \\
\operatorname{Tr}\left(C^{\prime}\right)=\sum_{i=1}^{n} c_{i i}^{\prime}=\sum_{i=1}^{n} \sum_{j=1}^{n} b_{i j} a_{j i}= \\
=\sum_{i=1}^{n} \sum_{j=1}^{n} a_{j i} b_{i j}=\operatorname{Tr}(C)
\end{gathered}
$$

## Non-canonical Isomorphism

## Remark

If $V$ is a finite dimensional vector space then the vector spaces $V$ and $V^{*}$ are isomorphic, however there is no canonical isomorphism.

## Example

For example, let $V=\mathbb{R}^{2}$ and let $F, G: V \rightarrow V^{*}$ be linear transformations given by the conditions

$$
\begin{array}{ll}
F\left(\varepsilon_{1}\right)=\varepsilon_{1}^{*}, & G\left(\alpha_{1}\right)=\alpha_{1}^{*} \\
F\left(\varepsilon_{2}\right)=\varepsilon_{2}^{*}, & G\left(\alpha_{2}\right)=\alpha_{2}^{*}
\end{array}
$$

where $\alpha_{1}=(1,2), \alpha_{2}=(1,3)$ then

$$
\begin{gathered}
F\left(\alpha_{1}\right)=F\left(\varepsilon_{1}+2 \varepsilon_{2}\right)=\varepsilon_{1}^{*}+2 \varepsilon_{2}^{*} \neq \alpha_{1}^{*}= \\
=3 \varepsilon_{1}^{*}-\varepsilon_{2}^{*} .
\end{gathered}
$$

## Non-canonical Isomorphism

Remark
If $V$ is infinite dimensional then $V$ is not isomorphic to $V^{*}$ (the reason is purely set-theoretical). For example

$$
(\mathbb{R} \oplus \mathbb{R} \oplus \ldots)^{*} \simeq(\mathbb{R} \times \mathbb{R} \times \ldots)
$$

## The Bidual Space

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For any vector space $V$, the bidual space is equal to

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$$

## Proposition

If $V$ is finite dimensional, there exists a canonical isomorphism of $V$ and $V^{* *}$ (which does not depend on the particular choice of a basis)

$$
F: V \ni v \mapsto\left(V^{*} \ni f \mapsto f(v) \in \mathbb{R}\right) \in\left(V^{*}\right)^{*}=V^{* *} .
$$

## The Bidual Space

## Definition

For any vector space $V$, the bidual space is equal to

$$
V^{* *}=\left(V^{*}\right)^{*} .
$$

## Proposition

If $V$ is finite dimensional, there exists a canonical isomorphism of $V$ and $V^{* *}$ (which does not depend on the particular choice of a basis)

$$
F: V \ni v \mapsto\left(V^{*} \ni f \mapsto f(v) \in \mathbb{R}\right) \in\left(V^{*}\right)^{*}=V^{* *}
$$

Proof.

$$
v \in \operatorname{ker} F \Leftrightarrow f(v)=0 \text { for all } f \in V^{*} \Leftrightarrow v=\mathbf{0},
$$

since any non-zero vector can be completed to basis, and then $v^{*}(v)=1$.

## Frobenius Norm

Proposition
The bilinear (real, Frobenius) form

$$
\langle\cdot, \cdot\rangle: M(m \times n ; \mathbb{R}) \times M(m \times n ; \mathbb{R}) \ni(A, B) \mapsto \operatorname{Tr}\left(A^{\top} B\right) \in \mathbb{R},
$$

is
i) symmetric, i.e.,

$$
\langle A, B\rangle=\langle B, A\rangle,
$$

ii) positive definite, i.e.,

$$
\langle A, A\rangle>0,
$$

if $A \neq \mathbf{0}$,

## Frobenius Norm (continued)

## Proposition

iii) non-degenerate, i.e., the linear transformation

$$
\langle A, \cdot\rangle: M(m \times n ; \mathbb{R}) \ni B \mapsto\langle A, B\rangle \in \mathbb{R},
$$

is non-zero if and only if $A \neq \mathbf{0}$,
iv) invariant under the left and the right multiplication by an orthogonal matrix, i.e., if $Q \in M(m \times m ; \mathbb{R})$ satisfies $Q^{\top} Q=Q Q^{\top}=I_{m}$, then

$$
\langle Q A, Q B\rangle=\langle A, B\rangle,
$$

and if $P \in M(n \times n ; \mathbb{R})$ satisfies $P^{\top} P=P P^{\top}=I_{n}$, then

$$
\langle A P, B P\rangle=\langle A, B\rangle .
$$

## Frobenius Norm (continued)

## Proof.

i) $\operatorname{Tr}\left(A^{\top} B\right)=\operatorname{Tr}\left(\left(A^{\top} B\right)^{\top}\right)=\operatorname{Tr}\left(B^{\top} A\right)$,
ii) if $A=\left[a_{i j}\right] \neq \mathbf{0}$, then
$\operatorname{Tr}\left(A^{\top} A\right)=\sum_{i=1}^{n} \sum_{j=1}^{n} a_{j i} a_{j i}=\sum_{i, j=1}^{n} a_{i j}^{2}>0$,
iii) follows from ii) (substitute $B=A$ ),
iv) $\operatorname{Tr}\left((Q A)^{\top} Q B\right)=\operatorname{Tr}\left(A^{\top} Q^{\top} Q B\right)=\operatorname{Tr}\left(A^{\top} B\right)$ and $\operatorname{Tr}\left((A P)^{\top} B P\right)=\operatorname{Tr}\left(P^{\top} A^{\top} B P\right)=\operatorname{Tr}\left(A^{\top} B\right)$ since $P^{\top}=P^{-1}$.

## Frobenius Norm (continued)

Definition
For any matrix $A \in M(m \times n ; \mathbb{R})$ the Frobenius norm of $A$ is equal to

$$
\|A\|_{F}=\sqrt{\langle A, A\rangle}=\sqrt{\operatorname{Tr}\left(A^{\top} A\right)}
$$

## Frobenius Norm (continued)

## Proposition

i) $\|A\|_{F}=0 \quad \Leftrightarrow \quad A=\mathbf{0}$,
ii) $\|\alpha A\|_{F}=|\alpha|\|A\|_{F}$ for any scalar $\alpha \in \mathbb{R}$,
iii) $\|A+B\|_{F} \leqslant\|A\|_{F}+\|B\|_{F}$, for any matrices
$A, B \in M(n \times m ; \mathbb{R})$
iv) $\|A\|_{F}=\left\|A^{\top}\right\|_{F}$,
v) $\|A B\|_{F} \leqslant\|A\|_{F}\|B\|_{F}$, for any matrix $A \in M(m \times n ; \mathbb{R})$ and any matrix $B \in M(n \times k ; \mathbb{R})$,
vi) $\|Q A P\|_{F}=\|A\|_{F}$ for any matrix $A \in M(n \times m ; \mathbb{R})$, any orthogonal matrix $Q \in M(m \times m ; \mathbb{R})$ and any orthogonal matrix $P \in M(n \times n ; \mathbb{R})$

## Frobenius Norm (continued)

## Proof.

i) the Frobenius form is positive definite,
ii) obvious,
iii) by the Cauchy-Schwarz inequality for the Frobenius form $|\langle A, B\rangle| \leqslant\|A\|_{F}\|B\|_{F}$, then

$$
\|A+B\|_{F}^{2}=\|A\|_{F}^{2}+2\langle A, B\rangle+\|B\|_{F}^{2} \leqslant\left(\|A\|_{F}+\|B\|_{F}\right)^{2}
$$

iv) obvious,

## Frobenius Norm (continued)

## Proof.

v) from the Cauchy-Schwarz inequality for the standard scalar product in $\mathbb{R}^{n}$, if $A B=\left[c_{p q}\right]$,

$$
\begin{gathered}
\|A B\|_{F}^{2}=\sum_{p=1}^{m} \sum_{q=1}^{k} c_{p q}^{2}= \\
\sum_{p=1}^{m} \sum_{q=1}^{k}\left(\sum_{r=1}^{n} a_{p r} b_{r q}\right)^{2} \leqslant \sum_{p=1}^{m} \sum_{q=1}^{n}\left(\sum_{r=1}^{n} a_{p r}^{2}\right)\left(\sum_{r^{\prime}=1}^{n} b_{q r^{\prime}}^{2}\right)= \\
=\left(\sum_{p=1}^{m}\left(\sum_{r=1}^{n} a_{p r}^{2}\right)\right)\left(\sum_{q=1}^{k}\left(\sum_{r^{\prime}=1}^{n} b_{q r^{\prime}}^{2}\right)\right)=\|A\|_{F}^{2}\|B\|_{F}^{2} .
\end{gathered}
$$

vi) follows from the properties of the Frobenius form.

## Frobenius Norm (continued)

Remark
$\|A B\|_{F}=\|A\|_{F}\|B\|_{F}$ holds if and only if $i-$ th row of $A$ is linearly dependent with $i$-th column of $B$ for any $i$ (equality in standard Cauchy-Schwarz inequality).
$\|A+B\|_{F}=\|A\|_{F}+\|B\|_{F}$ holds if and only if $\langle A, B\rangle=\|A\|\|B\|$,
i.e. $A=\lambda B$ or $B=\lambda A$ for some $\lambda \geqslant 0$ (as in the standard Minkowski inequality).

## Frobenius Norm (continued)

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By the SVD decomposition if $\sigma_{1}(A), \ldots, \sigma_{r}(A) \in \mathbb{R}$ denote the singular values of $A$ (i.e., square roots of the non-zero eigenvalues of the matrix $A^{\top} A$ ) then

$$
\|A\|_{F}=\sqrt{\sigma_{1}^{2}(A)+\ldots+\sigma_{r}^{2}(A)} .
$$

## Frobenius Norm (continued)

$$
\begin{aligned}
\|A\|_{F}= & \sqrt{\sigma_{1}^{2}(A)+\ldots+\sigma_{r}^{2}(A)}, \\
& \|A\|_{2}=\sigma_{1}(A),
\end{aligned}
$$

therefore for any matrix $A \in M(m \times n ; \mathbb{C})$

$$
\|A\|_{2} \leqslant\|A\|_{F} \leqslant \sqrt{\min \{m, n\}}\|A\|_{2} .
$$

Note that any two norms in a finite-dimensional space are equivalent (i.e. they induce the same topology and convergence in one norm is equivalent to the convergence in the other).

## Matrix Multiplication as a Sum of Rank 1 Matrices

## Remark

In some contexts (large sparse matrices) it is useful to interpret the matrix product in the following way. Let $A \in M(m \times n ; \mathbb{R})$ and let $B \in M(n \times l ; \mathbb{R})$. The matrix product of $A$ by $B$ is equal to the sum rank 1 matrices

$$
A B=\sum_{s=1}^{n} C_{i} R_{i}=\left[\begin{array}{c}
a_{1 i} \\
a_{2 i} \\
\vdots \\
a_{n i}
\end{array}\right]\left[\begin{array}{llll}
b_{i 1} & b_{i 2} & \ldots & b_{i n}
\end{array}\right] \text {. }
$$

Proof.
The $(i, j)$ entry of $C_{s} R_{s}$ is equal to $a_{i s} b_{s j}$.

