Linear Algebra

Lecture 4 - Linear Transformations

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For example, the function $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $\varphi((x_1, x_2, x_3)) = (x_1 + 2x_2 - x_3, x_2 + x_3)$ is a linear transformation.

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In general, the following theorem holds:

Linear Transformations (continued)

Theorem

A function $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if and only if

$$\varphi((x_1, \ldots, x_n)) = (a_{11}x_1 + \ldots + a_{1n}x_n, a_{21}x_1 + \ldots + a_{2n}x_n, \ldots$$

$$\ldots, a_{m1}x_1 + \ldots + a_{mn}x_n), \text{ where } a_{ij} \in \mathbb{R}.$$

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Proof.

Assume φ is a linear transformation. Let

$$\varphi(\varepsilon_i) = (a_{1i}, a_{2i}, \ldots, a_{mi}).$$

Then, by linearity,
$\varphi(x_1, \ldots, x_n) = \varphi(x_1\varepsilon_1 + \ldots + x_n\varepsilon_n) = x_1\varphi(\varepsilon_1) + \ldots + x_n\varphi(\varepsilon_n) =$
$(a_{11}x_1 + \ldots + a_{1n}x_n, a_{21}x_1 + \ldots + a_{2n}x_n, \ldots, a_{m1}x_1 + \ldots + a_{mn}x_n)$. $\square$

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The transformation $\varphi\colon \mathcal{F}(\mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\varphi(f) = f(1)$ is linear.

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The transformation $\varphi\colon \mathcal{D}(\mathbb{R},\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R})$, where $\mathcal{D}(\mathbb{R},\mathbb{R})$ denotes differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\varphi(f) = f'$ is linear.

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iii) the set $\varphi(V) = \{\varphi(v) \mid v \in V\}$ is a subspace of W called the **image** of V, if $v_1, \ldots, v_k$ span V then $\varphi(v_1), \ldots, \varphi(v_k)$ span $\varphi(V)$,

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v) if φ is injective and the vectors $v_1, \dots, v_k$ are linearly independent then $\varphi(v_1), \dots, \varphi(v_k)$ are linearly independent too.

Properties (continued)

Theorem

Let V, W be vector spaces. For any basis $v_1, \ldots, v_n \in V$ and any vectors $w_1, \ldots, w_n \in W$ there exists a unique linear transformation $\varphi\colon V \longrightarrow W$ such that $\varphi(v_i) = w_i$ for $i = 1, \ldots, n$.

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Proof.

For $v = \sum_{i=1}^{n} \alpha_i v_i$ set $\varphi(v) = \sum_{i=1}^{n} \alpha_i w_i$. It is easy to check that φ is a linear transformation (by the uniqueness of coordinates relative to a basis) and it is unique, since any other linear transformation sending v_i to w_i satisfies the same conditions. $\square$

Example

Let $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $\varphi((1,3)) = (1,1,1)$ and $\varphi((0,1)) = (-1,0,2)$.

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Then $\varphi((1,0)) = \varphi((1,3) - 3(0,1)) = \varphi((1,3)) - 3\varphi((0,1)) = (1,1,1) - 3(-1,0,2) = (4,1,-5)$.

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Therefore, $\varphi((x_1,x_2)) = \varphi(x_1\varepsilon_1 + x_2\varepsilon_2) = x_1(4,1,-5) + x_2(-1,0,2) = (4x_1 - x_2, x_1, -5x_1 + 2x_2)$.

Representation of Transformation by Matrices

Recall that by $M(m \times n; \mathbb{R})$ we denote the set of real matrices with m rows and n columns.

Definition

Let V, W be vector spaces and let $\mathcal{A} = (v_1, \ldots, v_n), \mathcal{B} = (w_1, \ldots, w_m)$ be their ordered bases, respectively. The **matrix of a linear transformation** $\varphi \colon V \longrightarrow W$ relative to the pair of ordered bases $\mathcal{A}$ and $\mathcal{B}$ is the matrix $M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = [a_{ij}] \in M(m \times n; \mathbb{R})$ given by the conditions $\varphi(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ for $j = 1, \ldots, n$.

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That is, columns of $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ consist of coefficients of the vectors $\varphi(v_1), \ldots, \varphi(v_n)$ relative to the basis $\mathcal{B}$.

Examples

Let $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by $\varphi((x_1, x_2, x_3)) = (2x_1 - x_2 + x_3, x_1 + x_3)$ and let $\mathcal{A} = ((1,0,1),(2,0,3),(0,1,1))$, $\mathcal{B} = ((1,1),(0,1))$ be the ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Then

$$\begin{aligned}
\varphi((1,0,1)) &= (3,2) = 3(1,1) - 1(0,1)\\
\varphi((2,0,3)) &= (7,5) = 7(1,1) - 2(0,1)\\
\varphi((0,1,1)) &= (0,1) = 0(1,1) + 1(0,1).
\end{aligned}$$

The matrix of φ relative to the (ordered) bases $\mathcal{A}, \mathcal{B}$ is

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 3 & 7 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Examples (continued)

Notation

By st we will denote the standard basis of $\mathbb{R}^n$*, i.e.*
$st = (\varepsilon_1, \ldots, \varepsilon_n)$.

For example, the matrix of the linear transformation given by $\varphi((x_1, x_2, x_3)) = (2x_1 - x_2 + 3x_3, x_1 + x_3)$ relative to the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^2$ is

$$M(\varphi)_{st}^{st} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

since

$$\begin{aligned}
\varphi(\varepsilon_1) &= \varphi((1,0,0)) &= (2,1) = 2(1,0) + 1(0,1) = &\quad 2\varepsilon_1 + 1\varepsilon_2 \\
\varphi(\varepsilon_2) &= \varphi((0,1,0)) &= (-1,0) = -1(1,0) + 0(0,1) = &-1\varepsilon_1 + 0\varepsilon_2 \\
\varphi(\varepsilon_3) &= \varphi((0,0,1)) &= (3,1) = 3(1,0) + 1(0,1) = &\quad 3\varepsilon_1 + 1\varepsilon_2.
\end{aligned}$$

Elementary Operations and Matrices of Linear Transformations

Proposition

Let V, W *be vector spaces and let* $\mathcal{A} = (v_1, \dots, v_n)$, $\mathcal{B} = (w_1, \dots, w_m)$ *be their ordered bases, respectively. Let* $A = [a_{ij}] = M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ *be the matrix of a linear transformation* $\varphi \colon V \longrightarrow W$ *relative to the bases* $\mathcal{A}$ *and* $\mathcal{B}$*. If* $\mathcal{A}' = (v_1 + v_2, v_2, \dots, v_n), \mathcal{A}'' = (\alpha v_1, v_2, \dots, v_n), \mathcal{B}' = (w_1 + w_2, w_2 \dots, w_m), \mathcal{B}'' = (\alpha w_1, w_2 \dots, w_m)$ *for some* $\alpha \neq 0$ *then*

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$$M(\varphi)_{\mathcal{A}'}^{\mathcal{B}} = \begin{bmatrix} a_{11} + a_{12} & a_{12} & \ldots & a_{1n} \\ a_{21} + a_{22} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + a_{m2} & a_{m2} & \ldots & a_{mn} \end{bmatrix},$$

Elementary Operations and Matrices of Linear Transformations (continued)

$$M(\varphi)^{\mathcal{B}}_{\mathcal{A}''} = \begin{bmatrix} \alpha a_{11} & a_{12} & \dots & a_{1n} \\ \alpha a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

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$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}'} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} - a_{11} & a_{22} - a_{12} & \dots & a_{2n} - a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

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$$M(\varphi)^{\mathcal{B}'}_{\mathcal{A}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} - a_{11} & a_{22} - a_{12} & \dots & a_{2n} - a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

$$M(\varphi)^{\mathcal{B}''}_{\mathcal{A}} = \begin{bmatrix} a_{11}/\alpha & a_{12}/\alpha & \dots & a_{1n}/\alpha \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Example

Let $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by the formula

$$\varphi((x_1, x_2, x_3)) = (3x_1 + 7x_2 + 4x_3, x_1 + 2x_2 + x_3).$$

Find bases $\mathcal{A}$ of $\mathbb{R}^3$ and $\mathcal{B}$ of $\mathbb{R}^2$ such that

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

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$$M(\varphi)_{st}^{st} = \begin{array}{c} \begin{array}{ccc} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{array} \\ \begin{bmatrix} 3 & 7 & 4 \\ 1 & 2 & 1 \end{bmatrix} \end{array} \xrightarrow[c_3 - c_1]{c_2 - 2c_1} \begin{array}{c} \begin{array}{ccc} \varepsilon_1 & -2\varepsilon_1 + \varepsilon_2 & -\varepsilon_1 + \varepsilon_3 \end{array} \\ \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow[2c_2]{c_1 - c_3}$$

$$\longrightarrow \begin{array}{c} \begin{array}{ccc} 2\varepsilon_1 - \varepsilon_3 & -4\varepsilon_1 + 2\varepsilon_2 & -\varepsilon_1 + \varepsilon_3 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}.$$

$$\mathcal{A} = ((2, 0, -1), (-4, 2, 0), (-1, 0, 1)), \quad \mathcal{B} = st.$$

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$$M(\varphi)_{\mathcal{A}}^{st} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Rank

Definition

Let $c_1, \ldots, c_n \in \mathbb{R}^m$ denote the vectors corresponding to columns of matrix $A \in M(m \times n; \mathbb{R})$. The dimension of the space $\text{lin}(c_1, \ldots, c_n) \subset \mathbb{R}^m$ will be called **rank** of the matrix A and denoted by $r(A)$.

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Proposition

Let $\varphi\colon V \longrightarrow W$ be a linear transformation and let $\mathcal{A}, \mathcal{B}$ are ordered bases of V and W, respectively. Then the rank of the matrix $M(\varphi)_{\mathcal{A}}^{\mathcal{B}}$ is equal to $\dim \varphi(V)$ and hence it does not depend on the bases $\mathcal{A}, \mathcal{B}$.

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Corollary

Elementary row operations on matrix A do not change its rank. Therefore, the rank of matrix A is equal to the rank of its (reduced) echelon form A', which is equal to the number of non-zero rows in A'.

Rank–nullity Theorem

Theorem

Let V, W be vector spaces. Let $\varphi \colon V \to W$ be a linear transformation. Assume V is finite–dimensional. Then

$$\dim \ker \varphi + \dim \varphi(V) = \dim V.$$

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Proof.

Without loss of generality one can assume $W = \varphi(V)$ is finite–dimensional and there exist bases $\mathcal{A} = (v_1, \ldots, v_n)$ of V and $\mathcal{B} = (w_1, \ldots, w_r)$ of W such that

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \left[\begin{array}{c|c} I_r & \mathbf{0} \end{array}\right],$$

where $r = r(A) = \dim \varphi(V)$. It follows that $\ker \varphi = \mathrm{lin}(v_{r+1}, \ldots, v_n)$ and $\dim \ker \varphi = n - r$. $\square$

Rank decomposition/factorisation

Proposition

For any matrix $A \in M(m \times n; \mathbb{R})$ *of rank* $r = r(A)$ *there exist matrices* $S \in M(m \times r; \mathbb{R})$ *and* $T \in M(r \times n; \mathbb{R})$ *such that* $r = r(A) = r(S) = r(T)$ *and*

$$A = ST.$$

Proof.

Let B be the reduced echelon form of A. Let S consist of columns of A (in the same order) in which there is a pivot in matrix B (this is exactly the basis $\mathcal{B}$ of the column space). Let T consists of non–zero r rows of matrix B. Each column of T contains coordinates of the corresponding column of matrix A relative to the basis $\mathcal{B}$. $\square$

Example

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 2 \\ 3 & 8 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns number 3 and 4 contain no pivot hence

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 8 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

Note that this decomposition is not unique as for any non-singular $G \in M(r \times r; \mathbb{R})$

$$A = (SG)(G^{-1}T).$$

Row rank is equal to column rank

Proposition

For any matrix $A \in M(m \times n; \mathbb{R})$

$$r(A^\intercal) = r(A).$$

Proof.

Let $A = ST$ be a rank decomposition. Then

$$A^\intercal = T^\intercal S^\intercal.$$

Columns of $A^\intercal$ are linear combinations of r columns of $T^\intercal$, hence, by Steinitz's Lemma,

$$r(A^\intercal) \leqslant r = r(A).$$

Replacing A with $A^\intercal$ in the same argument gives

$$r(A) \leqslant r(A^\intercal).$$

A vs $A^\intercal$ for square matrices

Proposition

For any $A \in M(n \times n; \mathbb{R})$ there exists an invertible matrix $C \in M(n \times n; \mathbb{R})$ such that

$$A = CA^\intercal C^{-1},$$

i.e., matrices A and $A^\intercal$ are similar.

Proof.

Omitted. Easy if you know Jordan decomposition. $\square$