

Linear Algebra

Lecture 3 - Linear Independence and Bases

Oskar Kędzierski

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Linearly (In)dependent Vectors

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Vectors $v_1, \dots, v_k \in V$ are said to be **linearly dependent** if there exist real numbers $\alpha_1, \dots, \alpha_k$, not all of which are 0 such that

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Linear independence does not depend on the order of vectors hence we may talk about independent (finite) sets. We assume that empty set is linearly independent.

Examples

- i) vectors $(1, 1, 2), (1, 1, 0), (2, 2, 1) \in \mathbb{R}^3$ are linearly dependent because $(1, 1, 2) + 3(1, 1, 0) - 2(2, 2, 1) = (0, 0, 0)$,
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- iv) vectors $(1, 2), (2, 4) \in \mathbb{R}^2$ are linearly dependent,
- v) vector $\varepsilon_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{R}^n$ with 1 at the i -th coordinate and 0 elsewhere is called **unit vector**. Vectors $\varepsilon_1, \dots, \varepsilon_n$ are linearly independent.

Properties

Proposition

Single vector $v \in V$ is linearly independent if and only if $v \neq \mathbf{0}$.

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A set of at least two vectors is linearly dependent if and only if one vector is a linear combination of the others.

Steinitz's (Exchange) Theorem

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If vectors $w_1, \dots, w_m \in \text{lin}(v_1, \dots, v_n)$ are linearly independent then $m \leq n$.

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If vectors $w_1, \dots, w_m \in \text{lin}(v_1, \dots, v_n)$ are linearly independent then $m \leq n$.

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For example, since $\mathbb{R}^n = \text{lin}(\varepsilon_1, \dots, \varepsilon_n)$ any independent set of vectors in \mathbb{R}^n has at most n elements.

Basis

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In general, a vector space can have many bases.

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- iii) vectors $\varepsilon_1, \dots, \varepsilon_n$ form a basis of \mathbb{R}^n . It is called the **standard basis**,
- iv) the set of solutions of a homogeneous system of linear equations is a vector space, its basis can be computed by substituting subsequently each free variable with 1 and the other free variables with 0's.

Example

Consider the following general solution of a homogeneous system

of linear equations:
$$\begin{cases} x_1 = 2x_2 + 4x_4 + x_5 \\ x_3 = - 3x_4 - x_5 \end{cases}$$

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$$\begin{cases} x_1 &= 2x_2 &+ 4x_4 &+ x_5 \\ x_3 &= &- 3x_4 &- x_5 \end{cases}$$

The free variables are x_2, x_4 and x_5 . By substituting $x_2 = 1, x_4 = x_5 = 0$ and then $x_4 = 1, x_2 = x_5 = 0$ and $x_5 = 1, x_2 = x_4 = 0$ we get three vectors $(2, 1, 0, 0, 0), (4, 0, -3, 1, 0), (1, 0, -1, 0, 1)$ which form a basis of the space of all solution.

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$$(2x_2 + 4x_4 + x_5, x_2, -3x_4 - x_5, x_4, x_5) = x_2(2, 1, 0, 0, 0) + x_4(4, 0, -3, 1, 0) + x_5(1, 0, -1, 0, 1), \quad x_2, x_4, x_5 \in \mathbb{R}.$$

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Definition

A vector space V is said to be n -dimensional if it has basis consisting of n vectors. We write $\dim V = n$ and say n is dimension of V . It is assumed that $\dim\{\mathbf{0}\} = 0$. A finite-dimensional vector space is a space of dimension $0, 1, 2, \dots$, otherwise it is infinite-dimensional and we write $\dim V = \infty$.

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- ii) if $V_U \subset \mathbb{R}^n$ is a subspace consisting of solutions of a homogeneous system of linear equations U then $\dim V_U =$ the number of free variables,
- iii) $\dim \mathbb{R}^\infty = \infty$ since it contains arbitrarily many independent vectors.

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Proposition

Let $v_1, \dots, v_{k+1} \in V$ and let v_1, \dots, v_k be linearly independent vectors. Then

v_1, \dots, v_{k+1} are linearly independent $\Leftrightarrow v_{k+1} \notin \text{lin}(v_1, \dots, v_k)$.

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Proof.

(\Leftarrow) Assume that $\alpha_1 v_1 + \dots + \alpha_{k+1} v_{k+1} = \mathbf{0}$. Then $\alpha_{k+1} = 0$, by the assumption. Vectors v_1, \dots, v_k are linearly independent hence $\alpha_1 = \dots = \alpha_k = 0$.

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(\Leftarrow) Assume that $\alpha_1 v_1 + \dots + \alpha_{k+1} v_{k+1} = \mathbf{0}$. Then $\alpha_{k+1} = 0$, by the assumption. Vectors v_1, \dots, v_k are linearly independent hence $\alpha_1 = \dots = \alpha_k = 0$. (\Rightarrow) if $v_{k+1} \in \text{lin}(v_1, \dots, v_k)$ then there exists a linear combination of v_1, \dots, v_k, v_{k+1} equal to $\mathbf{0}$ with non-zero coefficients. □

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- i) \Rightarrow ii) basis is a set spanning V , if removing say v_n , makes it a smaller set spanning V , then by the previous Proposition*
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iii) \Rightarrow i) it is enough to show that v_1, \dots, v_n span V , if they do not, by the previous Proposition, you could make it bigger contradicting maximality.

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- i) by the Steinitz's Theorem,
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- iii) as in ii),
- iv) by definition.

Subspaces of \mathbb{R}^n

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Let $V \subset \mathbb{R}^n$, $V \neq \{\mathbf{0}\}$ be a subspace. Then there exist linearly independent $v_1, \dots, v_k \in V$ such that $\text{lin}(v_1, \dots, v_k) = V$.

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Proof.

Since $V \neq \{\mathbf{0}\}$ there exists $v_1 \in V$, $v_1 \neq \mathbf{0}$ and $\text{lin}(v_1) \subset V$. Assume vectors $v_1, \dots, v_r \in V$ are linearly independent and $\text{lin}(v_1, \dots, v_r) \subsetneq V$.

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Corollary

Each subspace of \mathbb{R}^n has a basis.

Subspaces of \mathbb{R}^n (continued)

Proposition

Let $V = \text{lin}(v_1, \dots, v_k)$ and $V \neq \{\mathbf{0}\}$. Then there exist numbers $1 \leq j_1 < j_2 < \dots < j_m \leq k$ such that

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Proof.

Let $1 \leq j_1 \leq k$ be the smallest number such that $v_{j_1} \neq \mathbf{0}$.

Subspaces of \mathbb{R}^n (continued)

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Let $V = \text{lin}(v_1, \dots, v_k)$ and $V \neq \{\mathbf{0}\}$. Then there exist numbers $1 \leq j_1 < j_2 < \dots < j_m \leq k$ such that

i) $V = \text{lin}(v_{j_1}, v_{j_2}, \dots, v_{j_m}),$

ii) vectors $v_{j_1}, v_{j_2}, \dots, v_{j_m}$ are linearly independent,

i.e. $v_{j_1}, v_{j_2}, \dots, v_{j_m}$ is a basis of V .

Proof.

Let $1 \leq j_1 \leq k$ be the smallest number such that $v_{j_1} \neq \mathbf{0}$. Assume that there exist numbers $1 \leq j_1 < j_2 < \dots < j_m < n$ such that vectors $v_{j_1}, v_{j_2}, \dots, v_{j_m}$ are linearly independent and $v_i \in \text{lin}(v_{j_1}, \dots, v_{j_m})$ for $i \leq j_m$.

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$$v_i \in \text{lin}(v_{j_1}, \dots, v_{j_m}) \text{ for any } 1 \leq i \leq k,$$

i.e. $V = \text{lin}(v_{j_1}, v_{j_2}, \dots, v_{j_m})$ and the proof is finished.



Coordinates

Proposition

Vectors v_1, \dots, v_n form a basis of V if and only if any vector $v \in V$ can be uniquely written (up to the order of summands) as

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

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(\Rightarrow) basis spans the vector space V , hence any vector $v \in V$ is a linear combination of v_1, \dots, v_n . If $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ and $v = \beta_1 v_1 + \dots + \beta_n v_n$ then $\mathbf{0} = (\alpha_1 - \beta_1)v_1 + \dots + (\alpha_n - \beta_n)v_n$. This gives $\alpha_i = \beta_i$ for $i = 1, \dots, n$.

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(\Leftarrow) By assumption v_1, \dots, v_n span the vector space V . To prove they are linearly independent take $v = \mathbf{0}$. □

Coordinates (continued)

Definition

Let $\mathcal{B} = (v_1, \dots, v_n)$ be an ordered basis of V . If $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ the unique numbers $\alpha_1, \dots, \alpha_n$ are called the **coordinates** of v relative to the basis \mathcal{B} .

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For example, let $\mathcal{B} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\mathcal{B}' = (\varepsilon_2, \varepsilon_3, \varepsilon_1)$ and $\mathcal{B}'' = ((0, 0, 3), (0, 2, 0), (1, 0, 0))$ be three bases of \mathbb{R}^3 .

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$$(1, 2, 3) = 1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1),$$

$$(1, 2, 3) = 2(0, 1, 0) + 3(0, 0, 1) + 1(1, 0, 0),$$

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Linear Independence and Elementary Operations

Let V be a vector space.

Proposition

Assume that vectors $v_1, v_2, \dots, v_k \in V$ are linearly independent and $\alpha \in \mathbb{R} - \{0\}$. Then

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Proof.

Assume that v_1, \dots, v_k are linearly independent. The expression $\alpha_1(v_1 + v_2) + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_k v_k = \mathbf{0}$ can be rewritten as $\alpha_1 v_1 + (\alpha_1 + \alpha_2) v_2 + \alpha_3 v_3 + \dots + \alpha_k v_k = \mathbf{0}$. By assumption $\alpha_1 = \alpha_1 + \alpha_2 = \alpha_3 = \dots = \alpha_k = 0$ so $\alpha_i = 0$. The third case can be proven in a similar way. □

Linear Independence and Elementary Operations (continued)

Corollary

Let $v_1, \dots, v_n \in V$ and $\alpha \in \mathbb{R}$. The vectors v_1, \dots, v_n form a basis of V if and only if the vectors $v_1 + \alpha v_2, v_2, v_3, \dots, v_n$ form a basis of V .

Example

Find a basis of the subspace $V \subset \mathbb{R}^4$ given by

$$V = \text{lin}((1, 2, 1, 3), (2, 5, 6, 7), (4, 9, 8, 13)).$$

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$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 6 & 7 \\ 4 & 9 & 8 & 13 \end{bmatrix} &\xrightarrow[r_3 - 4r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} \longrightarrow \\ \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 1 \end{bmatrix} &\xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 0 & -7 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} \end{aligned}$$

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The vectors $(1, 0, -7, 1), (0, 1, 4, 1)$ are linearly independent. From the previous lecture it follows that

$$V = \text{lin}((1, 0, -7, 1), (0, 1, 4, 1)),$$

therefore $\mathcal{B} = ((1, 0, -7, 1), (0, 1, 4, 1))$ is a basis of the subspace V .

Example (continued)

Note that the vectors $(1, 2, 1, 3)$, $(2, 5, 6, 7)$, $(4, 9, 8, 13)$ **do not form a basis** of V since they are linearly dependent

$$2(1, 2, 1, 3) + (2, 5, 6, 7) - (4, 9, 8, 13) = (0, 0, 0, 0),$$

or equivalently, the reduced echelon form of a matrix with rows equal to vectors $(1, 2, 1, 3)$, $(2, 5, 6, 7)$, $(4, 9, 8, 13)$ has a zero row.

Row and Column Spaces

Definition

For any matrix $A \in M(m \times n; \mathbb{R})$, where $A = [a_{ij}]$, the **row space** of matrix A is the subspace of \mathbb{R}^n spanned by the **rows** of A , i.e.

$$\text{rowsp}(A) = \text{lin}((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, \\ \dots, (a_{m1}, a_{m2}, \dots, a_{mn})) \subset \mathbb{R}^n.$$

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The **column space** of matrix A is the subspace of \mathbb{R}^m spanned by the **columns** of A , i.e.

$$\text{colsp}(A) = \text{lin}((a_{11}, a_{21}, \dots, a_{m1}), (a_{12}, a_{22}, \dots, a_{m2}), \dots \\ \dots, (a_{1n}, a_{2n}, \dots, a_{mn})) \subset \mathbb{R}^m.$$

Null Space

Definition

The **null space** (or **nullspace**) of matrix A is the subspace $N(A) \subset \mathbb{R}^n$ equal to the set of solutions of a homogeneous system of linear equations given by A , i.e.

$$U: \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

$$N(A) = \{v \in \mathbb{R}^n \mid v \text{ is a solution of } U\}.$$

Row and Column Spaces, Null Space (continued)

Example

If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix},$$

then

$$\text{rowsp}(A) = \text{lin}((1, 2, 3), (3, 5, 7)) \subset \mathbb{R}^3,$$

$$\text{colsp}(A) = \text{lin}((1, 3), (2, 5), (3, 7)) \subset \mathbb{R}^2,$$

$$N(A) = \text{lin}((1, -2, 1)) \subset \mathbb{R}^3.$$

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That is, one can extend linearly independent vectors $w_1, \dots, w_m \in V$ to a basis of V using vectors spanning it.

Proof.

Without loss of generality one can assume that vectors v_1, \dots, v_n are linearly independent (by removing some of them). Assume that w_1, \dots, w_m are linearly independent and $m > n$. Let $a_{ij} \in \mathbb{R}$ be the numbers given by conditions

$$w_i = a_{i1}v_1 + a_{i2}v_2 + \dots + a_{in}v_n \text{ for } i = 1, \dots, m.$$

Let $A = [a_{ij}]$ for $i = 1, \dots, m, j = 1, \dots, n$ be an m -by- n matrix.

Proof of Steinitz's (Exchange) Theorem (continued)

Proof.

Elementary row operations on A correspond to elementary operations on vectors w_1, \dots, w_m .

Proof of Steinitz's (Exchange) Theorem (continued)

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Elementary row operations on A correspond to elementary operations on vectors w_1, \dots, w_m . Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A .

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Proof.

Elementary row operations on A correspond to elementary operations on vectors w_1, \dots, w_m . Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A . Matrix B has no zero row since that would contradict the assumption that w_1, \dots, w_m are linearly independent.

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Elementary row operations on A correspond to elementary operations on vectors w_1, \dots, w_m . Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A . Matrix B has no zero row since that would contradict the assumption that w_1, \dots, w_m are linearly independent. If $m > n$ then there are more rows than columns in B so it has a zero row, therefore $m \leq n$. Let

$1 \leq j_1 < j_2 < \dots < j_{n-m} \leq n$ be the numbers of columns in B **without** pivots. If we extend matrix B by rows $\varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{n-m}}$ to a square matrix then its reduced echelon form is equal to

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \text{ i.e. elementary row operations on}$$

$w_1, \dots, w_m, v_{j_1}, \dots, v_{j_{n-m}}$ lead to v_1, \dots, v_n hence both are bases of V . □

Bases and Elementary Operations

Proposition

If v_1, \dots, v_k and w_1, \dots, w_k are two bases of vector space V then one can be obtained by a sequence of elementary operations on the other.

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Proof.

Let $A \in M(n \times n; \mathbb{R})$ be the square matrix as in the above proof. Then its reduced echelon form has no zero rows so it is equal to I_n . □

Sum of Subspaces

Definition

Let $V, W \subset \mathbb{R}^n$ be subspaces. The set

$$V + W = \{v + w \in \mathbb{R}^n \mid v \in V, w \in W\} \subset \mathbb{R}^n,$$

is a subspace of \mathbb{R}^n called **the sum of V and W** .

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Proof.

Any subspace containing V and W contains, by definition, $V + W$.



Dimension of Sum of Subspaces

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Proof.

Let $u_1, \dots, u_r \in \mathbb{R}^n$ be a basis of $V \cap W$ which by (the Steinitz's theorem) can be extended by some vectors $v_1, \dots, v_s \in \mathbb{R}^n$ to a basis of V and by some vectors $w_1, \dots, w_t \in \mathbb{R}^n$ to a basis of W , i.e.

$$\dim V \cap W = r, \quad \dim V = r + s, \quad \dim W = r + t.$$

Obviously

$$\text{lin}(u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t) = V + W.$$

Dimension of Sum of Subspaces (continued)

Proof.

It is enough to show that vectors $u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t$ are linearly independent.

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Assume

$$\sum_{i=1}^r \alpha_i u_i + \sum_{i=1}^s \beta_i v_i + \sum_{i=1}^t \gamma_i w_i = \mathbf{0}.$$

Then

$$\sum_{i=1}^r \alpha_i u_i + \sum_{i=1}^s \beta_i v_i = - \sum_{i=1}^t \gamma_i w_i \in V \cap W,$$

i.e. it is equal to $\sum_{i=1}^r \alpha'_i u_i$ for some $\alpha'_i \in \mathbb{R}$, which implies $\beta_1 = \dots = \beta_s = 0$.

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Dimension of Sum of Subspaces (continued)

Proof.

Equivalently, consider linear transformation

$$s: V \times W \ni (v, w) \mapsto v + w \in \mathbb{R}^n.$$

Then

$$\ker s \cong V \cap W, \quad \operatorname{im} s = V + W,$$

hence

$$\begin{aligned} \dim(V \times W) &= \dim V + \dim W = \dim \ker s + \dim \operatorname{im} s = \\ &= \dim V \cap W + \dim(V + W). \end{aligned}$$



Direct Sum

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Let $V, W \subset \mathbb{R}^n$ be subspaces. The space \mathbb{R}^n is a **direct sum** of the subspaces V and W if

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Remark

If \mathbb{R}^n is a direct sum of V and W it is denoted by

$$V \oplus W = \mathbb{R}^n.$$

It follows that

$$\dim V + \dim W = n.$$

Direct Sum (continued)

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Proof.

Exercise.



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and if $A \in \text{Sym}(n \times n; \mathbb{R}) \cap \text{Skew}(n \times n; \mathbb{R})$ then

$$A = -A^T = -A \Rightarrow 2A = \mathbf{0}.$$

Direct Sum (continued)

Proposition

Let $V_1, \dots, V_n \subset V$ be subspaces of a vector space V . The following conditions are equivalent

- a) *for any $v \in V$ there exist unique vectors $v_1 \in V_1, \dots, v_n \in V_n$ such that*

$$v = v_1 + \dots + v_n.$$

- b) *the following conditions hold*

- i) $V_1 + \dots + V_n = V$,
- ii) $V_i \cap (V_1 + \dots + V_{i-1} + V_{i+1} + \dots + V_n) = \{\mathbf{0}\}$, for $i = 1, \dots, n$.

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If the above holds we say that V is a direct sum of its subspaces V_1, \dots, V_n and write

$$V = V_1 \oplus \dots \oplus V_n.$$

Direct Sum (continued)

Proof.

(\Rightarrow) exercise

Direct Sum (continued)

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(\Rightarrow) exercise

(\Leftarrow) it is enough to prove uniqueness. Fix $i \in \{1, \dots, n\}$. If

$$v = v_1 + \dots + v_n,$$

$$v = v'_1 + \dots + v'_n,$$

where $v_j, v'_j \in V$ for $j = 1, \dots, n$, then

$$v_i - v'_i = (v'_1 - v_1) + \dots + (v'_{i-1} - v_{i-1}) + (v'_{i+1} - v_{i+1}) + \dots + (v'_n - v_n),$$

therefore

$$v_i - v'_i \in V_i \cap (V_1 + \dots + V_{i-1} + V_{i+1} + \dots + V_n) = \{\mathbf{0}\},$$

hence

$$v_i = v'_i.$$

Matroid

Definition

Matroid M is an ordered pair (E, \mathcal{I}) consisting of a **finite** set E and a family \mathcal{I} of subsets of E such that

- i) $\emptyset \in \mathcal{I}$,
- ii) if $A \subset B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,
- iii) if $A, B \in \mathcal{I}$ and $|A| < |B|$ (i.e. there are less elements in A) then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

Elements of E are called **points**, the set E is called the **ground set**, sets $A \in \mathcal{I}$ are called **independent** sets and sets $A \notin \mathcal{I}$ are called **dependent** sets.

Example

Let $E = \{1, \dots, n\}$ and for some $m \geq 0$ let

$$\mathcal{I} = \{A \subset E \mid |A| \leq m\}.$$

This is the **uniform matroid** $U_{m,n}$.

Representable Matroid

Proposition

Let $A \in M(m \times n; \mathbb{R})$ be a matrix. Let $E = \{1, \dots, n\}$ and

$\mathcal{I} = \{J \subset E \mid \text{columns of } A \text{ indexed by } J \text{ are linearly independent}\}.$

Then (E, \mathcal{I}) is a matroid.

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Then (E, \mathcal{I}) is a matroid.

Proof.

Let $A = \{w_1, \dots, w_k\}$, $B = \{v_1, \dots, v_l\}$, where $w_i, v_i \in \mathbb{R}^m$ and $l > k$. By the Steinitz's Exchange Lemma applied to

$$w_1, \dots, w_k \in \text{lin}(w_1, \dots, w_k, v_1, \dots, v_l) = V,$$

linearly independent vectors w_1, \dots, w_k can be extended to a basis of V , by some vectors v_1, \dots, v_l where $\dim V \geq l$. □

Representable Matroid

Definition

Any matroid isomorphic (i.e. there exists a bijection of ground sets preserving independent sets) is called to be **representable** (or linear) over \mathbb{R} .

Remark

There exist matroids which are not representable over any field (see Vámos matroid or non-Pappus matroid). Therefore the notion of matroid generalizes the abstract notion of linear independence.

Bases and Circuits

Let $M = (E, \mathcal{I})$ be a matroid.

Definition

A set $B \in \mathcal{I}$ such that if $B \subset B', B' \in \mathcal{I}$ then $B = B'$ (that is maximal independent set) is called a **basis** of M . A set $C \notin \mathcal{I}$ such that if $C' \subset C$ then $C' \in \mathcal{I}$ (that is minimal dependent set) is called a **circuit** of M .

Proposition

If B, B' are bases of matroid M then $|B| = |B'|$.

Remark

In a representable matroid given by matrix A basis B correspond to columns which are basis of $\text{colsp}(A)$. In the uniform matroid $U_{m,n}$ basis is any subset of cardinality m .

Bases and Circuits (continued)

Proposition

Let $C, C' \subset E$ be two different circuits of the matroid $M = (E, \mathcal{I})$ and let $e \in C \cap C'$. Then there exists a circuit $C'' \subset (C \cup C') \setminus \{e\}$.

Bases and Circuits (continued)

Proof.

Assume on the contrary $(C \cup C') \setminus \{e\}$ does not contain any circuit, that is all its subsets are independent, in particular

$(C \cup C') \setminus \{e\} \in \mathcal{I}$. It is impossible that $C' \subset C$ so choose some $f \in C' \setminus C$. It follows that $C' \setminus \{f\} \in \mathcal{I}$ is independent. Consider a family

$$\{A \subset C \cup C' \mid C' \setminus \{f\} \subset A \text{ and } A \in \mathcal{I}\}.$$

This family is non-empty and finite so it contains a maximal element I . Obviously $f \notin I$ (otherwise $C' \subset I$ but I is independent and C' is not). It is impossible that $C \subset I$ so choose some $g \in C \setminus I$. Therefore $f, g \notin I$ but $f \notin C$ and $g \in C$ hence $f \neq g$ and

$$|I| \leq |(C \cup C') \setminus \{f, g\}| = |C \cup C'| - 2 < |(C \cup C') \setminus \{e\}|.$$

By the condition iii) I can be extended to an independent set by some element of $(C \cup C') \setminus \{e\}$ which contradicts its maximality. \square

Bases and Circuits (continued)

Proposition

Let $I \in \mathcal{I}$ and let $e \in E$ such that $I \cup \{e\} \notin \mathcal{I}$. Then there exists a unique circuit C such that $C \subset I \cup \{e\}$.

Proof.

It is clear that such circuit C exists and that for any such circuit $e \in C$. Assume that there are two circuits $C, C' \subset I \cup \{e\}$. Then $e \in C \cap C'$ and there exists a circuit $C'' \subset (C \cup C') \setminus \{e\}$ which is not possible as $C'' \not\subset I$.



Remark

*If $I = B$ is a basis then it is enough that $e \in E \setminus B$. In such case the unique C is called the **fundamental circuit** of e with respect to B .*

Greedy Algorithm

Assume there is a function $w: E \rightarrow \mathbb{R}$ on the ground set of some matroid $M = (E, \mathcal{I})$. For any $A \subset E$ let $w(A) = \sum_{e \in A} w(e)$. The following algorithm is called the **Greedy Algorithm**.

i) set $A = \emptyset$,

ii) let

$$F = \{e \in E \setminus A \mid A \cup \{e\} \in \mathcal{I}\},$$

if $F = \emptyset$ then STOP,

iii) choose $e \in F$ with maximal weight $w(e)$, assign $A \leftarrow A \cup \{e\}$ and go to step ii)

That is, at each step choose point e with maximal weight which preserves independence of A .

Greedy Algorithm (continued)

Proposition

The greedy algorithm returns a basis B of M with maximal weight $w(B)$, i.e. $w(B) \geq w(B')$ for any other basis B' .

Proof.

Say $B = \{e_1, \dots, e_r\}$ and $B' = \{f_1, \dots, f_r\}$ where $B = A$ was returned by the greedy algorithm and $w(f_i) \geq w(f_{i+1})$. Then $w(e_i) \geq w(f_i)$ for any i (which proves the statement). Assume otherwise there exists the least $k \geq 2$ such that $w(e_k) < w(f_k)$. Let $I_1 = \{e_1, \dots, e_{k-1}\}$, $I_2 = \{f_1, \dots, f_k\}$. There exists $f_j \in I_2 \setminus I_1$ such that $I_1 \cup \{f_j\} \in \mathcal{I}$. But $w(f_j) \geq w(f_k) > w(e_k)$. But then at the k -iteration the greedy algorithm would choose f_j instead of e_k (note that $f_k \notin I_1$). □

Greedy Algorithm (continued)

Proposition

Let \mathcal{I} be a family of subsets of a finite set E such that

- i) $\emptyset \in \mathcal{I}$,
- ii) if $A \subset B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,
- iii) for any weight function $w: E \rightarrow \mathbb{R}$ yields a maximal element of \mathcal{I} of maximal weight (among other maximal elements).

Then $M = (E, \mathcal{I})$ is a matroid¹.

Proof.

It is enough to check condition iii). Let $A, B \in \mathcal{I}$ and let $|A| < |B|$. It follows that

$$|A \setminus B| < |B \setminus A|.$$

Assume that for each $e \in B \setminus A$ the set $A \cup \{e\} \notin \mathcal{I}$ is dependent.

¹based on J. Oxley *Matroid Theory*

Greedy Algorithm (continued)

Proof.

Fix $\varepsilon > 0$ such that

$$\frac{|A \setminus B|}{|B \setminus A|} < \varepsilon < 1,$$

and weight function

$$w(e) = \begin{cases} 1 & e \in A \\ \varepsilon & e \in B \setminus A \\ 0 & \text{otherwise} \end{cases}$$

The greedy algorithm picks all elements from A and then, by the assumption, some elements of weight 0 so it yields the total weight equal to $w(A)$. Choose maximal independent set B' such that $B' \supset B$. Then

$$\begin{aligned} w(B') &\geq w(B) = |A \cap B| + \varepsilon |B \setminus A| > |A \cap B| + \frac{|A \setminus B|}{|B \setminus A|} |B \setminus A| = \\ &= |A \cap B| + |A \setminus B| = w(A). \end{aligned}$$

Rank and Span

Let $M = (E, \mathcal{I})$ be a matroid.

Definition

For any $B \subset E$ let

$$r(B) = \max\{|A| \mid A \subset B, A \in \mathcal{I}\},$$

be the **rank** of a subset B , i.e. the largest size of an independent subset.

For any $B \subset E$ let

$$\text{span}(B) = \{e \in E \mid r(B \cup \{e\}) = r(B)\},$$

be the **span** (or **closure**) of the set B .

Remark

In a representable matroid rank is equal to the dimension of the column space (columns indexed by B) and the span consists of all columns contained in the column space given by B .

Rank and Span (continued)

Proposition

For any $B \subset E$

$$r(\text{span}(B)) = r(B).$$

Proof.

Obviously $B \subset \text{span}(B)$ hence $r(B) \leq r(\text{span}(B))$ (independent subset of B are subsets of $\text{span}(B)$). Assume the inequality is strict, i.e. there exists independent set $A \subset \text{span}(B)$ such that $r(B) < |A|$. Let $A' \subset B$ be the independent subset of B such that $r(B) = |A'|$. By the condition iii) there exists $e \in A \setminus A'$ such that $B \cup \{e\} \in \mathcal{I}$. This contradicts that $e \in \text{span}(B)$ (adding e to B raises its rank). □

Spanning Set

Definition

A set $B \subset E$ is **spanning** if $r(B) = r(E)$.

Remark

Obviously $r(B) \leq r(E)$ for any $B \subset E$ and $r(E)$ is equal to a cardinality of any basis. Hence B is spanning if and only if it contains a basis.

Basis

Proposition

Let $B \subset E$, where $M = (E, \mathcal{I})$ is a matroid. The following conditions are equivalent

- i) B is a basis,
- ii) B is maximal independent set,
- iii) B is spanning independent set,
- iv) B is minimal spanning set.

Proof.

ii) is definition of i). ii) \Rightarrow iii) condition $r(B) < r(E)$ is impossible as any two bases are of the same cardinality, iii) \Rightarrow iv) any subset $B' \subset B, B' \neq B$ is independent hence $r(B') = |B'| < |B| = r(B) = r(E)$ so it cannot be spanning. iv) \Rightarrow i) B is spanning so it contains a basis B' but any basis is spanning so $B' = B$. □