Linear Algebra

Lecture 3 - Linear Independence and Bases

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Linearly (In)dependent Vectors

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Linear independence does not depend on the order of vectors hence we may talk about independent (finite) sets. We assume that empty set is linearly independent.

Examples

i) vectors $(1,1,2),(1,1,0),(2,2,1) \in \mathbb{R}^3$ are linearly dependent because $(1,1,2)+3(1,1,0)-2(2,2,1)=(0,0,0)$,

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v) vector $\varepsilon_i = (0,\dots,0,\overset{i}{\check{1}},0,\dots,0) \in \mathbb{R}^n$ with 1 at the i-th coordinate and 0 elsewhere is called **unit vector**. Vectors $\varepsilon_1,\dots,\varepsilon_n$ are linearly independent.

Properties

Proposition

Single vector $v \in V$ *is linearly independent if and only if* $v \neq \mathbf{0}$.

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A set of at least two vectors is linearly dependent if and only if one vector is a linear combination of the others.

Steinitz's (Exchange) Theorem

Theorem (Steinitz's Theorem)

If vectors $w_1, \ldots, w_m \in \mathrm{lin}(v_1, \ldots, v_n)$ *are linearly independent then* $m \leqslant n$*.*

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For example, since $\mathbb{R}^n = \mathrm{lin}(\varepsilon_1, \ldots, \varepsilon_n)$ any independent set of vectors in $\mathbb{R}^n$ has at most n elements.

Basis

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In general, a vector space can have many bases.

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iii) vectors $\varepsilon_1, \ldots, \varepsilon_n$ form a basis of $\mathbb{R}^n$. It is called the **standard basis**,

iv) the set of solutions of a homogeneous system of linear equations is a vector space, its basis can be computed by substituting subsequently each free variable with 1 and the other free variables with $0'$s.

Example

Consider the following general solution of a homogeneous system of linear equations: $\begin{cases} x_1 = 2x_2 + 4x_4 + x_5 \\ x_3 = - 3x_4 - x_5 \end{cases}$

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The free variables are x_2, x_4 and x_5. By substituting $x_2 = 1, x_4 = x_5 = 0$ and then $x_4 = 1, x_2 = x_5 = 0$ and $x_5 = 1, x_2 = x_4 = 0$ we get three vectors $(2, 1, 0, 0, 0), (4, 0, -3, 1, 0), (1, 0, -1, 0, 1)$ which form a basis of the space of all solution.

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$$(2x_2 + 4x_4 + x_5, x_2, -3x_4 - x_5, x_4, x_5) =$$
$$x_2(2, 1, 0, 0, 0) + x_4(4, 0, -3, 1, 0) + x_5(1, 0, -1, 0, 1), \; x_2, x_4, x_5 \in \mathbb{R}.$$

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Definition

A vector space V is said to be n-dimensional if it has basis consisting of n vectors. We write $\dim V = n$ and say n is dimension of V. It is assumed that $\dim\{\mathbf{0}\} = 0$. A finite-dimensional vector space is a space of dimension $0, 1, 2 \ldots$, otherwise it is infinite-dimensional and we write $\dim V = \infty$.

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ii) if $V_U \subset \mathbb{R}^n$ is a subspace consisting of solutions of a homogeneous system of linear equations U then $\dim V_U =$the number of free variables,

iii) $\dim \mathbb{R}^\infty = \infty$ since it contains arbitrarily many independent vectors.

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Let $v_1, \ldots, v_{k+1} \in V$ *and let* $v_1, \ldots, v_k$ *be linearly independent vectors. Then*

$$v_1, \ldots, v_{k+1} \text{ are linearly independent} \Leftrightarrow v_{k+1} \notin \operatorname{lin}(v_1, \ldots, v_k).$$

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Proof.

($\Leftarrow$) Assume that $\alpha_1 v_1 + \ldots + \alpha_{k+1} v_{k+1} = \mathbf{0}$. Then $\alpha_{k+1} = 0$, by the assumption. Vectors $v_1, \ldots, v_k$ are linearly independent hence $\alpha_1 = \ldots = \alpha_k = 0$.

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$iii) \Rightarrow i)$ it is enough to show that $v_1, \ldots, v_n$ span V, if they do not, by the previous Proposition, you could make it bigger contradicting maximality.

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iv) by definition.

Subspaces of $\mathbb{R}^n$

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Let $V \subset \mathbb{R}^n, V \neq \{\mathbf{0}\}$ *be a subspace. Then there exist linearly independent* $v_1, \ldots, v_k \in V$ *such that* $\operatorname{lin}(v_1, \ldots, v_k) = V$.

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Since $V \neq \{\mathbf{0}\}$ there exists $v_1 \in V, v_1 \neq \mathbf{0}$ and $\operatorname{lin}(v_1) \subset V$.
Assume vectors $v_1, \ldots, v_r \in V$ are linearly independent and $\operatorname{lin}(v_1, \ldots, v_r) \subsetneq V$.

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Corollary

Each subspace of $\mathbb{R}^n$ has a basis.

Subspaces of $\mathbb{R}^n$ (continued)

Proposition

Let $V = \text{lin}(v_1, \ldots, v_k)$ *and* $V \neq \{\mathbf{0}\}$. *Then there exist numbers* $1 \leqslant j_1 < j_2 < \ldots < j_m \leqslant k$ *such that*

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Proof.

Let $1 \leqslant j_1 \leqslant k$ be the smallest number such that $v_{j_1} \neq \mathbf{0}$.

Subspaces of $\mathbb{R}^n$ (continued)

Proposition

Let $V = \mathrm{lin}(v_1, \ldots, v_k)$ *and* $V \neq \{\mathbf{0}\}$. *Then there exist numbers* $1 \leqslant j_1 < j_2 < \ldots < j_m \leqslant k$ *such that*

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Proof.

Let $1 \leqslant j_1 \leqslant k$ be the smallest number such that $v_{j_1} \neq \mathbf{0}$. Assume that there exist numbers $1 \leqslant j_1 < j_2 < \ldots < j_m < n$ such that vectors $v_{j_1}, v_{j_2}, \ldots, v_{j_m}$ are linearly independent and $v_i \in \mathrm{lin}(v_{j_1}, \ldots, v_{j_m})$ for $i \leqslant j_m$.

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$$v_i \in \text{lin}(v_{j_1}, \ldots, v_{j_m}) \text{ for any } 1 \leqslant i \leqslant k,$$

i.e. $V = \text{lin}(v_{j_1}, v_{j_2}, \ldots, v_{j_m})$ and the proof is finished. □

Coordinates

Proposition

Vectors $v_1, \ldots, v_n$ form a basis of V if and only if any vector $v \in V$ can be uniquely written (up to the order of summands) as $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$.

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$(\Rightarrow)$ basis spans the vector space V, hence any vector $v \in V$ is a linear combination of $v_1, \ldots, v_n$. If $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and $v = \beta_1 v_1 + \ldots + \beta_n v_n$ then $\mathbf{0} = (\alpha_1 - \beta_1) v_1 + \ldots + (\alpha_n - \beta_n) v_n$. This gives $\alpha_i = \beta_i$ for $i = 1, \ldots, n$.

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($\Leftarrow$) By assumption $v_1, \ldots, v_n$ span the vector space V. To prove they are linearly independent take $v = \mathbf{0}$. $\square$

Coordinates (continued)

Definition

Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis of V. If $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ the unique numbers $\alpha_1, \ldots, \alpha_n$ are called the **coordinates** of v relative to the basis $\mathcal{B}$.

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For example, let $\mathcal{B} = (\varepsilon_1, \varepsilon_2, \varepsilon_3), \mathcal{B}' = (\varepsilon_2, \varepsilon_3, \varepsilon_1)$ and $\mathcal{B}'' = ((0,0,3),(0,2,0),(1,0,0))$ be three bases of $\mathbb{R}^3$.

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$$(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1),$$

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Linear Independence and Elementary Operations

Let V be a vector space.

Proposition

Assume that vectors $v_1, v_2, \ldots, v_k \in V$ *are linearly independent and* $\alpha \in \mathbb{R} - \{0\}$. *Then*

i) *vectors* $v_2, v_1, v_3, \ldots, v_k$ *are linearly independent,*

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Proof.

Assume that $v_1, \ldots, v_k$ are linearly independent. The expression $\alpha_1(v_1 + v_2) + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_k v_k = \mathbf{0}$ can be rewritten as $\alpha_1 v_1 + (\alpha_1 + \alpha_2) v_2 + \alpha_3 v_3 + \ldots + \alpha_k v_k = \mathbf{0}$. By assumption $\alpha_1 = \alpha_1 + \alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$ so $\alpha_i = 0$. The third case can be proven in a similar way. $\square$

Linear Independence and Elementary Operations (continued)

Corollary

Let $v_1, \ldots, v_n \in V$ *and* $\alpha \in \mathbb{R}$*. The vectors* $v_1, \ldots, v_n$ *form a basis of* V *if and only if the vectors* $v_1 + \alpha v_2, v_2, v_3, \ldots, v_n$ *form a basis of* V*.*

Example

Find a basis of the subspace $V \subset \mathbb{R}^4$ given by

$$V = \text{lin}((1,2,1,3),(2,5,6,7),(4,9,8,13)).$$

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The vectors $(1,0,-7,1),(0,1,4,1)$ are linearly independent. From the previous lecture it follows that

$$V = \mathrm{lin}((1,0,-7,1),(0,1,4,1)),$$

therefore $\mathcal{B} = ((1,0,-7,1),(0,1,4,1))$ is a basis of the subspace V.

Example (continued)

Note that the vectors $(1, 2, 1, 3), (2, 5, 6, 7), (4, 9, 8, 13)$ **do not form a basis** of V since they are linearly dependent

$$2(1, 2, 1, 3) + (2, 5, 6, 7) - (4, 9, 8, 13) = (0, 0, 0, 0),$$

or equivalently, the reduced echelon form of a matrix with rows equal to vectors $(1, 2, 1, 3), (2, 5, 6, 7), (4, 9, 8, 13)$ has a zero row.

Row and Column Spaces

Definition

For any matrix $A \in M(m \times n; \mathbb{R})$, where $A = [a_{ij}]$, the **row space** of matrix A is the subspace of $\mathbb{R}^n$ spanned by the **rows** of A, i.e.

$$\operatorname{rowsp}(A) = \operatorname{lin}\left((a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, \right.$$

$$\left. \ldots, (a_{m1}, a_{m2}, \ldots, a_{mn})\right) \subset \mathbb{R}^n.$$

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The **column space** of matrix A is the subspace of $\mathbb{R}^m$ spanned by the **columns** of A, i.e.

$$\operatorname{colsp}(A) = \operatorname{lin}\left((a_{11}, a_{21}, \ldots, a_{m1}), (a_{12}, a_{22}, \ldots, a_{m2}), \ldots\right.$$

$$\left.\ldots, (a_{1n}, a_{2n}, \ldots, a_{mn})\right) \subset \mathbb{R}^m.$$

Null Space

Definition

The **null space** (or **nullspace**) of matrix A is the subspace $\mathrm{N}(A) \subset \mathbb{R}^n$ equal to the set of solutions of a homogeneous system of linear equations given by A, i.e.

$$U\colon \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = 0 \end{cases}$$

$$N(A) = \{v \in \mathbb{R}^n \mid v \text{ is a solution of } U\}.$$

Row and Column Spaces, Null Space (continued)

Example

If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix},$$

then

$$\mathrm{rowsp}(A) = \mathrm{lin}((1,2,3),(3,5,7)) \subset \mathbb{R}^3,$$

$$\mathrm{colsp}(A) = \mathrm{lin}((1,3),(2,5),(3,7)) \subset \mathbb{R}^2,$$

$$\mathrm{N}(A) = \mathrm{lin}((1,-2,1)) \subset \mathbb{R}^3.$$

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Assume that vectors $w_1, \ldots, w_m \in \mathrm{lin}(v_1, \ldots, v_n) = V$ *are linearly independent.*

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That is, one can extend linearly independent vectors $w_1, \ldots, w_m \in V$ *to a basis of* V *using vectors spanning it.*

Proof.

Without loss of generality one can assume that vectors $v_1, \ldots, v_n$ are linearly independent (by removing some of them). Assume that $w_1, \ldots, w_m$ are linearly independent and $m > n$. Let $a_{ij} \in \mathbb{R}$ be the numbers given by conditions

$$w_i = a_{i1}v_1 + a_{i2}v_2 + \ldots + a_{in}v_n \text{ for } i = 1, \ldots, m.$$

Let $A = [a_{ij}]$ for $i = 1, \ldots, m, j = 1, \ldots, n$ be an m-by-n matrix.

Proof of Steinitz's (Exchange) Theorem (continued)

Proof.

Elementary row operations on A correspond to elementary operations on vectors $w_1, \ldots, w_m$.

Proof of Steinitz's (Exchange) Theorem (continued)

Proof.

Elementary row operations on A correspond to elementary operations on vectors $w_1, \ldots, w_m$. Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A.

Proof of Steinitz's (Exchange) Theorem (continued)

Proof.

Elementary row operations on A correspond to elementary operations on vectors $w_1, \ldots, w_m$. Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A. Matrix B has no zero row since that would contradict the assumption that $w_1, \ldots, w_m$ are linearly independent.

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Proof of Steinitz's (Exchange) Theorem (continued)

Proof.

Elementary row operations on A correspond to elementary operations on vectors $w_1, \ldots, w_m$. Let $B = [b_{ij}] \in M(m \times n; \mathbb{R})$ be the reduced echelon form of A. Matrix B has no zero row since that would contradict the assumption that $w_1, \ldots, w_m$ are linearly independent. If $m > n$ then there are more rows than columns in B so it has a zero row, therefore $m \leqslant n$. Let $1 \leqslant j_1 < j_2 < \ldots < j_{n-m} \leqslant n$ be the numbers of columns in B **without** pivots. If we extend matrix B by rows $\varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_{n-m}}$ to a square matrix then its reduced echelon form is equal to $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, i.e. elementary row operations on $w_1, \ldots, w_m, v_{j_1}, \ldots, v_{j_{n-m}}$ lead to $v_1, \ldots, v_n$ hence both are bases of V. □

Bases and Elementary Operations

Proposition

If $v_1, \ldots, v_k$ *and* $w_1, \ldots, w_k$ *are two bases of vector space* V *then one can be obtained by a sequence of elementary operations on the other.*

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Proposition

If $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are two bases of vector space V then one can be obtained by a sequence of elementary operations on the other.

Proof.

Let $A \in M(n \times n; \mathbb{R})$ be the square matrix as in the above proof. Then its reduced echelon form has no zero rows so it is equal to I_n. $\square$

Sum of Subspaces

Definition

Let $V, W \subset \mathbb{R}^n$ be subspaces. The set

$$V + W = \{v + w \in \mathbb{R}^n \mid v \in V,\ w \in W\} \subset \mathbb{R}^n,$$

is a subspace of $\mathbb{R}^n$ called **the sum of** V and W.

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The subspace $V + W$ is the smallest subspace of $\mathbb{R}^n$ containing subspace V and subspace W.

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Proof.

Any subspace containing V and W contains, by definition, $V + W$. $\square$

Dimension of Sum of Subspaces

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Let $V, W \subset \mathbb{R}^n$ *be subspaces. Then*

$$\dim(V + W) = \dim V + \dim W - \dim V \cap W.$$

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Proof.

Let $u_1, \ldots, u_r \in \mathbb{R}^n$ be a basis of $V \cap W$ which by (the Steinitz's theorem) can be extended by some vectors $v_1, \ldots, v_s \in \mathbb{R}^n$ to a basis of V and by some vectors $w_1, \ldots, w_t \in \mathbb{R}^n$ to a basis of W, i.e.

$$\dim V \cap W = r, \quad \dim V = r + s, \quad \dim W = r + t.$$

Obviously

$$\operatorname{lin}(u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t) = V + W.$$

Dimension of Sum of Subspaces (continued)

Proof.

It is enough to show that vectors $u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t$ are linearly independent.

Dimension of Sum of Subspaces (continued)

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It is enough to show that vectors $u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t$ are linearly independent.

Assume

$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{i=1}^{s} \beta_i v_i + \sum_{i=1}^{t} \gamma_i w_i = \mathbf{0}.$$

Then

$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{i=1}^{s} \beta_i v_i = -\sum_{i=1}^{t} \gamma_i w_i \in V \cap W,$$

i.e. it is equal to $\sum_{i=1}^{r} \alpha_i' u_i$ for some $\alpha_i' \in \mathbb{R}$, which implies $\beta_1 = \ldots = \beta_s = 0$.

Dimension of Sum of Subspaces (continued)

Proof.

It is enough to show that vectors $u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t$ are linearly independent.

Assume

$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{i=1}^{s} \beta_i v_i + \sum_{i=1}^{t} \gamma_i w_i = \mathbf{0}.$$

Then

$$\sum_{i=1}^{r} \alpha_i u_i + \sum_{i=1}^{s} \beta_i v_i = -\sum_{i=1}^{t} \gamma_i w_i \in V \cap W,$$

i.e. it is equal to $\sum_{i=1}^{r} \alpha_i' u_i$ for some $\alpha_i' \in \mathbb{R}$, which implies $\beta_1 = \ldots = \beta_s = 0$. Analogously $\gamma_1 = \ldots = \gamma_t = 0$ and finally $\alpha_1 = \ldots = \alpha_r = 0$. $\square$

Dimension of Sum of Subspaces (continued)

Proof.

Equivalently, consider linear transformation

$$s\colon V \times W \ni (v, w) \mapsto v + w \in \mathbb{R}^n.$$

Then

$$\ker s \cong V \cap W, \quad \operatorname{im} s = V + W,$$

hence

$$\dim(V \times W) = \dim V + \dim W = \dim \ker s + \dim \operatorname{im} s =$$

$$= \dim V \cap W + \dim(V + W).$$

$\square$

Direct Sum

Definition

Let $V, W \subset \mathbb{R}^n$ be subspaces. The space $\mathbb{R}^n$ is **a direct sum** of the subspaces V and W if

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Remark

If $\mathbb{R}^n$ is a direct sum of V and W it is denoted by

$$V \oplus W = \mathbb{R}^n.$$

It follows that

$$\dim V + \dim W = n.$$

Direct Sum (continued)

Proposition

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iii) *if* $(v_1, \dots, v_k)$ *is a basis of the subspace* V *and* $(w_1, \dots, w_{n-k})$ *is a basis of the subspace* W *then* $(v_1, \dots, v_k, w_1, \dots, w_{n-k})$ *is a basis of* $\mathbb{R}^n$.

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Proof.

Exercise. $\square$

Direct Sum – Examples

$$\mathbb{R}^2 = \mathrm{lin}((1,2)) \oplus \mathrm{lin}((1,3)),$$

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$$A = \frac{A + A^{\mathsf{T}}}{2} + \frac{A - A^{\mathsf{T}}}{2},$$

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$$A = \frac{A + A^\mathsf{T}}{2} + \frac{A - A^\mathsf{T}}{2},$$

and if $A \in \mathrm{Sym}(n \times n; \mathbb{R}) \cap \mathrm{Skew}(n \times n; \mathbb{R})$ then

$$A = -A^\mathsf{T} = -A \quad \Rightarrow \quad 2A = \mathbf{0}.$$

Direct Sum (continued)

Proposition

Let $V_1, \ldots, V_n \subset V$ be subspaces of a vector space V. The following conditions are equivalent

a) *for any $v \in V$ there exist unique vectors $v_1 \in V_1, \ldots, v_n \in V_n$ such that*
$$v = v_1 + \ldots + v_n.$$

b) *the following conditions hold*
   i) $V_1 + \ldots + V_n = V$,
   ii) $V_i \cap (V_1 + \ldots + V_{i-1} + V_{i+1} + \ldots + V_n) = \{\mathbf{0}\}$, *for* $i = 1, \ldots, n$.

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   ii) $V_i \cap (V_1 + \ldots + V_{i-1} + V_{i+1} + \ldots + V_n) = \{\mathbf{0}\}$, *for* $i = 1, \ldots, n$.

If the above holds we say that V is a direct sum of its subspaces $V_1, \ldots, V_n$ and write

$$V = V_1 \oplus \cdots \oplus V_n.$$

Direct Sum (continued)

Proof.

$(\Rightarrow)$ exercise

Direct Sum (continued)

Proof.

($\Rightarrow$) exercise

($\Leftarrow$) it is enough to prove uniqueness. Fix $i \in \{1, \ldots, n\}$. If

$$v = v_1 + \ldots + v_n,$$

$$v = v_1' + \ldots + v_n',$$

where $v_j, v_j' \in V$ for $j = 1, \ldots, n$, then

$$v_i - v_i' = (v_1' - v_1) + \ldots + (v_{i-1}' - v_{i-1}) + (v_{i+1}' - v_{i+1}) + \ldots + (v_n' - v_n),$$

therefore

$$v_i - v_i' \in V_i \cap (V_1 + \ldots + V_{i-1} + V_{i+1} + \ldots + V_n) = \{\mathbf{0}\},$$

hence

$$v_i = v_i'.$$

Matroid

Definition

Matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a **finite** set E and a family $\mathcal{I}$ of subsets of E such that

i) $\varnothing \in \mathcal{I}$,

ii) if $A \subset B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,

iii) if $A, B \in \mathcal{I}$ and $|A| < |B|$ (i.e. there are less elements in A) then there exists $e \in B \backslash A$ such that $A \cup \{e\} \in \mathcal{I}$.

Elements of E are called **points**, the set E is called the **ground set**, sets $A \in \mathcal{I}$ are called **independent** sets and sets $A \notin \mathcal{I}$ are called **dependent** sets.

Example

Let $E = \{1, \ldots, n\}$ and for some $m \geqslant 0$ let

$$\mathcal{I} = \{A \subset E \mid |E| \leqslant m\}.$$

This is the **uniform matroid** $U_{m,n}$.

Representable Matroid

Proposition

Let $A \in M(m \times n; \mathbb{R})$ *be a matrix. Let* $E = \{1, \ldots, n\}$ *and*

$$\mathcal{I} = \{J \subset E \mid \textit{columns of } A \textit{ indexed by } J \textit{ are linearly independent}\}.$$

Then $(E, \mathcal{I})$ *is a matroid.*

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Then $(E, \mathcal{I})$ *is a matroid.*

Proof.

Let $A = \{w_1, \ldots, w_k\}, B = \{v_1, \ldots, v_l\}$, where $w_i, v_i \in \mathbb{R}^m$ and $l > k$. By the Steinitz's Exchange Lemma applied to

$$w_1, \ldots, w_k \in \operatorname{lin}(w_1, \ldots, w_k, v_1, \ldots, v_l) = V,$$

linearly independent vectors $w_1, \ldots, w_k$ can be extended to a basis of V, by some vectors $v_1, \ldots, v_l$ where $\dim V \geqslant l$. $\square$

Representable Matroid

Definition

Any matroid isomorphic (i.e. there exists a bijection of ground sets preserving independent sets) is called to be **representable** (or linear) over $\mathbb{R}$.

Remark

There exist matroids which are not representable over any field (see Vámos matroid or non–Pappus matroid). Therefore the notion of matroid generalizes the abstract notion of linear independence.

Bases and Circuits

Let $M = (E, \mathcal{I})$ be a matroid.

Definition

A set $B \in \mathcal{I}$ such that if $B \subset B', B' \in \mathcal{I}$ then $B = B'$ (that is maximal independent set) is called a **basis** of M. A set $C \notin \mathcal{I}$ such that if $C' \subset C$ then $C' \in \mathcal{I}$ (that is minimal dependent set) is called a **circuit** of M.

Proposition

If B, B' are bases of matroid M then $|B| = |B'|$.

Remark

In a representable matroid given by matrix A basis B correspond to columns which are basis of $colsp(A)$. In the uniform matroid $U_{m,n}$ basis is any subset of cardinality m.

Bases and Circuits (continued)

Proposition

Let $C, C' \subset E$ be two different circuits of the matroid $M = (E, \mathcal{I})$ and let $e \in C \cap C'$. Then there exists a circuit $C'' \subset (C \cup C') \setminus \{e\}$.

Bases and Circuits (continued)

Proof.

Assume on the contrary $(C \cup C')\backslash\{e\}$ does not contain any circuit, that is all its subsets are independent, in particular $(C \cup C')\backslash\{e\} \in \mathcal{I}$. It is impossible that $C' \subset C$ so choose some $f \in C'\backslash C$. It follows that $C'\backslash\{f\} \in \mathcal{I}$ is independent. Consider a family

$$\left\{A \subset C \cup C' \mid C'\backslash\{f\} \subset A \text{ and } A \in \mathcal{I}\right\}.$$

This family is non–empty and finite so it contains a maximal element I. Obviously $f \notin I$ (otherwise $C' \subset I$ but I is independent and C' is not). It is impossible that $C \subset I$ so choose some $g \in C\backslash I$. Therefore $f, g \notin I$ but $f \notin C$ and $g \in C$ hence $f \neq g$ and

$$|I| \leqslant \left|(C \cup C')\backslash\{f, g\}\right| = \left|C \cup C'\right| - 2 < \left|(C \cup C')\backslash\{e\}\right|.$$

By the condition iii) I can be extended to an independent set by some element of $(C \cup C')\backslash\{e\}$ which contradicts it maximality. $\square$

Bases and Circuits (continued)

Proposition

Let $I \in \mathcal{I}$ and let $e \in E$ such that $I \cup \{e\} \notin \mathcal{I}$. Then there exists a unique circuit C such that $C \subset I \cup \{e\}$.

Proof.

It is clear that such circuit C exists and that for any such circuit $e \in C$. Assume that there a two circuits $C, C' \subset I \cup \{e\}$. Then $e \in C \cap C'$ and there exists a circuit $C'' \subset (C \cup C') \backslash \{e\}$ which is not possible as $C'' \not\subset I$.

$\square$

Remark

If $I = B$ is a basis then it is enough that $e \in E \backslash B$. In such case the unique C is called the **fundamental circuit** *of e with respect to B.*

Greedy Algorithm

Assume there is a function $w\colon E \to \mathbb{R}$ on the ground set of some matroid $M = (E, \mathcal{I})$. For any $A \subset E$ let $w(A) = \sum_{e \in A} w(e)$. The following algorithm is called the **Greedy Algorithm**.

i) set $A = \varnothing$,

ii) let
$$F = \{e \in E \backslash A \mid A \cup \{e\} \in \mathcal{I}\},$$
if $F = \varnothing$ then STOP,

iii) choose $e \in F$ with maximal weight $w(e)$, assign $A \leftarrow A \cup \{e\}$ and go to step ii)

That is, at each steep choose point e with maximal weight which preserves independence of A.

Greedy Algorithm (continued)

Proposition

The greedy algorithm returns a basis B of M with maximal weight $w(B)$, i.e. $w(B) \geqslant w(B')$ for any other basis B'.

Proof.

Say $B = \{e_1, \ldots, e_r\}$ and $B' = \{f_1, \ldots, f_r\}$ where $B = A$ was returned by the greedy algorithm and $w(f_i) \geqslant w_(f_{i+1})$. Then $w(e_i) \geqslant w(f_i)$ for any i (which proves the statement). Assume otherwise there exists the least $k \geqslant 2$ such that $w(e_k) < w(f_k)$. Let $I_1 = \{e_1, \ldots, e_{k-1}\}$, $I_2 = \{f_1, \ldots, f_k\}$. There exists $f_j \in I_2 \backslash I_1$ such that $I_1 \cup \{f_j\} \in \mathcal{I}$. But $w(f_j) \geqslant w(f_k) > w(e_k)$. But then at the k-iteration the greedy algorithm would choose f_j instead of e_k (note that $f_k \notin I_1$). $\square$

Greedy Algorithm (continued)

Proposition

Let $\mathcal{I}$ be a family of subsets of a finite set E such that

i) $\emptyset \in \mathcal{I}$,

ii) *if $A \subset B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,*

iii) *for any weight function $w\colon E \to \mathbb{R}$ yields a maximal element of $\mathcal{I}$ of maximal weight (among other maximal elements).*

Then $M = (E, \mathcal{I})$ is a matroid[1].

Proof.

It is enough to check condition iii). Let $A, B \in \mathcal{I}$ and let $|A| < |B|$. It follows that

$$|A \backslash B| < |B \backslash A|.$$

Assume that for each $e \in B \backslash A$ the set $A \cup \{e\} \notin \mathcal{I}$ is dependent.

[1] based on J. Oxley *Matroid Theory*

Greedy Algorithm (continued)

Proof.

Fix $\varepsilon > 0$ such that

$$\frac{|A \backslash B|}{|B \backslash A|} < \varepsilon < 1,$$

and weight function

$$w(e) = \begin{cases} 1 & e \in A \\ \varepsilon & e \in B \backslash A \\ 0 & \textit{otherwise} \end{cases}$$

The greedy algorithms picks all elements from A and then, by the assumption, some elements of weight 0 so it yields the total weight equal to $w(A)$. Choose maximal independent set B' such that $B' \supset B$. Then

$$w(B') \geqslant w(B) = |A \cap B| + \varepsilon |B \backslash A| > |A \cap B| + \frac{|A \backslash B|}{|B \backslash A|} |B \backslash A| =$$

$$= |A \cap B| + |A \backslash B| = w(A).$$

Rank and Span

Let $M = (E, \mathcal{I})$ be a matroid.

Definition

For any $B \subset E$ let

$$r(B) = \max\{|A| \mid A \subset B,\ A \in \mathcal{I}\},$$

be the **rank** of a subset B, i.e. the largest size of an independent subset.
For any $B \subset E$ let

$$\mathrm{span}(B) = \{e \in E \mid r(B \cup \{e\}) = r(B)\},$$

be the **span** (or **closure**) of the set B.

Remark

In a representable matroid rank is equal to the dimension of the column space (columns indexed by B) and the span consists of all columns contained in the column space given by B.

Rank and Span (continued)

Proposition

For any $B \subset E$

$$r(\text{span}(B)) = r(B).$$

Proof.

Obviously $B \subset \text{span}(B)$ hence $r(B) \leqslant r(\text{span}(B))$ (independent subset of B are subsets of $\text{span}(B)$). Assume the inequality is strict, i.e. there exists independent set $A \subset \text{span}(B)$ such that $r(B) < |A|$. Let $A' \subset B$ be the independent subset of B such that $r(B) = |A'|$. By the condition iii) there exists $e \in A \backslash A'$ such that $B \cup \{e\} \in \mathcal{I}$. This contradicts that $e \in \text{span}(B)$ (adding e to B raises its rank). $\square$

Spanning Set

Definition

A set $B \subset E$ is **spanning** if $r(B) = r(E)$.

Remark

Obviously $r(B) \leqslant r(E)$ *for any* $B \subset E$ *and* $r(E)$ *is equal to a cardinality of any basis. Hence* B *is spanning if and only if it contains a basis.*

Basis

Proposition

Let $B \subset E$, where $M = (E, \mathcal{I})$ is a matroid. The following conditions are equivalent

i) *B is a basis,*

ii) *B is maximal independent set,*

iii) *B is spanning independent set,*

iv) *B is minimal spanning set.*

Proof.

ii) is definition of i). ii)⇒iii) condition $r(B) < r(E)$ is impossible as any two bases are of the same cardinality, iii)⇒iv) any subset $B' \subset B, B' \neq B$ is independent hence
$r(B') = |B'| < |B| = r(B) = r(E)$ so it cannot be spanning.
iv)⇒i) B is spanning so it contains a basis B' but any basis is spanning so $B' = B$. □