Linear Algebra

Lecture 2 - Vector Spaces

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satisfying the following rules:

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viii) $1v = v$ for any $v \in V$.

A Few Facts

The following facts are direct consequences of these rules:

i) The element $\mathbf{0} \in V$ is unique. Suppose there is another $\mathbf{0}' \in V$, then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$.

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You may try to prove in a similar fashion that $\alpha\mathbf{0} = \mathbf{0}$ or that $\alpha v = \mathbf{0}$ implies $\alpha = 0$ or $v = \mathbf{0}$.

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ii) the n-tuple space $\mathbb{R}^n$, with addition $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$, multiplication $\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$ and the zero vector $\mathbf{0} = (0, \ldots, 0)$, in particular $\mathbb{R}$ =line, $\mathbb{R}^2$ =plane, $\mathbb{R}^3$ =three-dimensional space,

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iii) the space $\mathbb{R}^\infty$ of infinite sequences of real numbers, with addition $(x_i) + (y_i) = (x_i + y_i)$, multiplication $\alpha(x_i) = (\alpha x_i)$ and the zero vector $\mathbf{0} = (0, 0, \ldots)$,

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iv) the space of real functions on any non–empty set X $\mathcal{F}(X, \mathbb{R}) = \{f : X \longrightarrow \mathbb{R}\}$ with addition and multiplication defined pointwise: $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$. The zero vector is the constant function admitting 0 everywhere on X.

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A subspace W of V is called **proper** if $W \neq V$. Any subspace is a vector space.

Examples

The set of solutions of any homogeneous system of linear equations in n unknowns is a subspace of $\mathbb{R}^n$

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = 0 \end{cases}$$

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It can be shown that any subspace of $\mathbb{R}^n$ is of that form. Every subspace contains $\mathbf{0}$. Note that the set of solutions of a non-homogeneous system of linear equations is not a subspace since it does not contain $\mathbf{0}$.

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Let $x_0 \in X$. Then $\{f \in \mathcal{F}(X, \mathbb{R}) \mid f(x_0) = 0\}$ is a subspace of $\mathcal{F}(X, \mathbb{R})$.

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If $U, V \subset W$ are subspaces of vector space W, then $U \cap V$ is a subspace of W. You may try to prove that $U \cup V$ is a subspace of W if and only if $U \subset V$ or $V \subset U$.

Linear Combinations

Let V be a vector space. The **linear combination** of vectors $v_1, \ldots, v_k \in V$ with coefficients $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ is the vector $\alpha_1 v_1 + \ldots + \alpha_k v_k = \sum_{i=1}^{k} \alpha_i v_i \in V$.

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For example, the vector $(4, 1, 3)$ is a linear combination of vectors $v_1 = (1, 0, 1), v_2 = (0, 1, 0), v_3 = (1, -1, 0) \in \mathbb{R}^3$ with coefficients $3, 2, 1$, because $(4, 1, 3) = 3(1, 0, 1) + 2(0, 1, 0) + (1, -1, 0)$.

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Let $v = \alpha_1 v_1 + \ldots \alpha_k v_k$ and $w = \beta_1 v_1 + \ldots \beta_k v_k$. Then $v + w = (\alpha_1 + \beta_1) v_1 + \ldots + (\alpha_k + \beta_k) v_k$. $\square$

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Definition

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Corollary

If $w_1, \ldots, w_l \in \text{lin}(v_1, \ldots, v_k)$ *then*

$$\text{lin}(w_1, \ldots, w_l) \subset \text{lin}(v_1, \ldots, v_k)$$

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For any $v_1, \ldots, v_k \in V$ and $\alpha \in \mathbb{R} - \{0\}$ the following hold:

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ii) $\mathrm{lin}(v_1, v_2, \ldots, v_k) = \mathrm{lin}(\alpha v_1, v_2, v_3, \ldots, v_k)$,

iii) $\mathrm{lin}(v_1, v_2, \ldots, v_k) = \mathrm{lin}(v_1 + v_2, v_2, v_3, \ldots, v_k)$.

Corollary

We have

$$\mathrm{lin}(v_1, \ldots, v_k) = \mathrm{lin}(v_1 + \alpha v_2, v_2, \ldots, v_k),$$

that is, elementary operations on vectors do not change the spanned subspace.

Linear Span (continued)

Proof.

i) $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = \alpha_2 v_2 + \alpha_1 v_1 + \ldots + \alpha_k v_k,$

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Subspaces of $\mathbb{R}^n$ and Homogenous Systems of Linear Equations

Proposition

Let $V = \operatorname{lin}(v_1, \ldots, v_k) \subset \mathbb{R}^n$ *be the linear span of vectors* $v_1, \ldots, v_k \in \mathbb{R}^n$ *where* $k \geq 1$. *Then there exists a homogeneous system of linear equations in* n *variables whose set of solutions is equal to* V.

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Proof.

Let $A = [a_{ij}] \in M(k \times n; \mathbb{R})$ be a matrix whose rows are equal to $v_1, \dots, v_k$, i.e. $v_1 = (a_{11}, \dots, a_{1n}), \dots, v_k = (a_{k1}, \dots, a_{kn})$. Let $B = [b_{ij}] \in M(k \times n; \mathbb{R})$ be a matrix equal to the reduced echelon form of A, where $w_1, \dots, w_k \in \mathbb{R}^n$ are rows of B, i.e. $w_1 = (b_{11}, \dots, b_{1n}), \dots, w_k = (b_{k1}, \dots, b_{kn})$.

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$$V = \operatorname{lin}(v_1, \ldots, v_k) = \operatorname{lin}(w_1, \ldots, w_k).$$

Subspaces of $\mathbb{R}^n$ and Homogenous Systems of Linear Equations (continued)

Proof.

For simplicity assume that pivots appear in columns of numbers $1, 2, \ldots, m$, where $m \leq k$, i.e.

$$w_1 = (1, 0, 0, \ldots, 0, 0, b_{1m}, b_{1(m+1)}, \ldots, b_{1n}),$$

$$w_2 = (0, 1, 0, \ldots, 0, 0, b_{2m}, b_{2(m+1)}, \ldots, b_{2n}),$$

$$\vdots$$

$$w_m = (0, 0, 0, \ldots, 0, 1, b_{km}, b_{k(m+1)}, \ldots, b_{1n}),$$

$$\text{and } w_{m+1} = \ldots = w_k = \mathbf{0}.$$

Subspaces of $\mathbb{R}^n$ and Homogenous Systems of Linear Equations (continued)

Proof.

Then

$$V = \text{lin}(v_1, \dots, v_k) = \text{lin}(w_1, \dots, w_m) =$$

$$= \{x_1 w_1 + \dots + x_m w_m \in \mathbb{R}^n \mid x_1, \dots, x_m \in \mathbb{R}\} =$$

$$\{(x_1, x_2, \dots, x_m, b_{1m}x_1 + b_{2m}x_2 + \dots + b_{km}x_m,$$

$$, b_{1(m+1)}x_1 + b_{2(m+1)m}x_2 + \dots + b_{k(m+1)}x_m, \dots$$

$$\dots, b_{1n}x_1 + b_{2n}x_2 + \dots + b_{kn}x_m) \mid x_1, \dots, x_m \in \mathbb{R}\},$$

which is equal to the set of solutions of the system

$$\begin{cases} x_{m+1} & = & b_{1m}x_1 & + & b_{2m}x_2 & + & \dots & + & b_{km}x_m, \\ x_{m+2} & = & b_{1(m+1)}x_1 & + & b_{2(m+1)}x_2 & + & \dots & + & b_{k(m+1)}x_m, \\ & \vdots & & & & & & & \\ x_n & = & b_{1n}x_1 & + & b_{2n}x_2 & + & \dots & + & b_{kn}x_m, \end{cases}$$

of $n - m$ equations with free variables $x_1, \dots, x_m \in \mathbb{R}$. $\square$

Subspaces and Homogenous Systems of Linear Equations (continued)

Proof.

In general, if the numbers of columns with pivots are equal to $1 \leq j_1 < j_2 < \ldots < j_m \leq n$ then one should consider vector

$$x_{j_1} w_1 + x_{j_2} w_2 + \ldots + x_{j_m} w_m,$$

which leads to a homogeneous system of $n - m$ equations in n unknowns and free variables $x_{j_1}, \ldots, x_{j_m} \in \mathbb{R}$. $\square$

Example

Let $V = \mathrm{lin}((1,2,1,0),(0,2,1,1),(1,4,2,1),(3,8,4,1)) \subset \mathbb{R}^4$ be a subspace of $\mathbb{R}^4$. Find a system of linear equations which set of solutions is equal to V.

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Put vectors horizontally in a matrix and perform elementary row operations to get the reduced echelon form (up to column permutation).

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 3 & 8 & 4 & 1 \end{bmatrix} \xrightarrow[r_4-3r_1]{r_3-r_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

Example (continued)

Any vector in the space V is equal to $x_1(1,2,1,0) + x_4(0,2,1,1) = (x_1, 2x_1 + 2x_4, x_1 + x_4, x_4)$ for some $x_1, x_4 \in \mathbb{R}$. This is a general solution of the following system of linear equations

$$\begin{cases} x_2 = 2x_1 + 2x_4 \\ x_3 = x_1 + x_4 \end{cases}$$

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$$\begin{cases} x_2 &= 2x_1 &+ 2x_4 \\ x_3 &= x_1 &+ x_4 \end{cases}$$

The system is equal to

$$\begin{cases} 2x_1 - x_2 + 2x_4 = 0 \\ x_1 - x_3 + x_4 = 0 \end{cases}$$

Remarks

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The system corresponding to the subspace $V = \operatorname{lin}(v_1, \ldots, v_k) \subset \mathbb{R}^n$ is **not unique**.

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So far we have shown that

$$\left\{\begin{array}{c}\textit{subspaces}\\ \textit{of } \mathbb{R}^n\end{array}\right\} \supset \left\{\begin{array}{c}\textit{linear spans}\\ \textit{of } v_1,\ldots,v_k\in\mathbb{R}^n\end{array}\right\} =$$

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Later we will show that any subspace $V \subset \mathbb{R}^n$ *is equal to a linear span of some vectors* $v_1, \dots, v_k \in \mathbb{R}^n$*, that is the first inclusion is an equality.*

Hölder and Minkowski Inequalities[1]

Proposition (Hölder)

For any $p, q > 1$ *such that*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(or* $\in \mathbb{C}^n$*)*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$

[1] For proofs see W. Rudin, *Functional Analysis*

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Corollary (Minkowski)

For any $p \geq 1$ *and any* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(or* $\mathbb{C}^n$*)*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}.$$

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The Spaces ℓ^1 and ℓ^2

Definition

Let

$$\ell^1 = \left\{ (x_i) \in \mathbb{R}^\infty \mid \sum_{i=1}^{\infty} |x_i| < +\infty \right\},$$

$$\ell^2 = \left\{ (x_i) \in \mathbb{R}^\infty \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty \right\},$$

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Corollary

The sets $\ell^1, \ell^2, \ell^\infty \subset \mathbb{R}^\infty$ are subspaces and

$$\ell^1 \subsetneq \ell^2 \subsetneq \ell^\infty \subsetneq \mathbb{R}^\infty.$$

The Spaces ℓ^1 and ℓ^2 (continued)

Proof.

If $(x_i), (y_i) \in \ell^\infty$ then $|x_i| < M,\ |y_i| < N$, therefore

$$|x_i + y_i| \leq |x_i| + |y_i| < M + N,$$

$$|\alpha x_i| < |\alpha| M,$$

for any $\alpha \neq 0$.

The Spaces ℓ^1 and ℓ^2 (continued)

Proof.

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for any $\alpha \neq 0$. For ℓ^1 and ℓ^2, since

$$\sum_{i=1}^{\infty} x_i = \lim_{n \to +\infty} \sum_{i=1}^{n} x_i,$$

it is enough to take limits in the Minkowski inequality for $p = 1$ and $p = 2$, respectively.

The Spaces ℓ^1 and ℓ^2 (continued)

Proof.

If $\sum_{i=1}^{\infty} |x_i| < \infty$ then $\lim\limits_{i \to +\infty} |x_i| = 0$ and the sequence (x_i) is bounded.

The Spaces ℓ^1 and ℓ^2 (continued)

Proof.

If $\sum_{i=1}^{\infty} |x_i| < \infty$ then $\lim_{i \to +\infty} |x_i| = 0$ and the sequence (x_i) is bounded. Therefore for $i > N$, where N is large enough, $|x_i| < 1$ and

$$|x_i|^2 \leq |x_i|,$$

$$\sum_{i=N+1}^{\infty} |x_i|^2 \leq \sum_{i=N+1}^{\infty} |x_i| < \infty,$$

which implies

$$\ell^1 \subset \ell^2 \subset \ell^{\infty}.$$

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If $x_i = 1$ then $(x_i) \in \ell^\infty$ but $(x_i) \notin \ell^2$.

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which implies

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If $x_i = 1$ then $(x_i) \in \ell^\infty$ but $(x_i) \notin \ell^2$. If $y_i = \frac{1}{i}$ then $(y_i) \in \ell^2$ but $(y_i) \notin \ell^1$. $\square$

Harmonic Functions/Laplace Equation

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Harmonic Functions/Laplace Equation

Let $\Omega \subset \mathbb{R}^n$ be an open set. If

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then

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are subspaces of $\mathbb{F}(\Omega, \mathbb{R})$.

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are subspaces of $\mathbb{F}(\Omega, \mathbb{R})$. The first inclusion is a theorem in the theory of harmonic functions.

Homogeneous (Ordinary) Linear Differential Equations

If $X = (a, b) \subset \mathbb{R}$ and

$$V = \{x \in \mathcal{C}^n(X) \mid x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = 0\},$$

i.e., V is the set of all functions $x\colon X \to \mathbb{R}$ such that $x \in \mathcal{C}^n(X)$ and

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \ldots + a_n(t)x(t) = 0,$$

for all $t \in X$, where

$$a_i\colon X \to \mathbb{R},$$

are continuous functions then

$$V \subset \mathcal{C}^n(X),$$

is a subspace of dimension n.

Grassmannian

This material is meant to be read at the end of the course.

Definition

Grassmanian $\mathrm{Gr}(k,n)$ is the set of all subspaces of dimension k in the vector space $\mathbb{R}^n$, that is

$$\mathrm{Gr}(k,n) = \{V \subset \mathbb{R}^n \mid \dim V = k\}.$$

It is possible to identify this set with a smooth algebraic variety (you might think of it a special version of a submanifold of $\mathbb{R}^{n^2}$)

$$\mathrm{Gr}(k,n) = \{A \in M(n \times n; \mathbb{R}) \mid A^2 = A, \quad A^\intercal = A, \quad \mathrm{rk}\, A = k\}.$$

There exists a bijection between subspaces of dimension k of $\mathbb{R}^n$ and matrices as above (exercise). Note that Grassmanian is given by a system of polynomial equations in the entries a_{ij} of matrix $A = [a_{ij}]$.

Grassmannian (continued)

There are several methods of inducing a topology on $G(k, n)$. A metric on $Gr(k, n)$ can be defined by

$$d(V, W) = \|P_V - P_W\|,$$

where P_V, P_W are (matrices) of orthogonal projections onto k-dimensional subspaces V and W, respectively and $\|\cdot\|$ is (some) matrix norm.

Using the SVD decomposition it is possible to give interpretation of those distances in the terms of principal angles between the subspaces V and W for $\|\cdot\| = \|\cdot\|_2$ and $\|\cdot\| = \|\cdot\|_F$ (exercise).