Linear Algebra

Lecture 13 - Simplex Method

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Simplex Method

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A linear programming problem in a standard form is a task of minimizing the objective function

$$f((x_1, \dots, x_n)) = c_1 x_1 + \dots + c_n x_n \longrightarrow min$$

under the constraints

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = b_m \end{cases}, \quad x_1, \dots, x_n \geqslant 0$$

Simplex Method

Set

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We assume that $r(A) = m$.

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We assume that $r(A) = m$.
Let $X \subset \mathbb{R}^n$ be a convex polytope defined by the conditions $Ax = b, \; x \geqslant 0$. Recall that if there is an optimal solution to the problem (i.e. a point $\overline{x} \in X$ in which f admits its minimum over X) then it can be chosen to be a vertex of X.

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Vertices of X correspond to basic feasible solutions of the problem. They are given by basic feasible sets $\mathcal{B} \subset \{1, \ldots, n\}$ of $m = r(A)$ elements, such that the system of linear equations $Ax = b, \ x_i = 0$ for $i \notin \mathcal{B}$ has a unique non-negative solution.

Simplex Method

Simplex method starts from a basic feasible solutions. Then one moves to another basic feasible solution by replacing one element in the basic set $\mathcal{B}$ in order to decrease the value of the objective function f.

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

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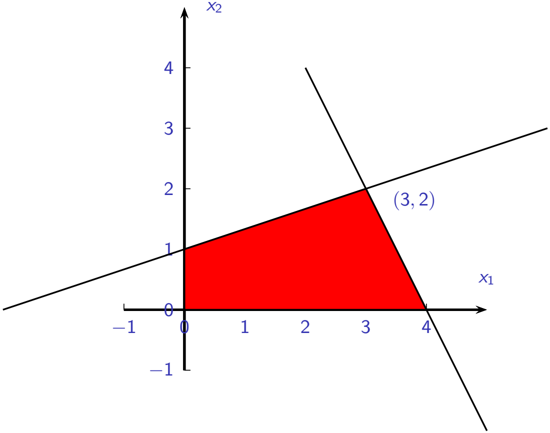
Express this problem in a standard form

$$-x_1 - 2x_2 \longrightarrow min$$

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 8 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.

Example



maximize $x_1 + 2x_2$

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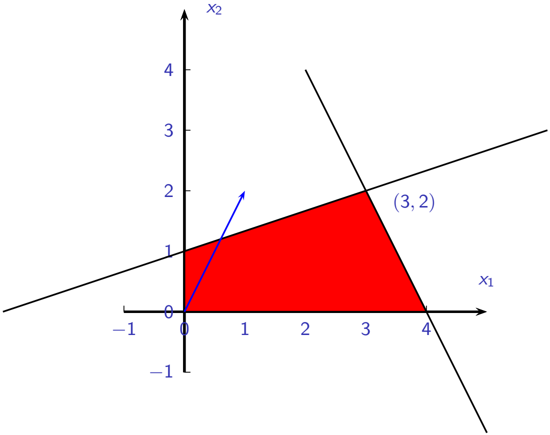

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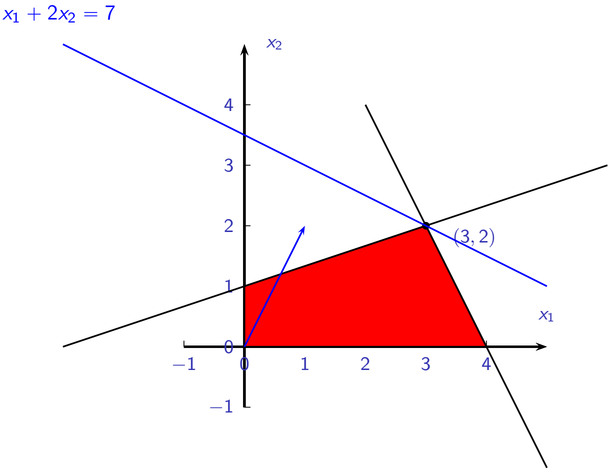

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optimal solution is $(3, 2)$

Example

We start from the basic feasible set $\mathcal{B}_1 = \{3, 4\}$. The basic variables are x_3, x_4 and the non-basic ones are x_1, x_2.

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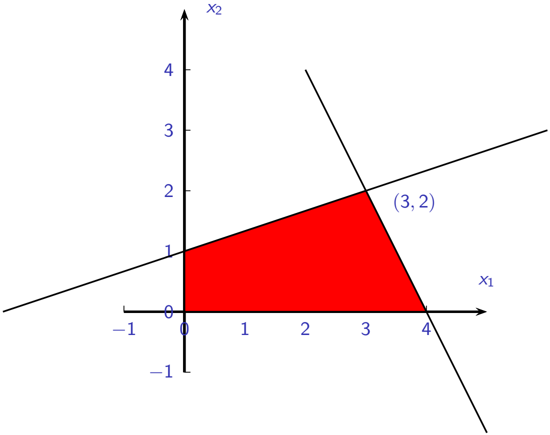
by setting $x_1 = x_2 = 0$.
Since $f(x) = -x_1 - 2x_2$ therefore $f(\overline{x}_{\mathcal{B}_1}) = 0$. We could decrease it by making either x_1 or x_2 non-zero. By a heuristic rule we choose x_2 since the coefficient -2 is smaller than -1. Assume $s = 2$ will enter the new basic (feasible) set $\mathcal{B}_2$.

Example



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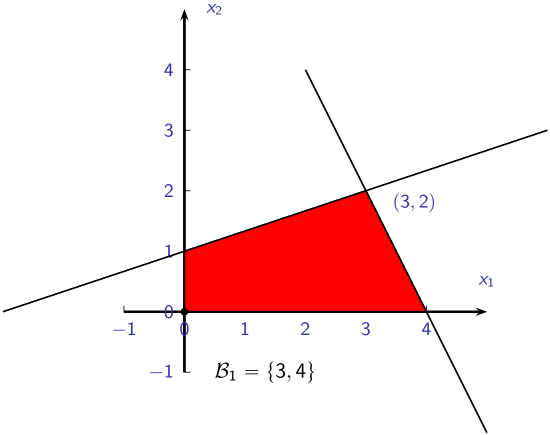

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$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -\frac{7}{3}x_1 - x_3 + \frac{1}{3}x_4 = -7 \end{cases}$$

and $\overline{x}_{\{2,4\}} = (0, 8, 0, -21)$.

Example

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Subtract the second equation from the first one to make x_2, x_3 basic variables. This means 4 leaves the basic set $\mathcal{B}_1$.

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$$\left\{\begin{array}{rcrcrcrcl} \frac{7}{3}x_1 & & & + & x_3 & - & \frac{1}{3}x_4 & = & 7 \\ -\frac{1}{3}x_1 & + & x_2 & & & + & \frac{1}{3}x_4 & = & 1 \end{array}\right.$$

$$\text{and} \qquad \overline{x}_{\{2,3\}} = (0, 1, 7, 0).$$

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$$\begin{cases} \frac{7}{3}x_1 + x_3 - \frac{1}{3}x_4 = 7 \\ -\frac{1}{3}x_1 + x_2 + \frac{1}{3}x_4 = 1 \end{cases}$$

$$\text{and} \quad \overline{x}_{\{2,3\}} = (0, 1, 7, 0).$$

Both sets $\{2, 3\}$ and $\{2, 4\}$ are basic but only $\{2, 3\}$ is feasible since $\overline{x}_{\{2,3\}} = (0, 1, 7, 0) \geqslant 0$ and $\overline{x}_{\{2,4\}} = (0, 8, 0, -21) \ngeqslant 0$

Recall

$$\begin{cases} 2x_1 + 1x_2 + x_3 = 8 \\ -x_1 + 3x_2 + x_4 = 3 \end{cases}$$

Observe $\frac{8}{1} \geqslant \frac{3}{3}$. The crucial point is to subtract **smaller** ratio from the bigger one to get a positive number.

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For $\mathcal{B}_2 = \{2, 3\}$ the general solution with x_2, x_3 as basic variables is

$$\begin{cases} -\frac{1}{3}x_1 + x_2 + \frac{1}{3}x_4 = 1 \\ \frac{7}{3}x_1 + x_3 - \frac{1}{3}x_4 = 7 \end{cases}$$

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Substitute $x_2 = 1 + \frac{1}{3}x_1 - \frac{1}{3}x_4$ to $f(x)$

$$f(x) = -x_1 - 2x_2 = -2 - \frac{5}{3}x_1 + \frac{2}{3}x_4.$$

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Making x_1 non-zero will decrease f, i.e. $s = 1$ will enter the new basic set $\mathcal{B}_3$.

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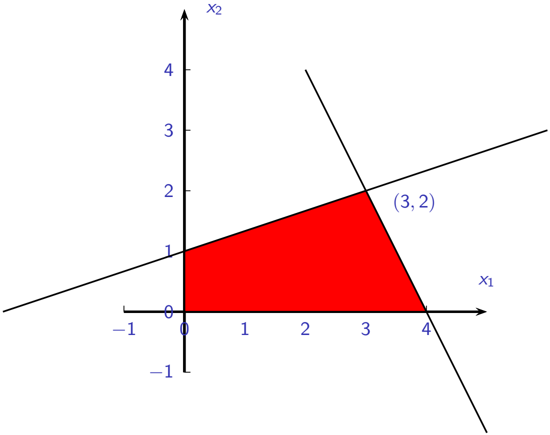

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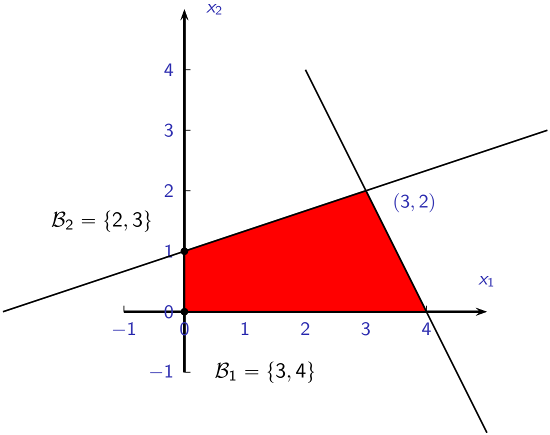

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Example

$$\begin{cases} -\frac{1}{3}x_1 + x_2 + \frac{1}{3}x_4 = 1 \\ \frac{7}{3}x_1 + x_3 - \frac{1}{3}x_4 = 7 \end{cases}$$

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Now $\frac{1}{-1/3} \leqslant \frac{7}{7/3}$ but unlike the previous case, subtracting the first equation from the second one leads to an infeasible basic set $\{1, 3\}$ with $\overline{x}_{\{1,3\}} = (-3, 0, 14, 0) \not\geqslant 0$. Therefore we need to choose **the smallest ratio among the positive ones**.

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Example

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & & & - & x_4 & = & -3 \\ x_1 & & & + & \frac{3}{7}x_3 & - & \frac{1}{7}x_4 & = & 3 \end{array}\right.$$

Example

$$\begin{cases} x_1 - 3x_2 \phantom{+ \frac{3}{7}x_3} - x_4 = -3 \\ x_1 + \frac{3}{7}x_3 - \frac{1}{7}x_4 = 3 \end{cases}$$

The new basic set is $\mathcal{B}_3 = \{1, 2\}$. Subtract the second equation from the first one

$$\begin{cases} x_1 + \frac{3}{7}x_3 - \frac{1}{7}x_4 = 3 \\ x_2 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = 2 \end{cases}$$

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and substitute the result to $f(x) = -x_1 - 2x_2$

$$f(x) = -7 + \frac{5}{7}x_3 + \frac{3}{7}x_4.$$

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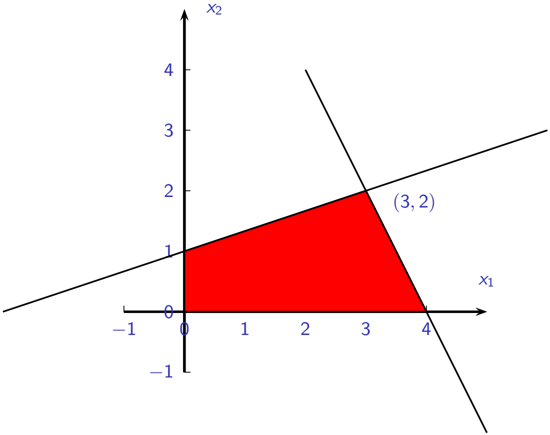
Making x_3 or x_4 a basic variable would increase the value of f.

Example

Therefore the basic set $\mathcal{B}_3 = \{1, 2\}$ corresponds to a vertex $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ in which function f attains minimum on the feasible region, i.e. $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ is an optimal solution.

Example



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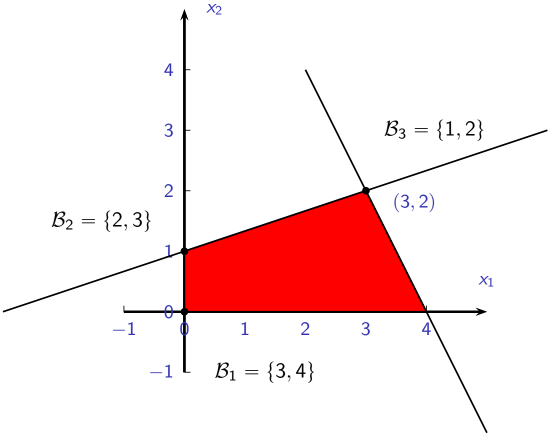

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Simplex Method

Given a linear programming problem in the standard form $f((x_1,\ldots,x_n)) = c_1x_1 + \ldots + c_nx_n \longrightarrow min$ under the constraints $Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and $r(A) = m$ proceed as follows.

Simplex Method

1) build a simplex tableau $\left[\begin{array}{cccc|c} c_1 & c_2 & \cdots & c_n & 0 \\ \hline a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ we will refer to the part above the horizontal line as the upper part and to the other as the lower part,

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2) find some basic feasible set $\mathcal{B} = \{i_1, \ldots, i_m\},\ i_1 < i_2 < \ldots < i_m,$

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2) find some basic feasible set $\mathcal{B} = \{i_1, \ldots, i_m\}, \; i_1 < i_2 < \ldots < i_m$,
3) using elementary row operations (adding or subtracting the upper row from rows in the lower part is not allowed) bring the simplex tableau to the form

Simplex Method

$$
\begin{array}{c}
\begin{array}{ccccccccccc}
1 & & i_1 & & i_2 & & i_{m-1} & & i_m & & n
\end{array} \\
\left[\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & 0 & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \ldots & 1 & \ldots & 0 & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{1n} & b'_1 \\
a'_{21} & \ldots & 0 & \ldots & 1 & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{2n} & b'_2 \\
a'_{31} & \ldots & 0 & \ldots & 0 & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \ldots & 0 & \ldots & 0 & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \ldots & 0 & \ldots & 0 & \ldots\ldots & 1 & \ldots & 0 & \ldots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \ldots & 0 & \ldots & 0 & \ldots\ldots & 0 & \ldots & 1 & \ldots & a'_{mn} & b'_m
\end{array}\right]
\end{array}
$$

[1]Some authors say the tableau is in **canonical form** (with respect to $\mathcal{B}$).

Simplex Method

$$
\begin{array}{c}
\begin{array}{ccccccccccccc}
1 & & i_1 & & i_2 & & i_{m-1} & & i_m & & n & & \\
\end{array} \\
\left[\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & 0 & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \dots & 1 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{1n} & b'_1 \\
a'_{21} & \dots & 0 & \dots & 1 & \dots\dots & 0 & \dots & 0 & \dots & a'_{2n} & b'_2 \\
a'_{31} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 0 & \dots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \dots & 0 & \dots & 0 & \dots\dots & 1 & \dots & 0 & \dots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \dots & 0 & \dots & 0 & \dots\dots & 0 & \dots & 1 & \dots & a'_{mn} & b'_m
\end{array}\right]
\end{array}
$$

i.e. the submatrix of the lower part of the simplex tableau consisting of columns $i_1, \dots, i_m$ is the identity matrix and the coefficients of the objective function corresponding to the basic variables $x_{i_1}, \dots, x_{i_m}$ are zero.[1]

[1] Some authors say the tableau is in **canonical form** (with respect to $\mathcal{B}$).

Simplex Method

4) let $s \in \{1, \ldots, n\}$ be such that $c'_s = \min\{c'_1, c'_2, \ldots, c'_n\}$, i.e. let s be the number of the column with **the smallest** coefficient c'_i,

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4) let $s \in \{1, \ldots, n\}$ be such that $c'_s = \min\{c'_1, c'_2, \ldots, c'_n\}$, i.e. let s be the number of the column with **the smallest** coefficient c'_i,
5) if $c'_s \geqslant 0$ (i.e. all c'_i are non-negative) then STOP, $-c'$ is the minimal value of the objective function and the optimal solution is $\overline{x}_{\mathcal{B}}$,

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5) if $c'_s \geqslant 0$ (i.e. all c'_i are non-negative) then STOP, $-c'$ is the minimal value of the objective function and the optimal solution is $\overline{x}_{\mathcal{B}}$,
6) if the set $\{a'_{is} \mid a'_{is} > 0, \ i = 1, \ldots, m\}$ is empty, i.e. all entries in the lower part of the s-th column of the simplex tableau are non-positive then STOP, the objective function attains no minimum on the feasible region,

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6) if the set $\{a'_{is} \mid a'_{is} > 0, \; i = 1, \dots, m\}$ is empty, i.e. all entries in the lower part of the s-th column of the simplex tableau are non-positive then STOP, the objective function attains no minimum on the feasible region,
7) let $r \in \{1, \dots, m\}$ be given by $\frac{b'_r}{a'_{rs}} = \min\left\{\frac{b'_i}{a'_{is}} \mid a'_{is} > 0, i = 1, \dots, m\right\}$, i.e. let r be the number of the equation in the simplex tableau with the smallest non–negative ratio $\frac{b'_i}{a'_{is}}$,

Simplex Method

$$
\begin{array}{c}
\begin{array}{ccccccccccccc}
1 & & i_1 & & s & & i_{m-1} & & i_m & & n & &
\end{array} \\
\left[\begin{array}{ccccccccccc|c}
c'_1 & & 0 & & c'_s & & 0 & & 0 & & c'_n & c' \\
\hline
a'_{11} & \ldots & 1 & \ldots & a'_{1s} & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{1n} & b'_1 \\
a'_{21} & \ldots & 0 & \ldots & a'_{2s} & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{2n} & b'_2 \\
a'_{31} & \ldots & 0 & \ldots & a'_{3s} & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{3n} & b'_3 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \ldots & 0 & \ldots & a'_{(m-2)s} & \ldots\ldots & 0 & \ldots & 0 & \ldots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \ldots & 0 & \ldots & a'_{(m-1)s} & \ldots\ldots & 1 & \ldots & 0 & \ldots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \ldots & 0 & \ldots & a'_{ms} & \ldots\ldots & 0 & \ldots & 1 & \ldots & a'_{mn} & b'_m
\end{array}\right]
\end{array}
$$

$$
\frac{b'_r}{a'_{rs}} = \min\left\{ \frac{b'_i}{a'_{is}} \mid a'_{is} > 0, i = 1, \ldots, m \right\}
$$

Simplex Method

8) the r-th element of $\mathcal{B}$ (i.e. i_r) is removed and s enters the basic set $\mathcal{B}$,

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9) go to step 3).

Example

Now we can redo our first example using simplex tableau.

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Recall

$$-x_1 - 2x_2 \longrightarrow min$$

$$\begin{cases} 2x_1 + x_2 + x_3 = 8 \\ -x_1 + 3x_2 + x_4 = 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.

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and $x_1, x_2, x_3, x_4 \geqslant 0$.
Choose basic feasible set $\mathcal{B} = \{3, 4\}$ and write the simplex tableau:

$$\begin{array}{c} \begin{array}{cccc} \;1\; & \;2\; & \;3\; & \;4\; \end{array} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \end{array}$$

It is already in the form from step 3) (i.e. in the upper row there are zeroes in the 3-th and 4-th column and the submatrix of the lower part consisting of columns $3, 4$ is the identity matrix).

Example

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & \\
\multicolumn{5}{c}{\left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right]}
\end{array}$$

The smallest coefficient of the objective function is $c_2' = -2$ and hence $s = 2$.

Example

$$
\begin{array}{cccc|c}
1 & 2 & 3 & 4 & \\
\end{array}
$$

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\end{array}\right]
$$

The smallest coefficient of the objective function is $c'_2 = -2$ and hence $s = 2$.
Compute ratios of the entries in the last column and in the second one.

$$
\begin{array}{cccc|c}
1 & 2 & 3 & 4 & \\
\end{array}
$$

$$
\left[\begin{array}{cccc|c}
-1 & -2 & 0 & 0 & 0 \\
\hline
2 & 1 & 1 & 0 & 8 \\
-1 & 3 & 0 & 1 & 3
\end{array}\right]
$$

$$
\frac{b'_2}{a'_{s2}} = \frac{3}{3} = \min\left\{\frac{8}{1}, \frac{3}{3}\right\}
$$

Example

$$\frac{b'_2}{a'_{s2}} = \frac{3}{3} = \min\left\{\frac{8}{1}, \frac{3}{3}\right\}$$

The smallest ratio is provided by the second row so $r = 2$. Therefore the second element of $\mathcal{B} = \{3, 4\}$ leaves and $s = 2$ enters the basic set.

Example

$$\frac{b'_2}{a'_{s2}} = \frac{3}{3} = \min\left\{\frac{8}{1}, \frac{3}{3}\right\}$$

The smallest ratio is provided by the second row so $r = 2$. Therefore the second element of $\mathcal{B} = \{3, 4\}$ leaves and $s = 2$ enters the basic set. For $\mathcal{B} = \{2, 3\}$ bring the simplex tableau into the form described in step 3).

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \end{array} \xrightarrow{r_2/3} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -1 & -2 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 8 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array} \xrightarrow[r_1-r_2]{r_0+2r_2}$$

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \end{array}\right] \end{array} \xrightarrow{r_1 \leftrightarrow r_2} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & \end{matrix} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\ \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \end{array}\right] \end{array}$$

Example

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & & \\
\multicolumn{6}{c}{
\left[\begin{array}{cccc|c}
-\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\
\hline
-\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\
\color{red}{\frac{7}{3}} & 0 & 1 & -\frac{1}{3} & \color{red}{7}
\end{array}\right]}
\end{array}
$$

Now $c'_1 = -\frac{5}{3} < c'_4 = \frac{2}{3}$ hence $s = 1$.

Example

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc|c} -\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline -\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\ \frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7 \end{array}\right] \end{array}$$

Now $c'_1 = -\frac{5}{3} < c'_4 = \frac{2}{3}$ hence $s = 1$.
In the first column only one number is positive, that is the smallest ratio is $\frac{7}{7/3}$ hence $r = 2$. The second element from $\mathcal{B} = \{2, 3\}$ leaves and $s = 1$ enters the basic set.

Example

$$\begin{array}{c}
\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\
\left[\begin{array}{cccc|c}
-\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline
-\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\
\frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7
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Now $c'_1 = -\frac{5}{3} < c'_4 = \frac{2}{3}$ hence $s = 1$.
In the first column only one number is positive, that is the smallest ratio is $\frac{7}{7/3}$ hence $r = 2$. The second element from $\mathcal{B} = \{2, 3\}$ leaves and $s = 1$ enters the basic set.
Now $\mathcal{B} = \{1, 2\}$.

$$\begin{array}{c}
\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\
\left[\begin{array}{cccc|c}
-\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline
-\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1 \\
\frac{7}{3} & 0 & 1 & -\frac{1}{3} & 7
\end{array}\right]
\end{array}
\xrightarrow[r_1 \leftrightarrow r_2]{\frac{3}{7} r_2}
\begin{array}{c}
\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\
\left[\begin{array}{cccc|c}
-\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\ \hline
1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\
-\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1
\end{array}\right]
\end{array}$$

Example

$$
\begin{array}{c}
\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\
\left[\begin{array}{cccc|c}
-\frac{5}{3} & 0 & 0 & \frac{2}{3} & 2 \\
\hline
1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\
-\frac{1}{3} & 1 & 0 & \frac{1}{3} & 1
\end{array}\right]
\end{array}
\xrightarrow[r_2+\frac{1}{3}r_1]{r_0+\frac{5}{3}r_1}
\begin{array}{c}
\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \end{array} \\
\left[\begin{array}{cccc|c}
0 & 0 & \frac{5}{7} & \frac{3}{7} & 7 \\
\hline
1 & 0 & \frac{3}{7} & -\frac{1}{7} & 3 \\
0 & 1 & \frac{1}{7} & \frac{2}{7} & 2
\end{array}\right]
\end{array}
$$

Since $c_i' \geqslant 0$ for $i = 1, 2, 3, 4$ we have arrived at an optimal solution which is $\overline{x}_{\{1,2\}} = (3, 2, 0, 0)$ and the minimal value is -7.

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v) at any step the objective function is equal to

$$f((x_1, \dots, x_n)) = c'_1 x_1 + \dots + c'_n x_n - c',$$

where $c'_{i_j} = 0$ for $j = 1, \dots, m$ (i.e. $c'_i = 0$ for $i \in \mathcal{B}$).

Remarks

$$
\begin{array}{ccccccccccccc|c}
1 & & i_1 & & s & & i_{m-1} & & i_m & & n & & \\
\end{array}
$$

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\left[\begin{array}{ccccccccccc|c}
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\hline
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a'_{21} & \dots & 0 & \dots & a'_{2s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{2n} & b'_2 \\
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\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\
a'_{(m-2)1} & \dots & 0 & \dots & a'_{(m-2)s} & \dots\dots & 0 & \dots & 0 & \dots & a'_{(m-2)n} & b'_{m-2} \\
a'_{(m-1)1} & \dots & 0 & \dots & a'_{(m-1)s} & \dots\dots & 1 & \dots & 0 & \dots & a'_{(m-1)n} & b'_{m-1} \\
a'_{m1} & \dots & 0 & \dots & a'_{ms} & \dots\dots & 0 & \dots & 1 & \dots & a'_{mn} & b'_m
\end{array}\right]
$$

Move terms involving x_s to the right hand side of all equations. Set $x_i = 0$ for $i \notin \mathcal{B} \cup \{s\}$. For any positive value of x_s the system of linear equations in variables $x_{i_1}, \dots, x_{i_m}$ has a non-negative solution.

Remarks

$$
\begin{array}{ccccccccccccc|c}
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$$

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\left[\begin{array}{ccccccccccc|c}
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Remarks – Global Minimum

If $c'_i \geqslant 0$ for $i \notin \mathcal{B}$ it is easy to see that $-c'$ is the global minimum (attained at $\overline{x}_{\mathcal{B}}$). If $x = (x_1, \ldots, x_n) \in X$ is any other feasible solution then

$$f(x) = c'^{\mathsf{T}} x - c' = \sum_{i \notin \mathcal{B}} c'_i x_i - c',$$

while

$$f(\overline{x}_{\mathcal{B}}) = c'^{\mathsf{T}} \overline{x}_{\mathcal{B}} - c' = -c'.$$

Therefore, if for some $i \notin \mathcal{B}$ we have $x_i > 0$ then

$$f(\overline{x}_{\mathcal{B}}) = -c' \leqslant c'^{\mathsf{T}} x - c' = f(x).$$

Otherwise, i.e. if $x_i = 0$ for all $i \notin \mathcal{B}$ then $x = \overline{x}_{\mathcal{B}}$.

Remarks – New Set is Basic

The determinant of square submatrix consisting of columns $s, i_1, \ldots, i_{r-1}, r_{r+1}, \ldots, i_m$ is equal to $\pm a_{rs} \neq 0$ (by definition $a_{rs} > 0$).

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ -x_1 & + & x_2 & \leqslant & 1 \end{cases}$$

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The standard form of this linear programming problem is $f(x_1, x_2, x_3) = -x_1 - 2x_2 \longrightarrow min$ under the constraints

$$\begin{cases} -x_1 & + & x_2 & + & x_3 & = & 1 \end{cases}, \; x_1, x_2, x_3 \geqslant 0$$

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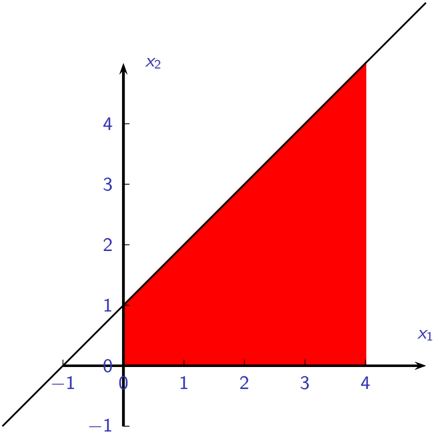
Build the simplex tableau

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & \end{array} \\ \left[\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ \hline -1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

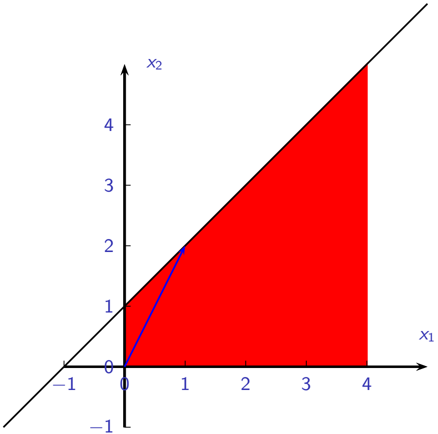
Example



$$\text{maximize } x_1 + 2x_2$$

$$\begin{cases} x_1 \geqslant 0 \\ x_2 \geqslant 0 \\ -x_1 + x_2 \leqslant 1 \end{cases}$$

Example



maximize $x_1 + 2x_2$

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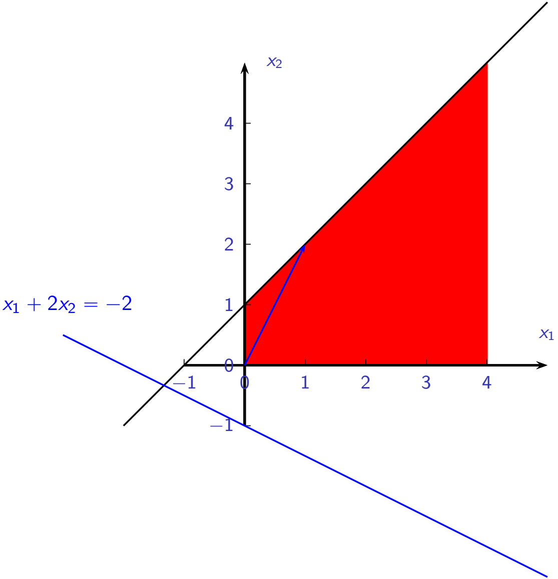
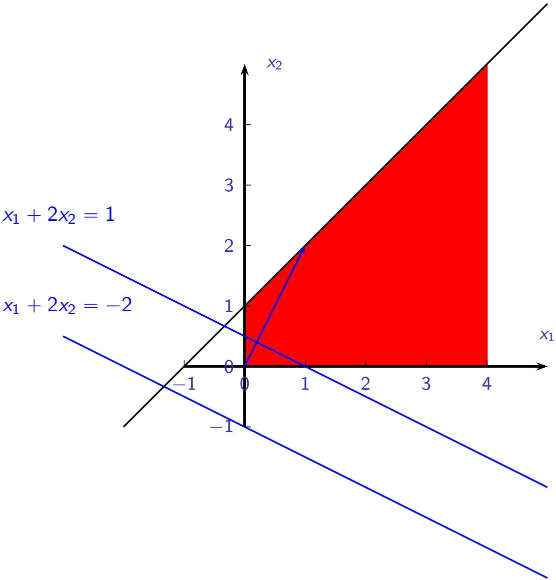
Example



maximize $x_1 + 2x_2$

$$\left\{\begin{array}{rcrcl} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ -x_1 & + & x_2 & \leqslant & 1 \end{array}\right.$$

Example



maximize $x_1 + 2x_2$

$$\begin{cases} x_1 \geqslant 0 \\ x_2 \geqslant 0 \\ -x_1 + x_2 \leqslant 1 \end{cases}$$

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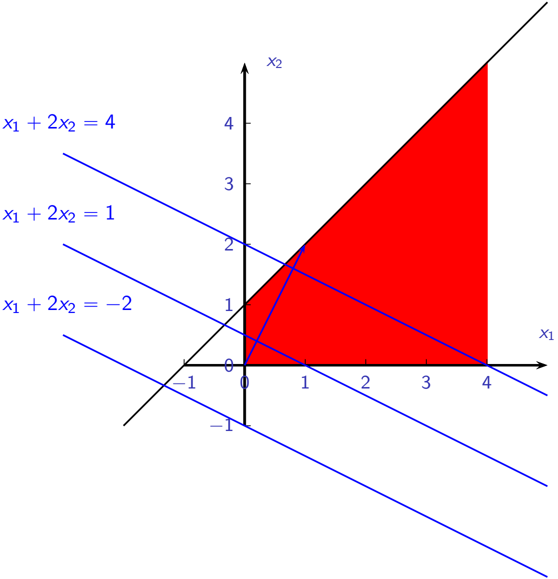

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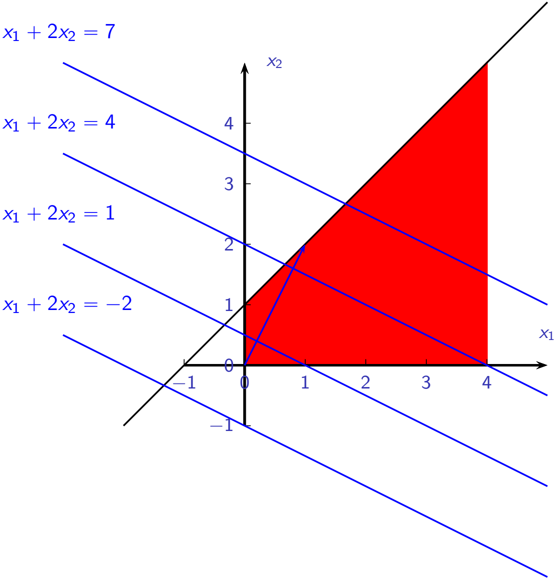

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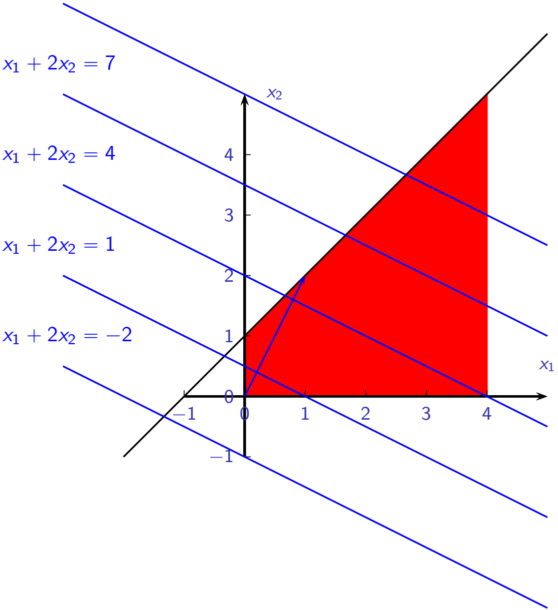

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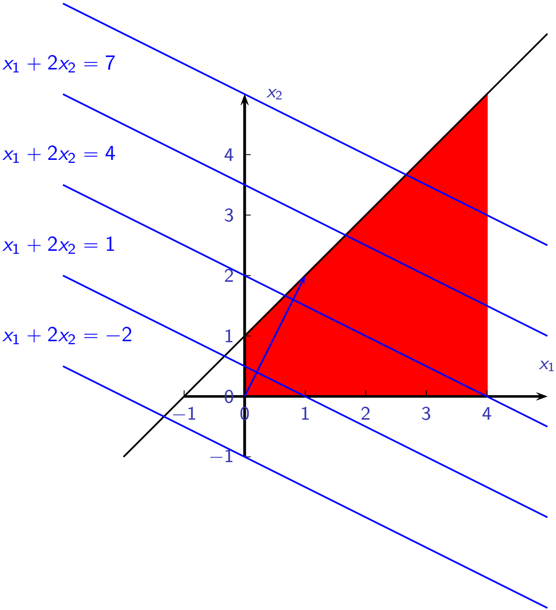

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no optimal solution

Example

Let $\mathcal{B} = \{3\}$ be a basic feasible set.

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Then $s = 2$ since $c'_2 = -2 < -1 = c'_1$. In the second column, in the lower part, there is only one positive element therefore $r = 1$.

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Then $s = 1$ and in the first column, in the lower part, all entries are non-positive. Therefore the objective function does not admit its minimum over the feasible region. In other words, there is no optimal solution.

Example

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When x_1 grows to $+\infty$ the objective function decreases to $-\infty$.

How to Find A Basic Feasible Set?

Given a linear programming problem in the standard form $f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n \longrightarrow min$ under the constraints $Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

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$$A' = [A|I_m] \in M(m \times (n+m); \mathbb{R}) \text{ and } x' = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{bmatrix},$$

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If the minimum of the function g is non-zero then the feasible region of the original problem is empty (there are no vertices).

Otherwise, the feasible region is non-empty and $y_1 = \ldots = y_m = 0$. Let $\mathcal{B}$ be the basic feasible set corresponding to an optimal solution of the auxiliary problem.

How to Find A Basic Feasible Set? (continued)

There are two separate cases:

i) $\mathcal{B} \subset \{1, \ldots, n\}$, i.e. the basic feasible set $\mathcal{B}$ is also a basic feasible set of the original problem,

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If $a'_{lj} = 0$ for all $j \in \{1, \ldots, n\}$ then $r(A) < m$ which contradicts the assumption.

Repeating step *ii*) followed with point 3) of the algorithm one can make all auxiliary variables non-basic.

Example

Find a basic feasible solution of the problem

$$\begin{cases} x_1 + x_2 \geqslant 4 \\ -3x_1 + 2x_2 \geqslant 8 \\ x_1 - x_2 \leqslant 0 \end{cases}$$

After putting it into standard form we use the above method starting from $\mathcal{B} = \{6, 7, 8\}$.

$$\left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 4 \\ -3 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccccccc|c} 1 & -2 & 1 & 1 & -1 & 0 & 0 & 0 & -12 \\ \hline 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 4 \\ -3 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow$$

$s = 2, r = 1, \mathcal{B} = \{2, 7, 8\}$

Example (continued)

$\mathcal{B} = \{2, 7, 8\}$

$$\left[\begin{array}{rrrrrrrr|r} 3 & 0 & -1 & 1 & -1 & 2 & 0 & 0 & -4 \\ \hline 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 4 \\ -5 & 0 & 2 & -1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 4 \end{array}\right] \longrightarrow$$

$s = 3, r = 2, \mathcal{B} = \{2, 3, 8\}$

$$\left[\begin{array}{rrrrrrrr|r} \frac{1}{2} & 0 & 0 & \frac{1}{2} & -1 & 1 & \frac{1}{2} & 0 & -4 \\ \hline -\frac{3}{2} & 1 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 4 \\ -\frac{5}{2} & 0 & 1 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 & 4 \end{array}\right] \xrightarrow{r_0 + r_3}$$

$s = 5, r = 3, \mathcal{B} = \{2, 3, 5\}$

Example (continued)

$\mathcal{B} = \{2, 3, 5\}$

$$\left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline -\frac{3}{2} & 1 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 4 \\ -\frac{5}{2} & 0 & 1 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 & 4 \end{array}\right]$$

Since the minimum is equal to 0, the set $\mathcal{B} = \{2, 3, 5\}$ is basic feasible for the original problem too (and it corresponds to the vertex $\overline{x}_{\mathcal{B}} = (0, 4, 0, 0, 4)$ of the original problem and to the vertex $\overline{x}'_{\mathcal{B}} = (0, 4, 0, 0, 4, 0, 0, 0)$ of the auxiliary problem). Note that for the sake of brevity most elementary row operations were omitted.

Degenerate Linear Programming Problem

Definition

A linear programming problem in the standard form is called **non–degenerate** if for each basic feasible set $\mathcal{B}$

$$x_i > 0 \text{ for } i \in \mathcal{B},$$

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Proof.

There is a finite number of basic feasible solutions and with each step of the algorithm the objective function strictly decreases. $\square$

Cycling

The following example comes from the MIT OpenCourseWare Optimization Methods in Management Science/Operations Research.

$\mathcal{B}_1 = \{5, 6, 7\}, s = 1, r = 1$

$$\left[\begin{array}{ccccccc|c} -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 & 0 \\ \hline \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_2 = \{1, 6, 7\}, s = 2, r = 2$

$$\left[\begin{array}{ccccccc|c} 0 & -4 & -\frac{7}{2} & 33 & 3 & 0 & 0 & 0 \\ \hline 1 & -32 & -4 & 36 & 4 & 0 & 0 & 0 \\ 0 & 4 & \frac{3}{2} & -15 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

Cycling (continued)

$\mathcal{B}_3 = \{1, 2, 7\}, s = 3, r = 1$

$$\left[\begin{array}{ccccccc|c} 0 & 0 & -2 & 18 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 8 & -84 & -12 & 8 & 0 & 0 \\ 0 & 1 & \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_4 = \{2, 3, 7\}, s = 4, r = 1$

$$\left[\begin{array}{ccccccc|c} \frac{1}{4} & 0 & 0 & -3 & -2 & 3 & 0 & 0 \\ \hline -\frac{3}{64} & 1 & 0 & \frac{3}{16} & \frac{1}{16} & -\frac{1}{8} & 0 & 0 \\ \frac{1}{8} & 0 & 1 & -\frac{21}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow$$

Cycling (continued)

$\mathcal{B}_5 = \{3, 4, 7\}, s = 5, r = 1$

$$\left[\begin{array}{ccccccc|c} -\frac{1}{2} & 16 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -\frac{5}{2} & 56 & 1 & 0 & 2 & -6 & 0 & 0 \\ -\frac{1}{4} & \frac{16}{3} & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ \frac{1}{4} & -\frac{16}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 1 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_6 = \{4, 5, 7\}, s = 6, r = 1$

$$\left[\begin{array}{ccccccc|c} -\frac{7}{4} & 44 & \frac{1}{2} & 0 & 0 & -2 & 0 & 0 \\ \hline \frac{1}{6} & -4 & -\frac{1}{6} & 1 & 0 & \frac{1}{3} & 0 & 0 \\ -\frac{5}{4} & 28 & \frac{1}{2} & 0 & 1 & -3 & 0 & 0 \\ -\frac{1}{6} & 4 & \frac{1}{6} & 0 & 0 & -\frac{1}{3} & 1 & 1 \end{array}\right] \longrightarrow$$

Cycling (continued)

$\mathcal{B}_7 = \mathcal{B}_1 = \{5, 6, 7\}, s = 1, r = 1$

$$\left[\begin{array}{rrrrrrr|r} -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 & 0 \\ \hline \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \cdots$$

which is the same basic set we started with, i.e. cycling occurs.

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Proof.

Assume on the contrary, with the Bland's rule cycling occurs and there is a sequence of basic feasible solutions

$$\mathcal{B}_1 \to \mathcal{B}_2 \to \ldots \to \mathcal{B}_l \to \mathcal{B}_1.$$

It follows that the objective function does not decrease and each entering variable is equal to 0 (i.e. the basic feasible solution $\overline{x}_{\mathcal{B}_i}$ remain constant).

Bland's Rule (continued)

Proof.

We call a variable x_i **fickle** if $x_i \in B_j$ and $x_i \notin B_{j'}$ for some $1 \leqslant j, j' \leqslant l$. Let x_t will be the fickle variable with the largest possible t. Let $1 \leqslant f \leqslant l$ be such number that

$$t = i_p \in B_f = \{i_1, \ldots, i_m\}, \quad t \notin B_{f+1},$$

that is x_t leaves the basic set B_f (where by convention $l+1$ means 1). Let c'_j, c', b'_i, a'_{ij} refer to the data of the simplex tableau from step 3) of the simplex algorithm for the basic feasible set B_f. Let $s \in B_{f+1} \backslash B_f$ be the entering variable in the step $B_f \to B_{f+1}$. Therefore

$$c'_s < 0, \text{ and } s < t.$$

Since t leaves B_f (and x_t is fickle)

$$a'_{ps} > 0, \quad b'_p = 0.$$

Since the p-th basic variable leaves, i.e. $x_{i_p} = x_t$ the p-th ratio is the smallest one. As x_t was fickle so $b'_p = 0$.

Bland's Rule (continued)

Proof.

At some step the variable x_t reenters some basic feasible set. Let B_g be a basic feasible set such that $t \in B_{g+1} \backslash B_g$, i.e. x_t is the entering variable in the step $\mathcal{B}_g \to \mathcal{B}_{g+1}$. Let $c_j^*, c^*, b_i^*, a_{ij}^*$ refer to the data of the simplex tableau from step 3) of the simplex algorithm for the basic feasible set B_g. Therefore

$$c_t^* < 0.$$

Consider a family of (possibly infeasible) solutions of the system $Ax = b$

$$\begin{cases} x_s = y, \\ x_i = 0 \text{ for } i \notin B_f \cup \{s\}, \\ x_{i_k} = b'_k - a'_{ks} y \text{ for } i_k \in B_f. \end{cases}$$

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Since two expressions for the objective function are the same on the set of all solutions $Ax = b$ (without the assumption $x \geqslant 0$), for any $y \in \mathbb{R}$

$$c'_s y - c' = c_s^* y + \sum_{\substack{k \notin B_g \\ k \neq s}} c_k^* x_k - c^* = c_s^* y + \sum_{i_k \in B_f \backslash B_g} c_{i_k}^* (b'_k - a'_{ks} y) - c^*,$$

Bland's Rule (continued)

Proof.

By comparing the left hand side (objective function expressed with the data for B_f) with the right hand side (objective function expressed with the data for B_g with values given by the family, in particular $x_i = 0$ for $i \notin B_f \cup \{s\}$)

$$c'_s y - c' = c^*_s y + \sum_{i_k \in B_f \setminus B_g} c^*_{i_k}(b'_k - a'_{ks} y) - c^*,$$

and rearranging ($c' = c^*$ as the value of the objective function does not change in the cycle)

$$\left(c'_s - c^*_s + \sum_{i_k \in B_f \setminus B_g} c^*_{i_k} a'_{ks} \right) y = \sum_{i_k \in B_f \setminus B_g} c^*_{i_k} b'_k,$$

we see that the right hand side does not depend on y hence the coefficient at y on the left hand side is equal to 0, i.e.

$$c'_s - c^*_s + \sum_{i_k \in B_f \setminus B_g} c^*_{i_k} a'_{ks} = 0.$$

Bland's Rule (continued)

Proof.

(note that $t \in B_f \backslash B_g$) which gives

$$c_s' - c_s^* + \sum_{i_k \in B_f \backslash B_g} c_{i_k}^* a_{ks}' = 0.$$

Bland's Rule (continued)

Proof.

(note that $t \in B_f \backslash B_g$) which gives

$$c'_s - c^*_s + \sum_{i_k \in B_f \backslash B_g} c^*_{i_k} a'_{ks} = 0.$$

Since x_s is not the entering variable in the step $\mathcal{B}_g \to \mathcal{B}_{g+1}$ and $s < t$ we have $c^*_s \geqslant 0$ (otherwise, by Bland's rule, x_s would enter the set $\mathcal{B}_{g+1}$). It was shown before that $c'_s < 0$, therefore for some $i_q \in B_f \backslash B_g$ (i.e. x_{i_q} is fickle)

$$c^*_{i_q} a'_{qs} > 0.$$

This implies that $c^*_{i_q} \neq 0$. We have seen that for $t = i_p$

$$c^*_{i_p} < 0 \text{ and } a'_{ps} > 0,$$

therefore $i_q \neq i_p = t$.

Bland's Rule (continued)

Proof.

By the choice of t

$$i_q < t = i_p$$

and x_{i_q} is not the entering variable in the step $\mathcal{B}_g \to \mathcal{B}_{g+1}$ (as x_t is), hence $c^*_{i_q} > 0$ (by the Bland's rule) and $q < p$ (as $i_q < i_p$). Variable x_{i_q} is fickle and we have shown

$$c^*_{i_q} a'_{qs} > 0,$$

which gives

$$a'_{qs} > 0 \text{ and } b'_q = 0.$$

Bland's Rule (continued)

Proof.

By the choice of t

$$i_q < t = i_p$$

and x_{i_q} is not the entering variable in the step $\mathcal{B}_g \to \mathcal{B}_{g+1}$ (as x_t is), hence $c^*_{i_q} > 0$ (by the Bland's rule) and $q < p$ (as $i_q < i_p$). Variable x_{i_q} is fickle and we have shown

$$c^*_{i_q} a'_{qs} > 0,$$

which gives

$$a'_{qs} > 0 \text{ and } b'_q = 0.$$

This leads to contradiction, as the ratios $\frac{b'_q}{a'_{qs}} = \frac{b'_p}{a'_{ps}} = 0$ are the smallest, therefore, in the step $\mathcal{B}_f \to \mathcal{B}_{f+1}$, the leaving variable should be x_{i_q} and not $x_{i_p} = x_t$.

Example with Cycling Revisited

Consider the previous example with cycling. Note that for the steps $\mathcal{B}_1 \to \ldots \to \mathcal{B}_5$ we have been using the Bland's rule.

Example with Cycling Revisited

Consider the previous example with cycling. Note that for the steps $\mathcal{B}_1 \to \ldots \to \mathcal{B}_5$ we have been using the Bland's rule.
$\mathcal{B}_5 = \{3, 4, 7\}, s = 1, r = 1$

$$\left[\begin{array}{rrrrrrr|r} -\frac{1}{2} & 16 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -\frac{5}{2} & 56 & 1 & 0 & 2 & -6 & 0 & 0 \\ -\frac{1}{4} & \frac{16}{3} & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ \frac{1}{4} & -\frac{16}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 1 & 1 \end{array}\right] \longrightarrow$$

Now choose $s = 1$ (Bland's rule) instead of $s = 5$.

$$\left[\begin{array}{rrrrrrr|r} 0 & \frac{16}{3} & 0 & 0 & -\frac{5}{3} & \frac{7}{3} & 2 & 2 \\ \hline 1 & -\frac{64}{3} & 0 & 0 & -\frac{4}{3} & \frac{8}{3} & 4 & 4 \\ 0 & \frac{8}{3} & 1 & 0 & -\frac{4}{3} & \frac{2}{3} & 10 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right].$$

The linear programming problem has no optimal solution.

Linear Programming Complexity/Klee–Minty Cube

The following linear programming problem may require $2^n - 1$ steps to finish (when starting from the basic feasible set coresponding to the vertex $(0, \ldots, 0)$)

$$\sum_{i=1}^{n} x_i \to \max$$

with constraints

$$\begin{cases} x_1 & \leqslant & 2^1 - 1, \\ 2x_1 + x_2 & \leqslant & 2^2 - 1, \\ 2x_1 + 2x_2 + x_3 & \leqslant & 2^3 - 1, \\ & \vdots & \\ 2x_1 + \ldots + 2x_{n-1} + x_n & \leqslant & 2^n - 1, \\ x_1, \ldots, x_n \geqslant 0. \end{cases}$$

This is a variant of so called Klee–Minty cube and comes from T. Kitahara and S. Mizuno.

Klee–Minty Cube for $n = 3$

$\mathcal{B}_1 = \{4, 5, 6\}, s = 1, r = 1$

$$\left[\begin{array}{rrrrrr|r} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & 3 \\ 2 & 2 & 1 & 0 & 0 & 1 & 7 \end{array}\right] \longrightarrow$$

$\mathcal{B}_2 = \{1, 5, 6\}, s = 2, r = 2$

$$\left[\begin{array}{rrrrrr|r} 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 2 & 1 & -2 & 0 & 1 & 5 \end{array}\right] \longrightarrow$$

$\mathcal{B}_3 = \{1, 2, 6\}, s = 4, r = 1$

$$\left[\begin{array}{rrrrrr|r} 0 & 0 & -1 & -1 & 1 & 0 & 2 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 & 3 \end{array}\right] \longrightarrow$$

Klee–Minty Cube for $n = 3$ (continued)

$\mathcal{B}_4 = \{2, 4, 6\}, s = 3, r = 3$

$$\left[\begin{array}{rrrrrr|r} 1 & 0 & -1 & 0 & 1 & 0 & 3 \\ \hline 2 & 1 & 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & -2 & 1 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_5 = \{2, 3, 4\}, s = 1, r = 3$

$$\left[\begin{array}{rrrrrr|r} -1 & 0 & 0 & 0 & -1 & 1 & 4 \\ \hline 2 & 1 & 0 & 0 & 1 & 0 & 3 \\ -2 & 0 & 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_6 = \{1, 2, 3\}, s = 5, r = 2$

$$\left[\begin{array}{rrrrrr|r} 0 & 0 & 0 & 1 & -1 & 1 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -2 & 1 & 3 \end{array}\right] \longrightarrow$$

Klee–Minty Cube for $n = 3$ (continued)

$\mathcal{B}_7 = \{1, 3, 5\}, s = 4, r = 1$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & -1 & 0 & 1 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -2 & 0 & 1 & 5 \\ 0 & 1 & 0 & -2 & 1 & 0 & 1 \end{array}\right] \longrightarrow$$

$\mathcal{B}_8 = \{3, 4, 5\}$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 1 & 7 \\ \hline 2 & 2 & 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \longrightarrow$$

the optimal solution is

$$\overline{x}_{\mathcal{B}_8} = (0, 0, 7, 1, 3, 0),$$

and $f(\overline{x}_{\mathcal{B}_8}) = 7$.

Klee–Minty Cube for $n = 3$ (no Bland's rule)

Remark

Note that using the Bland's rule the algorithm requires less steps, i.e.

$$\mathcal{B}_1 \to \mathcal{B}_2 \to \mathcal{B}_6 \to \mathcal{B}_7 \to \mathcal{B}_8.$$

However, there are known examples of exponential complexity for the Bland's rule.

Better Methods

The interior–point method (or barrier method) can be slower for small examples but for the big ones could be much faster than the simplex method. However, the solution is approximate.