Linear Algebra

Lecture 12 - Linear Programming

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13 January 2025

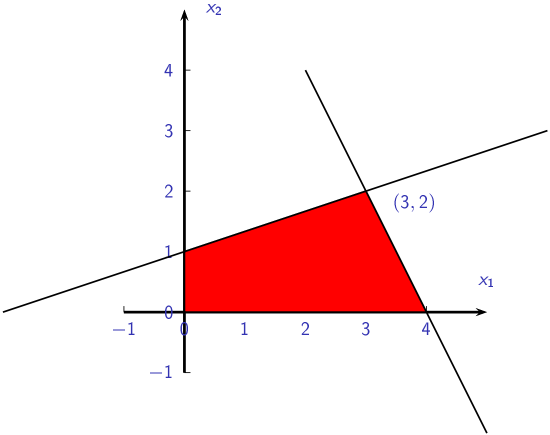
What is Linear Programming?

Example

Maximize the value $x_1 + 2x_2$ under the constraints

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

What is Linear Programming?



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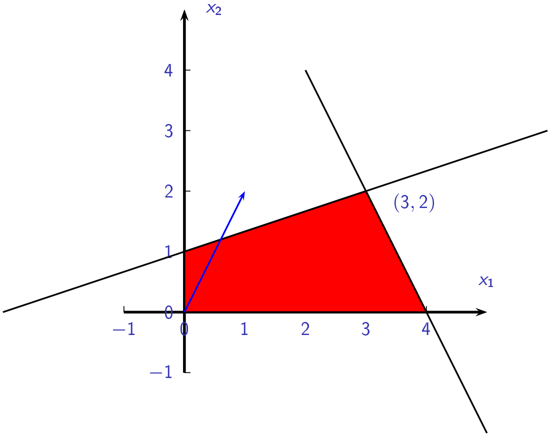

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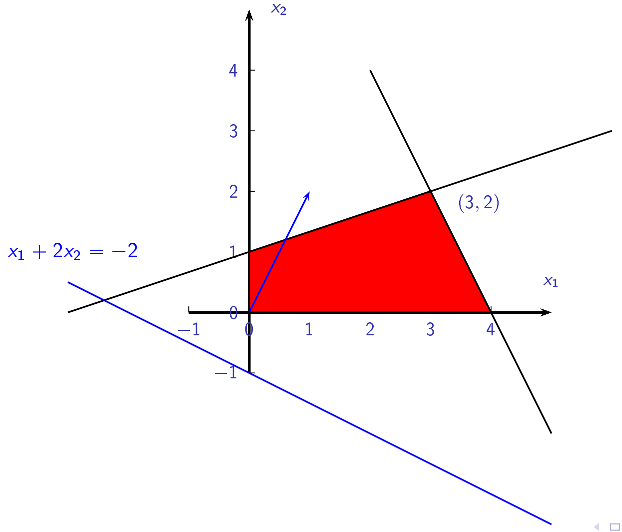

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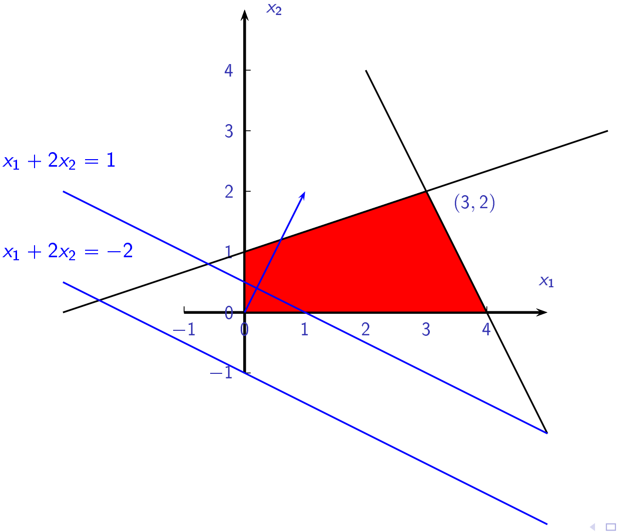

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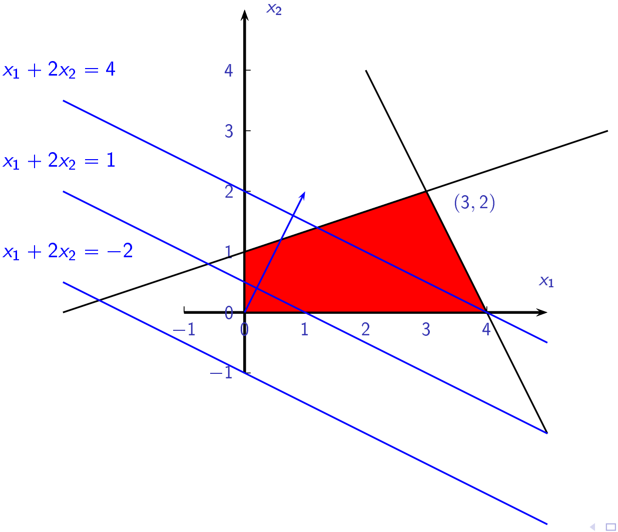

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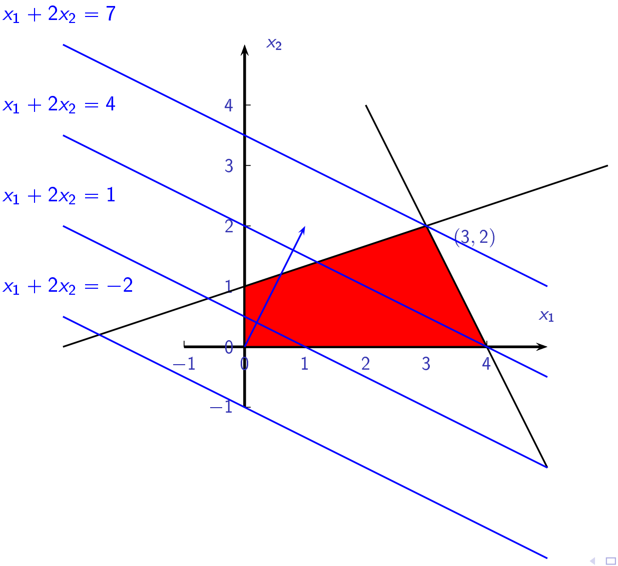

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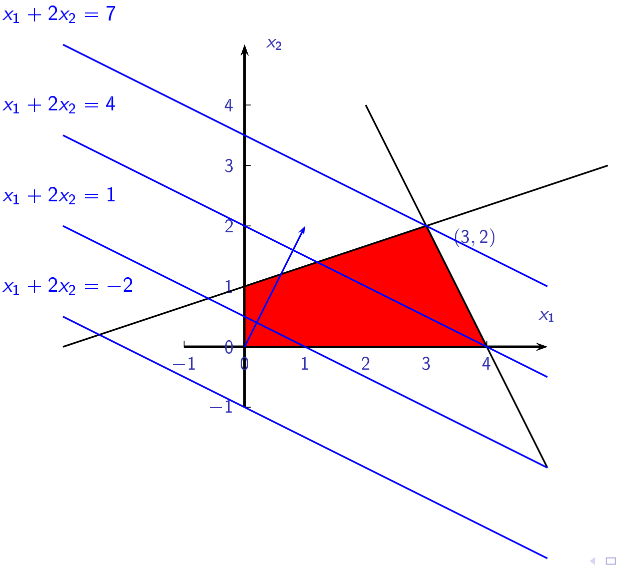

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optimal solution is $(3, 2)$

What is Linear Programming?

Example

Maximize the value $x_1 + 2x_2$ under the constraints

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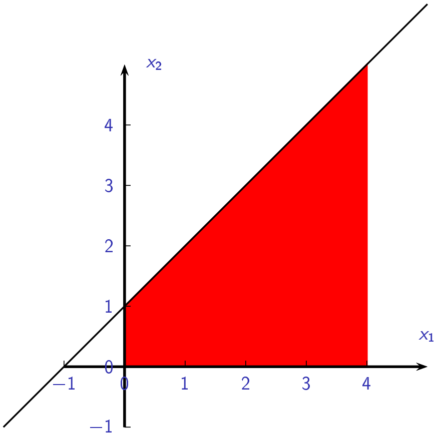

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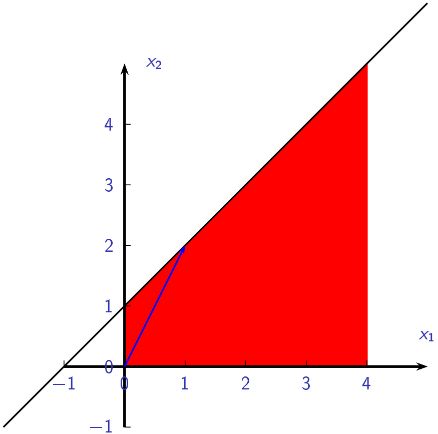

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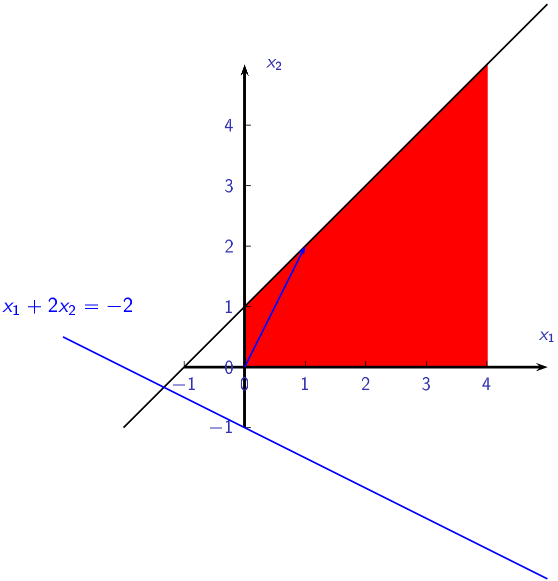

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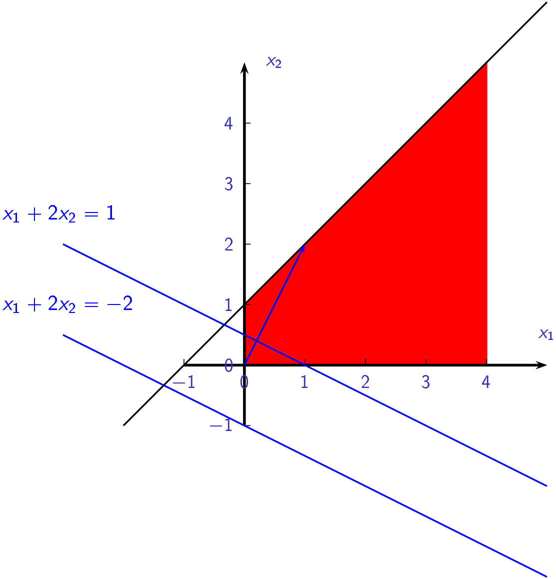

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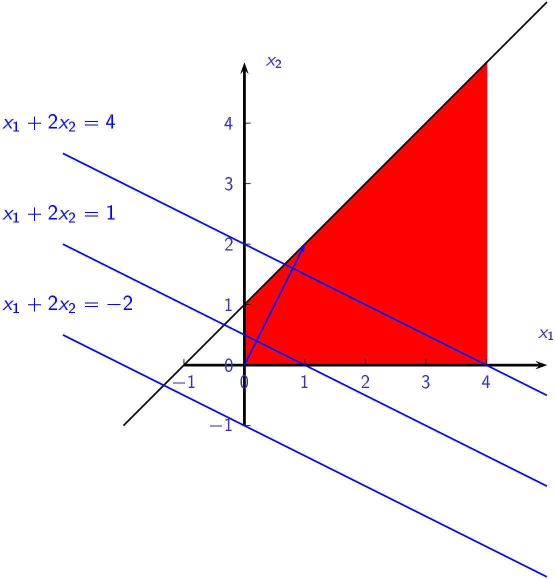

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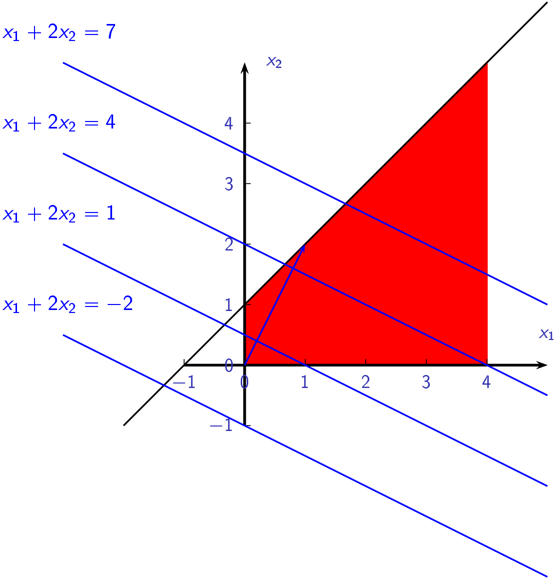

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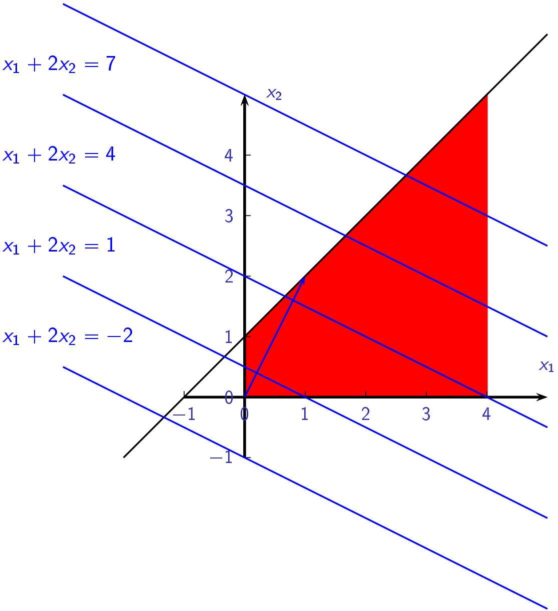

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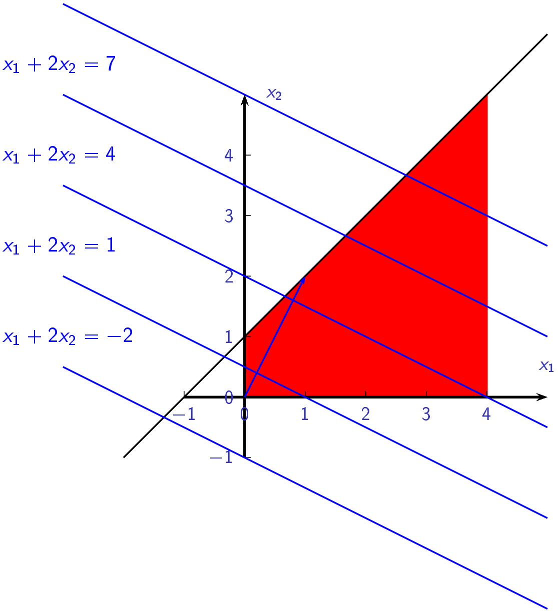

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$$\begin{cases} x_1 \geqslant 0 \\ x_2 \geqslant 0 \\ -x_1 + x_2 \leqslant 1 \end{cases}$$

no optimal solution

Economy and Economical

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Careful management of available resources.

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from Greek

oikonomia=household management, housekeeping

Linear Programming Problem

Definition

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That is, we look for the maximal or the minimal value of the function $f((x_1, x_2, \ldots, x_n)) = c_1x_1 + c_2x + \ldots + c_nx_n$ on the set $X \subset \mathbb{R}^n$ of points satisfying the system of linear inequalities, i.e.

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leqslant & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leqslant & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leqslant & b_m \end{cases}$$

Linear Programming Problem (continued)

Those conditions (also called **constraints**) can be written in a concise form. Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

This is an example of a global optimization problem with (inequality) constraints

Linear Programming Problem (continued)

The linear programming problem can be written in the form: maximize (or minimize) the linear function $f(x) = c^{\mathsf{T}}x$ over the set $X \subset \mathbb{R}^n$ given by

$$Ax \leqslant b.$$

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Remark

Since

$$a_1x_1 + \ldots + a_nx_n = b \Longleftrightarrow \begin{cases} a_1x_1 + \ldots + a_nx_n \leqslant & b \\ -a_1x_1 - \ldots - a_nx_n \leqslant & -b \end{cases}$$

a set given by a finite number of linear equations and finite number of inequalities can be expressed by a finite number of inequalities.

Real Life Applications - Transportation Problem

A firm stores some goods at l supply centers and ships those goods to k markets. The cost of transporting a unit of those goods from the i-th supply center to the j-th market is a_{ij}. Each market demands at least of b_j units of those goods. Each supply center produces at most w_i units of goods.

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Introduce $l \times k$ variables x_{ij} for $i = 1, \ldots, l$ and $j = 1, \ldots, k$ denoting the amount of the transport from the i-th supply center to the j-th market. We want to minimize the cost of transport and to satisfy demands of all markets.

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$$\begin{cases} x_{11} + x_{12} + x_{13} + \ldots + x_{1k} \leqslant w_1 \\ x_{21} + x_{22} + x_{23} + \ldots + x_{2k} \leqslant w_2 \\ \vdots \\ x_{l1} + x_{l2} + x_{l3} + \ldots + x_{lk} \leqslant w_l \end{cases}$$

i.e. no supply center cannot supply more than w_i of goods and

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$$\begin{cases} x_{11} + x_{21} + x_{31} + \ldots + x_{l1} \geqslant b_1 \\ x_{12} + x_{22} + x_{32} + \ldots + x_{l2} \geqslant b_2 \\ \vdots \\ x_{1k} + x_{2k} + x_{3k} + \ldots + x_{lk} \geqslant b_k \end{cases}$$

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i.e. the demand of each market is satisfied. We want to transport from a supply center to a market so we assume

$$x_{ij} \geqslant 0 \text{ for } i = 1, \ldots, l \text{ and } j = 1, \ldots, k.$$

Real Life Application - Diet Problem

Suppose there are n foods available. The cost of serving per j-th food is q_j. Assume there are k nutrients and each serving of j-th type of food contains z_{ij} units of the i-th nutrient.

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$$\begin{cases} z_{11}x_1 + z_{12}x_2 + z_{13}x_3 + \ldots + x_{1n}x_n \geqslant N_1 \\ z_{21}x_1 + z_{22}x_2 + z_{23}x_3 + \ldots + x_{2n}x_n \geqslant N_2 \\ \vdots \\ z_{k1}x_1 + z_{k2}x_2 + z_{k3}x_3 + \ldots + x_{kn}x_n \geqslant N_k \end{cases}$$

Real Life Applications - Diet Problem

If needed one may add another constraints for the minimal or maximal amount of servings of each type of food.

Real Life Applications - Diet Problem

If needed one may add another constraints for the minimal or maximal amount of servings of each type of food. A similar problem was considered in 1930s and 1940s in order to find an optimal diet for the US soldiers.

Real Life Applications

And many more: portfolio optimization, network design, vehicle routing.

Convex Set

Definition

For any $p, q \in \mathbb{R}^n$ the **line segment** joining p and q is the set

$$[p, q] = \{tp + (1 - t)q \in \mathbb{R}^n \mid t \in [0, 1]\}.$$

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Definition

A set $X \subset \mathbb{R}^n$ is **convex** if

$$[p, q] \subset X \text{ for any } p, q \in X.$$

Open and Closed Ball

Definition

An open ball with center $x \in \mathbb{R}^n$ and the radius $r > 0$ is the set

$$B(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}.$$

A closed ball with center $x \in \mathbb{R}^n$ and the radius $r > 0$ is the set

$$\overline{B}(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| \leqslant r\}.$$

Convex Set (continued)

Proposition

A ball is a convex set.

Convex Set (continued)

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Proof.

Let $p, q \in B(x, r)$. Then for any $t \in [0, 1]$

$$\|x - (tp + (1 - t)q)\| = \|t(x - p) + (1 - t)(x - q)\| \leqslant$$

$$\leqslant t\|x - p\| + (1 - t)\|x - q\| < r,$$

that is

$$[p, q] \subset B(x, r).$$

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The same proof works for a closed ball. $\square$

Convex Set (continued)

Proposition

Intersection of a family of convex sets is a convex set. In particular, if $X_1, \ldots, X_m \subset \mathbb{R}^n$ are convex sets then

$$X_1 \cap \ldots \cap X_m = \{x \in \mathbb{R}^n \mid x \in X_i \text{ for } i = 1, \ldots, m\},$$

is a convex set.

Half-space

Definition

A **half-space** $H_\leqslant \subset \mathbb{R}^n$ is a set given by a single inequality, that is

$$H_\leqslant = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \ldots + a_nx_n \leqslant b\}.$$

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Proposition

A half-space is a convex set.

Proof.

Let $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \in H_\leqslant$. Then for any $t \in [0, 1]$

$$a_1(tp_1 + (1-t)q_1) + a_2(tp_2 + (1-t)q_2) + \ldots + a_n(tp_n + (1-t)q_n) =$$

$$t(a_1 p_1 + a_2 p_2 + \ldots + a_n p_n) + (1-t)(a_1 q_1 + a_2 q_2 + \ldots + a_n q_n) \leqslant tb + (1-t)b = b,$$

i.e.

$$[p, q] \subset H_\leqslant.$$

$\square$

Open Sets, Closed Sets

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Example

An open ball is an open set. A closed ball is a closed set. A half-space is a closed set.

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Proof.

It is enough to show that $\mathbb{R}^n \backslash H_{\leqslant}$ is an open set. If $x \in \mathbb{R}^n \backslash H_{\leqslant}$ let $r = d(x, H)$ be the distance of x from H. Then

$$B(x, r) \subset \mathbb{R}^n \backslash H_{\leqslant}.$$

$\square$

Open Sets, Closed Sets (continued)

Proposition

Let $X_1, \ldots, X_m \subset \mathbb{R}^n$ *be open sets. Then the sets*

$$X_1 \cup \ldots \cup X_m \subset \mathbb{R}^n,$$

and

$$X_1 \cap \ldots \cap X_m \subset \mathbb{R}^n,$$

are open.

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Proof.

Let $x \in X_1 \cup \ldots \cup X_m \subset \mathbb{R}^n$. Then $x \in X_i$ for some i. Since X_i is open there exists $r > 0$ such that

$$B(x, r) \subset X_i \subset X_1 \cup \ldots \cup X_m.$$

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If $x \in X_i$ for $i = 1, \dots, m$, then there exist $r_i > 0$ such that $B(x, r_i) \subset X_i$ for $i = 1, \dots, m$. Let $r = \min\{r_1, \dots, r_m\}$. Then

$$B(x, r) \subset X_1 \cap \ldots \cap X_m.$$

Open Sets, Closed Sets (continued)

Corollary

Let $X_1, \ldots, X_m \subset \mathbb{R}^n$ *be closed sets. Then the sets*

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are closed.

Proof.

$$\mathbb{R}^n \backslash (X_1 \cup \ldots \cup X_m) = (\mathbb{R}^n \backslash X_1) \cap \ldots \cap (\mathbb{R}^n \backslash X_m),$$
$$\mathbb{R}^n \backslash (X_1 \cap \ldots \cap X_m) = (\mathbb{R}^n \backslash X_1) \cup \ldots \cup (\mathbb{R}^n \backslash X_m),$$

where $\mathbb{R}^n \backslash X_i$ are open. $\square$

Open Sets, Closed Sets (continued)

Corollary

Let $X_1, \ldots, X_m \subset \mathbb{R}^n$ *be closed sets. Then the sets*

$$X_1 \cup \ldots \cup X_m \subset \mathbb{R}^n,$$

and

$$X_1 \cap \ldots \cap X_m \subset \mathbb{R}^n,$$

are closed.

Proof.

$$\mathbb{R}^n \backslash (X_1 \cup \ldots \cup X_m) = (\mathbb{R}^n \backslash X_1) \cap \ldots \cap (\mathbb{R}^n \backslash X_m),$$
$$\mathbb{R}^n \backslash (X_1 \cap \ldots \cap X_m) = (\mathbb{R}^n \backslash X_1) \cup \ldots \cup (\mathbb{R}^n \backslash X_m),$$

where $\mathbb{R}^n \backslash X_i$ are open. □

Remark

In general, the union of any family of open sets is an open set and the intersection of any family of closed sets is a closed set.

Convex Polytopes

Definition

Convex polytope $X \subset \mathbb{R}^n$ is a non–empty set of solutions of a system of linear inequalities, i.e.

$$X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid Ax \leqslant b\},$$

where $A \in M(m \times n; \mathbb{R})$. Equivalently, it is a non–empty intersection of finite number of half-spaces.

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Convex polytope is a closed convex set.

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Proposition

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Proof.

It is an intersection of closed convex sets. $\square$

Polyhedra

Remark

Sometimes a different terminology is used: a **polyhedron** *(or a* **polyhedral set***) is a set of solutions of a system $Ax \leqslant b$ and a* **polytope** *is a bounded polyhedron.*

Compact Set

Definition

Set $X \subset \mathbb{R}^n$ is **bounded** if there exists $x \in \mathbb{R}^n$ and $r > 0$ such that

$$X \subset B(x, r).$$

Definition

Set $X \subset \mathbb{R}^n$ is **compact** if it is **closed** and **bounded**.

Extreme Value Theorem

Theorem

Let $X \subset \mathbb{R}^n$ be a compact set and let

$$f : X \to \mathbb{R},$$

be a continuous function. Let

$$m = \inf_{x \in X} f(x), \quad M = \sup_{x \in X} f(x).$$

Then there exist $x_m, x_M \in X$ such that

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Remark

Linear functions are continuous.

Convex Polytopes (continued)

Suppose we are given a linear programming problem with constraints $Ax \leqslant b$ and the objective function $f(x) = c^{\mathsf{T}}x \longrightarrow min$.

Convex Polytopes (continued)

Suppose we are given a linear programming problem with constraints $Ax \leqslant b$ and the objective function $f(x) = c^{\mathsf{T}}x \longrightarrow min$.

Definition

A **feasible region** (also a **feasible set)** is the set of all points $X \subset \mathbb{R}^n$ satisfying the conditions $Ax \leqslant b$. An optimal solution is any point $\overline{x} \in X$ such that $f(\overline{x}) \leqslant f(x)$ for any $x \in X$.

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A feasible region is a convex polytope.

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Remark

A feasible region is a convex polytope. If it is bounded (i.e. contained in a ball) then there exists an optimal solution. An optimal solution may not be unique.

Supporting Hyperplane

Definition

Let $X \subset \mathbb{R}^n$ be a convex closed set. A **supporting hyperplane** of X is a hyperplane

$$H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + \ldots + a_n x_n = b\},$$

such that $(a_1, \ldots, a_n) \neq 0$

$$H \cap X \neq \emptyset \text{ and } X \subset H_{\leqslant}.$$

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A **face** of a convex polytope X is the intersection of X with a supporting hyperplane.

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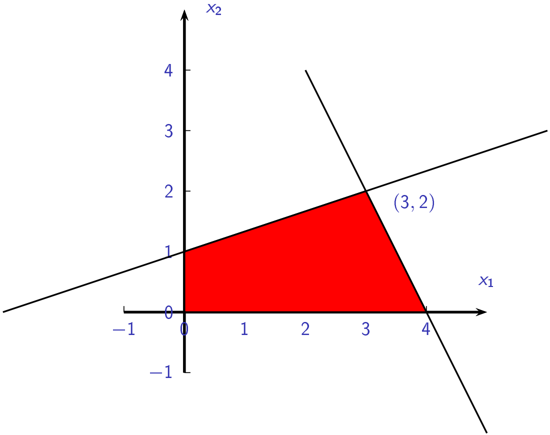
Definition

A **face** of a convex polytope X is the intersection of X with a supporting hyperplane.

Remark

A face of a convex polytope is a convex polytope.

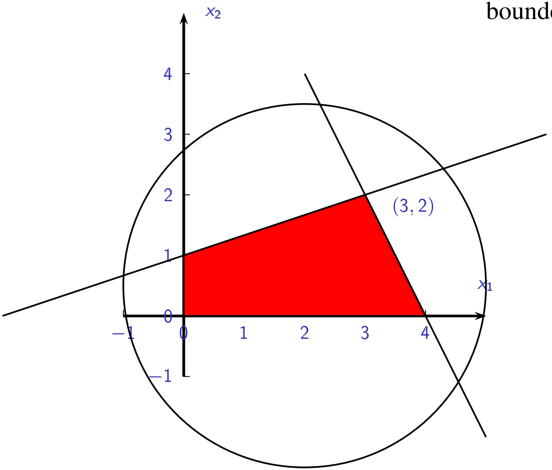
Example



$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

Example

bounded $\Rightarrow$ optimal solution exists



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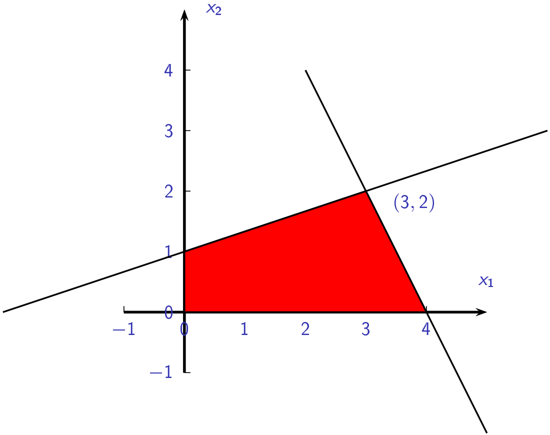

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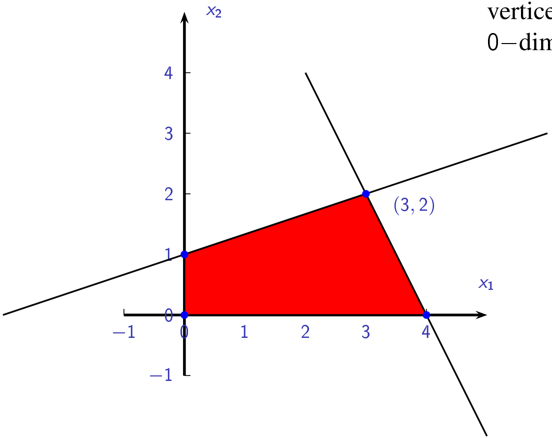

vertices, i.e.
0−dimensional faces

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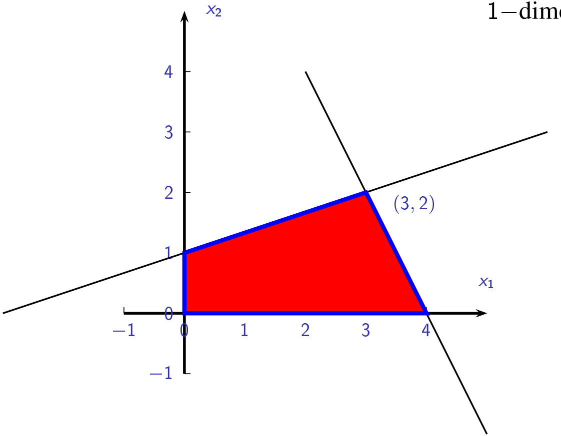

1−dimensional faces

$$\begin{cases} x_1 & \geqslant & 0 \\ x_2 & \geqslant & 0 \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

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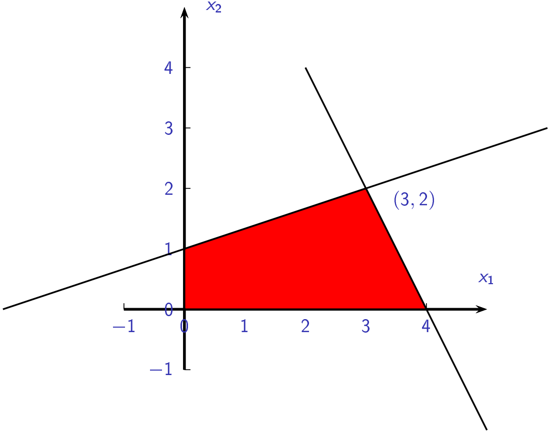

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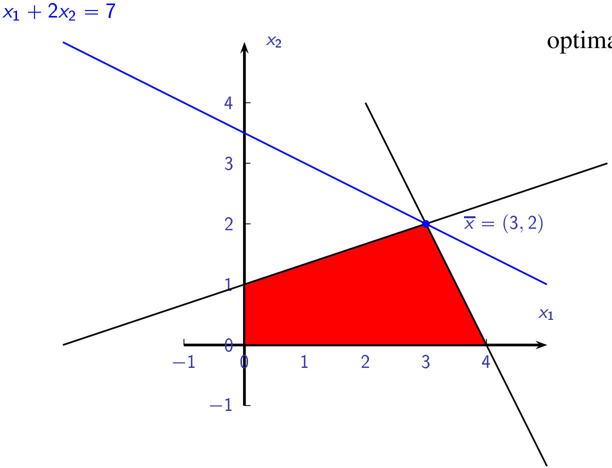

optimal solution is a vertex

maximize $x_1 + 2x_2$

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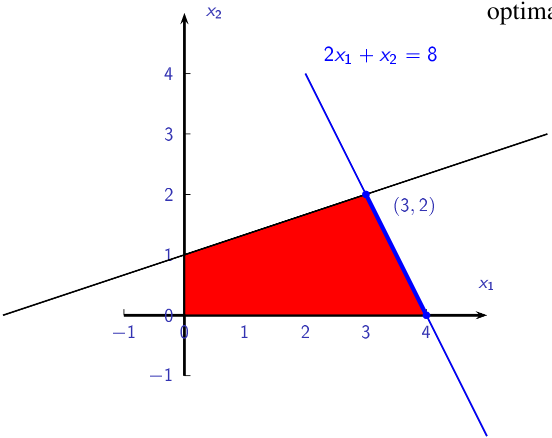

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Extreme Points

Definition

Let $X \subset \mathbb{R}^n$ be a convex set. Point $x \in \mathbb{R}^n$ is an **extreme point** of X if for any $p, q \in X$

$$\text{if } x \in [p, q] \text{ then } x = p \text{ or } x = q.$$

Extreme Points of a Convex Polytope

Definition

Let $X \subset \mathbb{R}^n$ be a convex polytope given by $Ax \leqslant b$, where $A \in M(m \times n; \mathbb{R})$, $b \in \mathbb{R}^m$. Let $a_1, \ldots, a_m \in \mathbb{R}^n$ denote the rows of matrix A. For any $p \in X$ denote by

$$J(p) = \{i \in \{1, \ldots, m\} \mid a_i^T p = b_i\}$$

the set of **active constraints**. Let $A_{J(p)}$ denote the submatrix of matrix A consisting of rows of A indexed by the set $J(p)$, the same for $b_{J(p)}$. In particular

$$A_{J(p)} p = b_{J(p)}.$$

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Proposition

Let $X \subset \mathbb{R}^n$ be a convex polytope given by $Ax \leqslant b$ and let $p \in X$ be a point. Then p is an extreme point of X if and only if $r(A_{J(p)}) = n$.

Extreme Points of a Convex Polytope (continued)

Proof.

Let $p \in X$ be an extreme point of X. Assume $r(A_{J(p)}) < n$. Then, by the Kronecker–Capelli theorem, there exists a non–zero solution $q \in \mathbb{R}^n$, that is

$$A_{J(p)} q = 0 \text{ and } q \neq 0.$$

Let $a_1, \ldots, a_m \in \mathbb{R}^n$ denote the rows of matrix A. For sufficiently small $t \in \mathbb{R}$, $t \neq 0$

$$\left| t(a_i^\top q) \right| < b_i - a_i^\top p \text{ for any } i \notin J(p),$$

which gives $p \pm tq \in X$ since $a_i^\top (p \pm tq) = b_i$ for $i \in J(p)$. Then $p \neq p \pm tq$ and $p \in [p - tq, p + tq]$ because

$$p = \frac{1}{2}(p - tq) + \frac{1}{2}(p + tq),$$

which leads to a contradiction with $p \in X$ being an extreme point.[1]

[1] Proof based on N. Lauritzen, *Lectures on Convex Sets*.

Extreme Points of a Convex Polytope (continued)

Proof.

Assume that $r(A_{J(p)}) = n$ and let $p = tp_1 + (1-t)p_2$ for some $t \in (0,1), p_1, p_2 \in X$, where $p_1 \neq p_2$. Then

$$b_{J(p)} = A_{J(p)}p = tA_{J(p)}p_1 + (1-t)A_{J(p)}p_2 \leqslant b_{J(p)},$$

which implies

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Assume that $r(A_{J(p)}) = n$ and let $p = tp_1 + (1-t)p_2$ for some $t \in (0,1)$, $p_1, p_2 \in X$, where $p_1 \neq p_2$. Then

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Since $r(A_{J(p)}) = n$ the system of linear equations $A_{J(p)}x = b_{J(p)}$ has a unique solution hence $p = p_1 = p_2$. By contradiction, either $t = 0$ or $t = 1$. $\square$

Extreme Points of a Convex Polytope (continued)

Corollary

Let $X \subset \mathbb{R}^n$ *be a convex polytope given by* $Ax \leqslant b$, *where* $A \in M(m \times n; \mathbb{R}), b \in \mathbb{R}^m$. *Then if* $X \neq \varnothing$

$$X \text{ has no extreme points} \iff r(A) < n.$$

Extreme Points of a Convex Polytope (continued)

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Let $X \subset \mathbb{R}^n$ be a convex polytope given by $Ax \leqslant b$, where $A \in M(m \times n; \mathbb{R}), b \in \mathbb{R}^m$. Then if $X \neq \varnothing$

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$(\Leftarrow)$ follows from the previous proposition,

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Proof.

($\Leftarrow$) follows from the previous proposition,

($\Rightarrow$) let $p \in X$ be some point, if $J(p) = \{1, \ldots, m\}$ then $A = A_{J(p)}$ and $r(A) < n$ since p in not an extreme point. If $J(p) \subsetneq \{1, \ldots, m\}$ then $r(A_{J(p)}) < n$ and there exist $q \in \mathbb{R}^n$, such that

$$A_{J(p)}q = 0 \text{ and } q \neq 0.$$

If $a_i^\intercal q = 0$ for $i \notin J(p)$ then $Aq = 0$ and $r(A) < n$. If $a_i^\intercal q < 0$ for all $i \notin J(p)$ then one can replace q with $-q$.

Extreme Points of a Convex Polytope (continued)

Proof.

Let

$$t = \min \left\{ \frac{b_i - a_i^\mathsf{T} p}{a_i^\mathsf{T} q} \in \mathbb{R} \mid i \notin J(p) \text{ and } a_i^\mathsf{T} q > 0 \right\}.$$

Then $t > 0$, $p + tq \in X$ and $J(p) \subsetneq J(p + tq)$. Eventually, by replacing p with $p + tq$ as above, one can find $p \in X$ such that $J(p) = \{1, \ldots, m\}$.[2] □

Corollary

If $m < n$ then the convex polytope X given by

$$Ax \leqslant b,$$

where $A \in M(m \times n; \mathbb{R})$, has no extreme points.

[2] Proof based on N. Lauritzen, *Lectures on Convex Sets*.

Vertices of Convex Polytopes

Definition

Let $X \subset \mathbb{R}^n$ be a convex polytope. Point $p \in X$ is a **vertex** of X if it is a face of X, i.e. there exists a half–space $H_{\leqslant} \subset \mathbb{R}^n$ such that

$$X \subset H_{\leqslant} \text{ and } X \cap H = \{p\}.$$

Proposition

Let $X \subset \mathbb{R}^n$ be a convex polytope given by the system of inequalities $Ax \leqslant b$. Let $p \in X$. Then

$$p \textit{ is an extreme point of } X \iff p \textit{ is a vertex of } X.$$

Vertices of Convex Polytopes (continued)

Proof.

($\Leftarrow$) Let

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_{m+1}^\mathsf{T} x = b_{m+1}\},$$

be the supporting hyperplane such that $X \cap H = \{p\}$. Since $X \subset H_\leqslant$ the polytope X is given by the system of inequalities $A'x \leqslant b'$ where

$$A' = \left[\frac{A}{a_{m+1}}\right], \quad b' = \left[\frac{b}{b_{m+1}}\right].$$

If $r(A'_{J(p)}) < n$ then, as in the previous proof, there exists $q \neq 0$ such that $A'_{J(p)}q = 0$ and $p + tq \in X$ for small $t \in \mathbb{R}$. Since $m + 1 \in J(p)$, that is $a_{m+1}^\mathsf{T} q = 0$

$$p + tq \in X \cap H,$$

which leads to a contradiction with $X \cap H = \{p\}$.

Vertices of Convex Polytopes (continued)

Proof.

($\Rightarrow$) Let $X \subset \mathbb{R}^n$ be given by $Ax \leqslant b$ and let $p \in X$ be an extreme point of X. Let

$$a_{m+1} = \sum_{i \in J'(p)} a_i, \quad b_{m+1} = \sum_{i \in J'(p)} b_i,$$

where

$$J'(p) = \{i_1, \dots, i_n\} \subset J(p),$$

and the n rows $a_{i_1}, \dots, a_{i_n} \in \mathbb{R}^n$ of $A_{J(p)}$ are linearly independent. Let

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_{m+1}^{\mathsf{T}} x = b_{m+1}\}.$$

By linear independence $a_{m+1} \neq 0$. Moreover, if $q \in X \cap H$, then $a_{i_j}^{\mathsf{T}} q = b_{i_j}$ for $j = 1, \dots, n$ (if sum is active then each summand is active too) which implies $q = p$. Therefore

$$X \subset H_{\leqslant}, \quad X \cap H = \{p\}.$$

$\square$

Standard Form

Definition

A linear programming problem in $\mathbb{R}^n$ is in the **standard form** if the constraints are given by a system of linear equations and all variables are non–negative, i.e

$$Ax = b, \; x_1, \ldots, x_n \geqslant 0,$$

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Remark

If $r(A) = r([A|b]) < m$ one can remove redundant equations. If $r(A) \neq r([A|b])$ then $X = \emptyset$.

Standard Form (continued)

Theorem

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i) the condition $f(x) \longrightarrow max$ can be replaced by

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ii) the inequality $a_1x_1 + \ldots + a_nx_n \leqslant b$ can replaced by $a_1x_1 + \ldots + a_nx_n + x_{n+1} = b$ and $x_{n+1} \geqslant 0$, the inequality $a_1x_1 + \ldots + a_nx_n \geqslant b$ can replaced by $a_1x_1 + \ldots + a_nx_n - x_{n+1} = b$ and $x_{n+1} \geqslant 0$, the newly introduced variable x_{n+1} is called **slack variable**,

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iii) the condition $x_i \leqslant 0$ can be replaced by $x_i' \geqslant 0$ and $x_i' = -x_i$,

iv) if there are no constraints on the variable x_i, one can introduce two slack variables $x_i^-, x_i^+ \geqslant 0$ and set $x_i = x_i^+ - x_i^-$.

Example

Bring to the standard form the following linear programming problem:

$x_1 + 2x_2 \longrightarrow max$

$$\begin{cases} x_1 & \geqslant & 0 & & \\ x_2 & \geqslant & 0 & & \\ 2x_1 & + & x_2 & \leqslant & 8 \\ -x_1 & + & 3x_2 & \leqslant & 3 \end{cases}$$

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A standard form: $-x_1 - 2x_2 \longrightarrow min$

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 8 \\ -x_1 & + & 3x_2 & & & + & x_4 & = & 3 \end{cases}$$

and $x_1, x_2, x_3, x_4 \geqslant 0$.

Example (continued)

Equivalently, it can be written $c^T x \longrightarrow min$, $Ax = b$, $x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \; c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

Example (continued)

Equivalently, it can be written $c^\mathsf{T} x \longrightarrow min, \;\; Ax = b, \;\; x \geqslant 0$ where

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The optimal solution is

$$\overline{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } c^\mathsf{T}\overline{x} = -7$$

Optimal Solution and Vertices

Proposition

If the (non–empty) convex polytope X is given by $Ax = b, x \geqslant 0$, where $A \in M(m \times n; \mathbb{R})$, then it has a vertex.

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Proof.

It can be given by a system of inequalities

$$\left[\begin{array}{c} A \\ \hline -A \\ \hline -I_n \end{array}\right] x \leqslant \left[\begin{array}{c} b \\ \hline -b \\ \hline 0 \end{array}\right]$$

where the matrix of coefficients has rank n. $\square$

Optimal Solution and Vertices (continued)

Proposition

Let the convex polytope X be given by $Ax = b, x \geqslant 0$ where $A \in M(m \times n; \mathbb{R})$. If $\overline{x} \in X$ is an optimal solution for the problem $f(x) = c^{\mathsf{T}}x \longrightarrow min$, $c \neq 0$ then there exists a vertex $\overline{x}'$ of X such that

$$f(\overline{x}') = f(\overline{x}).$$

That is, an optimal solution, if it exists, can be chosen to be a vertex of the feasible set.

Optimal Solution and Vertices (continued)

Proof.

If $\overline{x} \in X$ is an optimal solution then

$$H = \{x \in \mathbb{R}^n \mid c^\mathsf{T} x = c^\mathsf{T} \overline{x}\},$$

is a supporting hyperplane of X such that $Y = X \cap H$ is a face of X and the function f is constant on Y. Therefore Y is a convex polytope which can be described by

$$\left[\begin{array}{r} A \\ \hline -A \\ \hline -I_n \\ \hline c \\ \hline -c \end{array}\right] x \leqslant \left[\begin{array}{r} b \\ \hline -b \\ \hline 0 \\ \hline c^\mathsf{T}\overline{x} \\ \hline -c^\mathsf{T}\overline{x} \end{array}\right].$$

It follows that Y has a vertex $\overline{x}' \in Y$.

Optimal Solution and Vertices (continued)

Proof.

The point $\overline{x}' \in Y$ is also a vertex of X since the convex polytope X can be described by matrix of rank n

$$\left[\begin{array}{r} A \\ \hline -A \\ \hline -I_n \\ \hline -c \end{array}\right] x \leqslant \left[\begin{array}{r} b \\ \hline -b \\ \hline 0 \\ \hline -c^{\mathsf{T}}\overline{x} \end{array}\right].$$

and the rank of the submatrix given by active inequalities for $\overline{x}'$ has rank n (the same as for Y). □

Basic Set, Basic Variables

Consider a linear programming problem in the standard form $c^\mathsf{T} x \longrightarrow min, Ax = b, x \geqslant 0$ where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and $r(A) = r([A|b]) = m$.

Definition

A basic set $\mathcal{B} = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ is a set of m elements such that columns $c_{i_1}, \ldots, c_{i_m}$ of the matrix A are linearly independent (or equivalently, the determinant of square submatrix of A consisting of columns $c_{i_1}, \ldots, c_{i_m}$ is non–zero).

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Basic Solution and Basic Feasible Solution

Definition

Let $\mathcal{B}$ be a basic set. The unique solution $\overline{x}_{\mathcal{B}} \in \mathbb{R}^n$ of the system of linear equations $Ax = b$ with $x_i = 0$ for $i \notin \mathcal{B}$ is called a **basic solution**.

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Example

Consider a linear programming problem
$c^\mathsf{T} x \longrightarrow min, \; Ax = b, \; x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

There are $\binom{4}{2} = 6$ basic sets, i.e. every set of 2 elements is basic.

$$\begin{aligned}
\mathcal{B}_1 &= \{1,2\}, \; \overline{x}_{\mathcal{B}_1} = (3,2,0,0), \\
\mathcal{B}_2 &= \{1,3\}, \; \overline{x}_{\mathcal{B}_2} = (-3,0,14,0), \\
\mathcal{B}_3 &= \{1,4\}, \; \overline{x}_{\mathcal{B}_3} = (4,0,0,7), \\
\mathcal{B}_4 &= \{2,3\}, \; \overline{x}_{\mathcal{B}_4} = (0,1,7,0), \\
\mathcal{B}_5 &= \{2,4\}, \; \overline{x}_{\mathcal{B}_5} = (0,8,0,-21), \\
\mathcal{B}_6 &= \{3,4\}, \; \overline{x}_{\mathcal{B}_6} = (0,0,8,3),
\end{aligned}$$

The sets $\mathcal{B}_1, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_6$ are basic feasible, the sets $\mathcal{B}_2, \mathcal{B}_5$ are basic infeasible.

Example

Consider a linear programming problem
$c^\mathsf{T} x \longrightarrow min,\ \ Ax = b,\ \ x \geqslant 0$ where

$$A = \left[\begin{array}{rrrr} 2 & -6 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right],\ b = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

The set $\mathcal{B} = \{1, 2\}$ is **not** basic because $\det \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = 0.$

Vertices and the Standard Form

Proposition

Let $A \in M(m \times n; \mathbb{R})$ *be a matrix such that* $r(A) = m$ *and* $m \leqslant n$*. Let* $X \subset \mathbb{R}^n$ *be a convex polytope*

$$X = \{x \in \mathbb{R}^n \mid Ax = b, x_1, \ldots, x_n \geqslant 0\}.$$

Let $p \in X$*. Then*

p *is a vertex of* $X \iff$ *there exists a basic feasible set* $\mathcal{B}$ *such that* $p = \overline{x}_{\mathcal{B}}$.

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$$X = \{x \in \mathbb{R}^n \mid Ax = b, x_1, \ldots, x_n \geqslant 0\}.$$

Let $p \in X$. Then

p is a vertex of $X \iff$ there exists a basic feasible set $\mathcal{B}$ such that $p = \overline{x}_{\mathcal{B}}$.

Remark

This is not one–to–one correspondence. For example, different basic feasible sets $\mathcal{B} = \{1, 4\}$, $\mathcal{B}' = \{2, 4\}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, x_1, \ldots, x_4 \geqslant 0$$

give the same vertex $\overline{x}_{\mathcal{B}} = \overline{x}_{\mathcal{B}'} = (0, 0, 0, 3)$.

Vertices and the Standard Form (continued)

Proof.

($\Leftarrow$) Let

$$H_{\leqslant} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \notin \mathcal{B}} -x_i \leqslant 0\}.$$

Then

$$X \subset H_{\leqslant},$$

and for $p = (p_1, \ldots, x_n) \in X$

$$p \in X \cap H_{\leqslant} \Leftrightarrow p_i = 0 \text{ for } i \notin \mathcal{B} \Leftrightarrow p = \overline{x}_{\mathcal{B}}.$$

Vertices and the Standard Form (continued)

Proof.

($\Rightarrow$) Just a sketch. Let $p = (p_1, \ldots, p_n) \in X$ be an extreme point. Let

$$I = \{i \in \{1, \ldots, n\} \mid p_i > 0\}.$$

Columns c_i for $i \in I$ are linearly independent. Otherwise there exists a $q \in \mathbb{R}^n$ such that $q_i = 0$ for $i \notin I$ such that $p + tq \in X$ for small $|t| < \varepsilon$. It is now enough to observe that $|I| \leqslant m$ (exercise) and, if necessary, pick additional $m - |I|$ linearly independent columns.

Example

Consider a linear programming problem
$c^\top x \longrightarrow min,\ Ax = b,\ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

Example

Consider a linear programming problem
$c^{\mathsf{T}}x \longrightarrow min,\ \ Ax = b,\ \ x \geqslant 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix},\ \ b = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{3, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 8 & 3 \end{bmatrix}^{\mathsf{T}}$ is feasible since $\overline{x}_{\mathcal{B}} \geqslant 0$.

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The set $\mathcal{B} = \{2, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 8 & 0 & -21 \end{bmatrix}^{\mathsf{T}}$ is infeasible since $\overline{x}_{\mathcal{B}} \not\geqslant 0$.

Example

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The set $\mathcal{B} = \{3, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 8 & 3 \end{bmatrix}^{\mathsf{T}}$ is feasible since $\overline{x}_{\mathcal{B}} \geqslant 0$. It corresponds to the vertex $(0, 0)$ of a polytope given by the original problem.

The set $\mathcal{B} = \{2, 4\}$ is basic. The corresponding basic solution $\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 8 & 0 & -21 \end{bmatrix}^{\mathsf{T}}$ is infeasible since $\overline{x}_{\mathcal{B}} \ngeqslant 0$. The basic set $\mathcal{B} = \{2, 4\}$ is infeasible.

Basic Feasible Solution

Remark

Let $\mathcal{B} = \{i_1, \ldots, i_m\}$ be a basic set. Let

$$[A|b] \xrightarrow[\text{row operations}]{\text{elementary}} [A'|b'],$$

where the columns $i_1, \ldots, i_m$ of A' are equal to

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

respectively. Let $\overline{x}_{\mathcal{B}} = (x_1, \ldots, x_n)$. Then

$$x_i = 0 \text{ for } i \notin \mathcal{B},$$

$$x_{i_j} = b'_j \text{ for } j = 1, \ldots, m,$$

and $\mathcal{B}$ is feasible if and only if $b' \geqslant 0$.

Example

Consider a linear programming problem
$c^{\mathsf{T}}x \longrightarrow min, \;\; Ax = b, \;\; x \geqslant 0$ where

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Example

Consider a linear programming problem
$c^\intercal x \longrightarrow min,\ \ Ax = b,\ \ x \geqslant 0$ where

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The set $\mathcal{B} = \{2, 4\}$ is basic. We compute the basic solution by using elementary row operations on $[A|b]$ to get the 2−nd column equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the 4−th column equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 8 \\ -1 & 3 & 0 & 1 & 3 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 8 \\ -7 & 0 & -3 & 1 & -21 \end{array}\right]$$

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Therefore if $x_1 = x_3 = 0$ (non-basic variables) then
$x_2 = 8, \;\; x_4 = -21$ (basic variables).

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Therefore if $x_1 = x_3 = 0$ (non-basic variables) then
$x_2 = 8, \; x_4 = -21$ (basic variables). Since $x_4 < 0$ the basic solution
$\overline{x}_{\mathcal{B}} = \begin{bmatrix} 0 & 8 & 0 & -21 \end{bmatrix}^T$ is infeasible.

Next Lecture - Simplex Method

We will learn an algorithm, called simplex method, for finding an optimal solution.

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We will learn an algorithm, called simplex method, for finding an optimal solution. Simplex method starts from a basic feasible set and with each turn moves to another basic feasible set decreasing (possibly) the objective function.

Dual Linear Program

Definition

For given linear programming problem $c^{\mathsf{T}}x \to max, \ Ax \leqslant b$ the **dual linear program** is

$$b^{\mathsf{T}}y \to min, \ A^{\mathsf{T}}y = c, \ y \geqslant 0.$$

The original problem is called **primal** and the latter **dual**.

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The original problem is called **primal** and the latter **dual**.

Example

The linear programming problem dual to

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}^{\intercal} x \to max, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leqslant \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix},$$

is

Dual Linear Program (continued)

Example

$$\begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}^{\mathsf{T}} y \to min, \quad \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad y \geqslant 0.$$

Dual Linear Program (continued)

Remark

Some authors give a different definition.

For given linear programming problem $c^\intercal x \to max, \ Ax \leqslant b, \ x \geqslant 0$ *the* **dual linear program** *is*

$$b^\intercal y \to min, \ A^\intercal y \geqslant c, \ y \geqslant 0.$$

It is easy to see the definitions are equivalent. For example, in the above setting the primal is equivalent to
$c^\intercal x \to max, \ \left[\begin{array}{c} A \\ \hline -I \end{array}\right] x \leqslant \left[\begin{array}{c} b \\ \hline 0 \end{array}\right]$ *which is dual to*
$\left[\begin{array}{c} b \\ \hline 0 \end{array}\right]^\intercal \left[\begin{array}{c} y \\ z \end{array}\right] \to min, \ \left[\begin{array}{c|c} A^\intercal & -I \end{array}\right] \left[\begin{array}{c} y \\ z \end{array}\right] = c, \ y, z \geqslant 0$ *which in turn is equivalent to (z describes slack variables)*
$b^\intercal y \to min, \ Ay \geqslant c, \ y \geqslant 0.$

Weak Duality Theorem

Proposition

For any feasible (not necessarily basic) solution x of the **primal** *problem and for any feasible (not necessarily basic) solution y of the* **dual** *problem*

$$c^\mathsf{T} x \leqslant b^\mathsf{T} y.$$

Proof.

Note that since $y \geqslant 0$ and $b - Ax \geqslant 0$ then

$$0 \leqslant y^\mathsf{T}(b - Ax) = y^\mathsf{T} b - (A^\mathsf{T} y)^\mathsf{T} x = y^\mathsf{T} b - c^\mathsf{T} x.$$

$\square$

Weak Duality Theorem (continued)

Corollary

$$\begin{matrix}\textit{the } \textbf{primal} \textit{ problem is feasible but}\\ \textit{the objective function attains no maximum}\end{matrix} \implies \textit{the } \textbf{dual} \textit{ problem is infeasible}$$

$$\begin{matrix}\textit{the } \textbf{dual} \textit{ problem is feasible but}\\ \textit{the objective function attains no minimum}\end{matrix} \implies \textit{the } \textbf{primal} \textit{ problem is infeasible}$$

Remark

The converse does not hold in general. For example when

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

both primal and dual problems, i.e. $Ax \leqslant b$ *and* $A^{\mathsf{T}} y = c, \; y \geqslant 0$*, are infeasible.*

Strong Duality Theorem

Theorem

$$\begin{array}{c}\textit{x^* is an optimal solution}\\ \textit{of the \textbf{primal} problem}\end{array} \Longrightarrow \begin{array}{c}\textit{there exists y^* an optimal solution}\\ \textit{of the \textbf{dual} problem}\end{array}$$

Moreover

$$c^\mathsf{T} x^* = b^\mathsf{T} y^*.$$

Proof.

Omitted. □

Strong Duality Theorem (continued)

The linear programming problem

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}^{\mathsf{T}} x \to max, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} x \leqslant \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad x \geqslant 0,$$

has the optimal solution $x^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $c^{\mathsf{T}} x^* = 5$. The dual problem yields the tableaux

$$\left[\begin{array}{ccccc|c} 2 & 2 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{r_0 - 2r_1 - 2r_2} \left[\begin{array}{ccccc|c} 0 & 0 & -1 & 2 & 2 & -6 \\ \hline 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

For $\mathcal{B} = \{1, 2\}$ we have $s = 3$ and $r = 2$ therefore

$$\xrightarrow[r_1 - r_2]{r_0 + r_2} \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 1 & -5 \\ \hline 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Strong Duality Theorem (continued)

The optimal solution of the dual problem is $y^* = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and

$b^\intercal y^* = 5.$

Hyperplane Separation Theorem (for cones)

It is relatively easy to prove the Strong Duality Theorem using the Hyperplane Separation Theorem for a cone.

Theorem

For some $v_1, \ldots, v_k \in \mathbb{R}^n$ *let*

$$V = \operatorname{cone}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{k} t_i v_i \mid t_i \geqslant 0 \right\}.$$

Then $v \notin V$ *is and only if there exits* $d \in \mathbb{R}^n$ *such that* $d^\mathsf{T} v > 0$ *and*

$$d^\mathsf{T} v_i \leqslant 0 \quad \text{for} \quad i = 1, \ldots, k,$$

that is the hyperplane $d^\mathsf{T} x = 0$ *separates* V *(in particular the vectors* v_i*) from vector* v*.*

Proof.

If $v \in V$ and such d exists then $0 < d^\mathsf{T} v = \sum_{i=1}^{k} t_i \left(d^\mathsf{T} v_i\right) \leqslant 0$. We omit the converse. $\square$

Strong Duality Theorem – Proof

Let $x^* \in \mathbb{R}^n$ be an optimal solution of the primal problem, in particular $Ax^* \leqslant b$. Let $I = J(x^*)$ be the set of all active inequalities in $Ax^* \leqslant b$. Denote the rows of $A \in M(m \times n; \mathbb{R})$ by $a_1, \ldots, a_m \in \mathbb{R}^n$. Let $V = \operatorname{cone}(a_i)_{i \in I}$. Then $c \in V$. Otherwise, by the hyperplane separation theorem, there exists $d \in \mathbb{R}^n$ such that $d^\mathsf{T} c > 0$ and $d^\mathsf{T} a_i \leqslant 0$ for $i \in I$. Then for sufficiently small $\varepsilon > 0$ (active constraints are weakened and inactive still hold for sufficiently small $\varepsilon > 0$)

$$A(x^* + \varepsilon d) \leqslant b,$$

$$c^\mathsf{T}(x^* + \varepsilon d) > c^\mathsf{T} x^*.$$

This contradicts optimality of x^* hence $c \in V$, i.e. for $i \in I$ there exists $y_i \geqslant 0$ such that

$$c = \sum_{i \in I} y_i a_i.$$

Let $y^* \in \mathbb{R}^m$ be given by the above y_i's where $y_i = 0$ for $i \notin I$. It turns out that y^* is an optimal solution of the dual problem.

Strong Duality Theorem – Proof (continued)

Obviously $y^* \geqslant 0$. Moreover

$$A^\mathsf{T} y^* = \sum_{i \in I} y_i a_i = c.$$

That is y^* is feasible. The set I indexes all active constraints of the primal problem hence

$$b^\mathsf{T} y^* = \sum_{i \in I} b_i y_i = \sum_{i \in I} (a_i^\mathsf{T} x^*) y_i = \left(\sum_{i \in I} y_i a_i \right)^\mathsf{T} x^* = c^\mathsf{T} x^*.$$

By the Weak Duality, for any feasible y

$$c^\mathsf{T} x^* \leqslant b^\mathsf{T} y,$$

i.e., y^* is an optimal solution for the dual problem.

Complementary Slackness

Proposition

Let x, y be a feasible solutions of the primal and the dual problem respectively, i.e.

$$\begin{cases} c^\intercal x \to max, \\ Ax \leqslant b \end{cases} \quad \text{and} \quad \begin{cases} b^\intercal y \to min, \\ A^\intercal y = c, \\ y \geqslant 0 \end{cases}$$

Then

$$\underset{\text{are optimal solutions}}{x=x^*, y=y*} \iff y_i = 0 \text{ or } a_i x = b_i \text{ for } i = 1, \ldots, m,$$

where $A \in M(m \times n; \mathbb{R})$ and a_i denotes the $i-th$ row of A.

Proof.

By the Weak Duality, for feasible x, y

$$c^\intercal x = y^\intercal Ax \leqslant y^\intercal b.$$

Both solutions are optimal if and only if $y^\intercal Ax = y^\intercal b$. If $y_i > 0$ then $a_i x = b_i$. $\square$

Primal–Dual Method

There exists a method for solving a linear programming problem using any feasible solution of a primal to solve a smaller, related to dual problem and use it to improve the original solution. This is called the **Primal–Dual Method**.

Carathéodory's Theorem

Theorem

For any $v \in \operatorname{cone}(v_1, \dots, v_k)$ *there exist* $1 \leqslant i_1 < i_2 < \dots i_l \leqslant k$ *such that,*

i) *vectors* $v_{i_1}, \dots, v_{i_l}$ *are linearly independent,*

ii) $v \in \operatorname{cone}(v_{i_1}, \dots, v_{i_l})$.

Carathéodory's Theorem

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ii) $v \in \operatorname{cone}(v_{i_1}, \ldots, v_{i_l})$.

Corollary

Finitely generated cone is a union of finite number of **symplicial** *cones.*

Carathéodory's Theorem – Proof

Let k be the smallest positive number such that[3] (by changing the indices if necessary)

$$v = t_1 v_1 + \ldots + t_k v_k, \text{ for some } t_1, \ldots, t_k > 0.$$

Assume that $v_1, \ldots, v_k$ are linearly dependent. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, not all equal to 0, such that

$$\alpha_1 v_1 + \ldots + \alpha_k v_k = 0,$$

where $\alpha_i > 0$ for some i (multiply sidewise by -1 if necessary). Let

$$C = \min\left\{ \frac{t_i}{\alpha_i} \mid \alpha_i > 0 \right\}.$$

Then

$$v = (t_1 - C\alpha_1)v_1 + \ldots + (t_k - C\alpha_k)v_k,$$

where

$$t_i - C\alpha_i \begin{cases} \geqslant t_i & \text{if} \quad C < \frac{t_i}{\alpha_i}, \alpha_i \neq 0 \text{ or } \alpha_i = 0, \\ = 0 & \text{if} \quad C = \frac{t_i}{\alpha_i}, \alpha_i > 0. \end{cases}.$$

[3] This proof and the following ones based on N. Lauritzen, *Lectures on*

Carathéodory's Theorem (continued)

Proposition

Let $V = \operatorname{cone}(v_1, \dots, v_k) \subset \mathbb{R}^n$ *be a finitely generated cone. Then* V *is convex and closed.*

Proof.

By Carathéodory's Theorem it is enough to assume that V is symplicial. Complete $v_1, \dots, v_k$ to a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ and define linear homeomorphism $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ by the condition

$$\varphi(\varepsilon_i) = v_i.$$

Then $V = \varphi(\mathbb{R}^k_{\geqslant 0} \times 0)$ is the image of a closed set.

Convexity is left as an exercise. $\square$

Point Separation

Proposition

Let $X \subset \mathbb{R}^n$ be a non–empty, convex, closed set such that $0 \notin X$. Then there exists a unique $x_0 \in X$ such that

$$\|x_0\| = \inf_{x \in X} \|x\|.$$

Moreover $x_0 \neq 0$.

Point Separation

Proposition

Let $X \subset \mathbb{R}^n$ be a non–empty, convex, closed set such that $0 \notin X$. Then there exists a unique $x_0 \in X$ such that

$$\|x_0\| = \inf_{x \in X} \|x\|.$$

Moreover $x_0 \neq 0$.

Proof.

Without loss of generality one can assume that X bounded (exercise) hence compact. If a continuous function $\|\cdot\|$ attains on X minima at points $x_0, y_0 \in X$ then

$$\left\| \frac{1}{2} x_0 + \frac{1}{2} y_0 \right\| \leqslant \frac{1}{2}\|x_0\| + \frac{1}{2}\|y_0\| = \|x_0\|,$$

and $\frac{1}{2}x_0 + \frac{1}{2}y_0 \in X$ by convexity of X. Triangle inequality becomes equality if and only if $x_0 = ty_0$. As $\|x_0\| = \|y_0\|$ it follows that $t = \pm 1$. Since $0 \notin X$ we have $t = 1$.

Point Separation (continued)

Corollary

Let $X \subset \mathbb{R}^n$ be a non–empty, convex, closed set such that $0 \notin X$. Then there exists an affine hyperplane $H \subset \mathbb{R}^n$ separating (strictly) 0 from X, i.e. if H is given by the equation $d^\mathsf{T} x = c$ then

$$0 = d^\mathsf{T} 0 < c \text{ and } d^\mathsf{T} x > c \text{ for any } x \in X.$$

Point Separation (continued)

Corollary

Let $X \subset \mathbb{R}^n$ be a non–empty, convex, closed set such that $0 \notin X$. Then there exists an affine hyperplane $H \subset \mathbb{R}^n$ separating (strictly) 0 from X, i.e. if H is given by the equation $d^\mathsf{T} x = c$ then

$$0 = d^\mathsf{T} 0 < c \text{ and } d^\mathsf{T} x > c \text{ for any } x \in X.$$

Proof.

Let $x_0 \in X$ be a point as above. It is enough to take $d = x_0$ and $c = \frac{x_0^\mathsf{T} x_0}{2}$, i.e. hyperplane H is given by the formula

$$x_0^\mathsf{T} x = \frac{x_0^\mathsf{T} x_0}{2}.$$

Obviously $0 < \frac{\|x_0\|^2}{2}$ and for $x = x_0$ we have $d^\mathsf{T} x > c$. Assume there exists $x \in H \cap X$, i.e. $x_0^\mathsf{T} x = \frac{x_0^\mathsf{T} x_0}{2}$. Then the segment joining x and x_0 is contained in X.

Point Separation (continued)

Proof.

For $t \in [0,1]$

$$\|x_0\|^2 \leqslant \|(1-t)x_0 + tx\|^2 = (1-t)^2\|x_0\|^2 + 2t(1-t)x_0^\top x + t^2\|x\|^2 =$$

$$= (1-t)^2\|x_0\|^2 + t(1-t)\|x_0\|^2 + t^2\|x\|^2.$$

This is equivalent to

$$0 \leqslant -t\|x_0\|^2 + t^2\|x\|.$$

For $t \in (0,1]$

$$\|x_0\| \leqslant t\|x\|,$$

which contradicts that $0 \notin X$ (as 0 is not an accumulation point of X). $\square$

Point Separation (continued)

Corollary

For any non–empty, convex, closed set such that $X \subset \mathbb{R}^n$ *and* $v \notin X$ *there exists an affine hyperplane* $H \subset \mathbb{R}^n$ *separating (strictly)* v *from* X*, i.e. if* H *is given by the equation* $d^\intercal x = c$ *then*

$$d^\intercal v < c \text{ and } d^\intercal x > c \text{ for any } x \in X.$$

Proof.

Exercise. Consider the set $0 \notin X - v$ which is closed and convex too. $\square$

Hyperplane Separation Theorem (for cones) – Proof

Let $V = \mathrm{cone}(v_1, \ldots, v_k) \subset \mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a vector such that $v \notin V$. The set V is closed and convex hence there exists a hyperplane

$$H\colon d^\mathsf{T} x = c,$$

such that for any $x \in V$

$$d^\mathsf{T} x < c,$$

and (if necessary replace d, c with $-d, -c$)

$$d^\mathsf{T} v > c.$$

Since $0 \in V$ we have $0 < c$. Since for any $t \geqslant 0$

$$d^\mathsf{T}(tx) = t\,(d^\mathsf{T} x) < c,$$

it follows that $d^\mathsf{T} x \leqslant 0$, in particular, for $i = 1, \ldots, k$

$$d^\mathsf{T} x \leqslant 0.$$

Moreover

$$d^\mathsf{T} v > c > 0.$$

Farkas' Lemma

Corollary (Farkas' Lemma)

For $A \in M(m \times n; \mathbb{R})$, $b \in M(n \times 1; \mathbb{R})$ *exactly one of the following sentences is true*

i) *there exists* $x \in \mathbb{R}^n$ *such that* $Ax = b$, $x \geqslant 0$,

ii) *there exists* $y \in \mathbb{R}^m$ *such that* $A^\intercal y \leqslant 0$ *and* $y^\intercal b > 0$.

Remark

This is essentially reformulation of the Hyperplane Separation Theorem. Point i) says b *lies in the cone* V *generated by columns of* A *and point ii) says the hyperplane* $y^\intercal x = 0$ *separates the cone* V *from point* b. *There exist several equivalent variants of this lemma, for example with inequalities reversed in point ii).*

Remarks

The duality can be used in proofs of some results from combinatorial optimization and other theoretical considerations.

Lagrange Duality

Consider the problem $c^\mathsf{T} x \to max$, $Ax \leqslant b$ where $A \in M(m \times n; \mathbb{R})$ with an optimal solution x^*. For any $\lambda \in \mathbb{R}^m$, $\lambda \geqslant 0$ define the Lagrangian function

$$g(x, \lambda) = c^\mathsf{T} x + \lambda^\mathsf{T}(b - Ax).$$

By definition, for any feasible x

$$g(x, \lambda) \geqslant c^\mathsf{T} x.$$

In particular $g(x^*, \lambda) \geqslant c^\mathsf{T} x^*$. Set (a function possibly attaining infinity as a value)

$$g(\lambda) = \sup_{x \in \mathbb{R}^n} g(x, \lambda).$$

Then

$$g(\lambda) \geqslant c^\mathsf{T} x^*,$$

is an upper bound for the optimal value. Moreover, the lowest upper bound is

$$g^* = \min_{\lambda \geqslant 0} g(\lambda) \geqslant c^\mathsf{T} x^*.$$

Lagrange Duality (continued)

This is equivalent to

$$g^* = \min_{\lambda \geqslant 0} g(\lambda) = \min_{\lambda \geqslant 0} \sup_{x \in \mathbb{R}^n} \left(c^\mathsf{T} x + \lambda^\mathsf{T}(b - Ax)\right) =$$

$$= \min_{\lambda \geqslant 0} \left(\lambda^\mathsf{T} b + \sup_{x \in \mathbb{R}^n} \left(c^\mathsf{T} - \lambda^\mathsf{T} A\right) x \right).$$

If at least one entry of $c^\mathsf{T} - \lambda^\mathsf{T} A$ is non–zero then $g(\lambda) = +\infty$ which gives no finite upper bound. Hence one may restrict the domain of $g(\lambda)$ (as it does not change the minimum) to $\lambda' s$ such that $\lambda \geqslant 0$ and $A^\mathsf{T}\lambda - c = 0$, i.e.

$$g^* = \min_{\substack{\lambda \geqslant 0 \\ A^\mathsf{T}\lambda = c}} b^\mathsf{T}\lambda.$$

This is exactly the dual problem and the Strong Duality Theorem implies that $g^* = c^\mathsf{T} x^*$.

Maximum Matching/Minimum Cover

Let $G = (V, E)$ be an undirected graph.

Definition

A set $M \subset E$ is a **matching** in graph G if for any $e_1, e_2 \in M$ edges e_1, e_2 are not adjacent. A set $M \subset E$ is a **maximum (cardinality) matching** if it is a matching in G and for any other matching E' in G

$$|M'| \leqslant |M|.$$

Definition

A set $C \subset V$ is a **(vertex) cover** in graph G if any edge $e \in E$ has at least one of its vertices in C. A set $C \subset V$ is a **minimum (cardinality) cover** if it is a cover in G and for any other cover C' in G

$$|C| \leqslant |C'|.$$

Maximum Matching/Minimum Cover (continued)

Let $G = (V, E)$ be an undirected graph, where $V = \{v_1, \ldots, v_n\}, E = \{e_1, \ldots, e_m\}$. Let $B = B_G \in M(n \times m; \mathbb{R})$ be the incidence matrix of G. For any subset $C \in V$ let $v_C \in \mathbb{R}^n$ denote a vector with $i-$th coordinate equal to 1 if $v_i \in C$ and equal to 0 otherwise. For any subset $M \in E$ let $e_M \in \mathbb{R}^m$ denote a vector with $i-$th coordinate equal to 1 if $e_i \in M$ and equal to 0 otherwise.

Proposition

Set $M \subset E$ is a matching if and only if $e = e_M \in \{0, 1\}^m$ and

$$Be \leqslant \mathbb{1}_n.$$

Proof.

Components of Be are degrees of vertices $v_1, \ldots, v_n$ in a subgraph formed by edges from M. No two edges in a matching share a vertex. $\square$

Maximum Matching/Minimum Cover (continued)

Proposition

Set $C \subset V$ is a cover if and only if $v = v_C \in \{0,1\}^n$ and

$$B^\intercal v \geqslant \mathbb{1}_m.$$

Proof.

Components of $B^\intercal v$ are equal to either $0, 1$ or 2 (each row of $B^\intercal$ contains exactly two $1's$), which counts how many times the corresponding edge is covered by vertices from C. In a cover each edge should be covered by at least one vertex. $\square$

Maximum Matching/Minimum Cover (continued)

Proposition

An optimal solution of the following problem

$$e = e_M \in \{0,1\}^m,$$

$$\mathbb{1}_n^\top e \to max,$$

$$Be \leqslant \mathbb{1}_n,$$

is a maximum matching.

Proposition

An optimal solution of the following problem

$$v = v_C \in \{0,1\}^n,$$

$$\mathbb{1}_n^\top v \to min,$$

$$B^\top v \geqslant \mathbb{1}_m,$$

is a minimum cover.

Fractional Maximum Matching

Proposition

For any graph G both problems

$$\begin{cases} e = e_M \in \{0,1\}^m, \\ \mathbb{1}_m^\intercal e \to max \\ Be \leqslant \mathbb{1}_n. \end{cases} \quad \text{and} \quad \begin{cases} e \geqslant 0, \\ \mathbb{1}_m^\intercal e \to max \\ Be \leqslant \mathbb{1}_n. \end{cases}$$

have the same optimal **value**, *i.e. the cardinality of maximum matching.*

Proof.

The second problem possibly attains a bigger optimal value as $A \subset B \implies \sup_A f \leqslant \sup_B f$. Optimum value is attained at a vertex (of a feasible set/polytope) of the second problem. That vertex has integral components as it is a (unique) solution of a system of active inequalities in $Be \leqslant \mathbb{1}$ and B is a totally unimodular matrix. For any feasible solution $e = (e_1, \dots, e_n)$ of the second problem $e_1, \dots, e_n \leqslant 1$ and hence $e \in \{0,1\}^n$. An optimal solution of the second problem corresponds to a matching and therefore is also a solution of the first problem. $\square$

Fractional Minimum Cover

Proposition

If graph G has no isolated vertices then both problems

$$\begin{cases} v = v_C \in \{0,1\}^n, \\ \mathbb{1}_n^\intercal v \to min \\ B^\intercal v \geqslant \mathbb{1}_m. \end{cases} \quad and \quad \begin{cases} v \geqslant 0, \\ \mathbb{1}_n^\intercal v \to min \\ B^\intercal v \geqslant \mathbb{1}_m. \end{cases}$$

have the same optimal **value**, *i.e. the cardinality of minimum cover.*

Proof.

The second problem possibly attains a smaller optimal value as $A \subset B \implies \inf_B f \leqslant inf_A f$. As above, components of an optimal solution of the second problem are nonnegative integers. Assume that $v^* = (v_1^*, \dots, v_n^*)$ is an optimal solution of the second problem. If say $v_1^* \geqslant 2$ then $v' = (v_1^* - 1, \dots, v_n^*) \geqslant 0$ and $B^\intercal v' \geqslant \mathbb{1}$ but $\mathbb{1}^\intercal v' < \mathbb{1}^\intercal v^*$ (double vertex is wasteful). Therefore optimal solution of the second problem corresponds to a cover and therefore is also a solution of the first problem. $\square$

König's Theorem

Theorem

Let G be a bipartite (undirected) graph. Then the size of maximum matching is equal to the size of minimum cover.

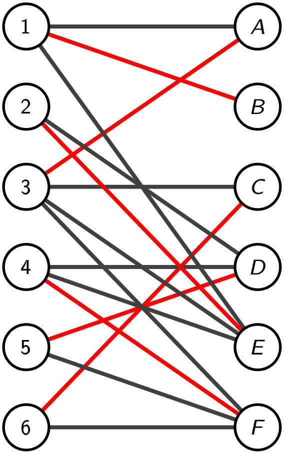
Proof.

By the Strong Duality Theorem both problems attain the same optimal **value**

$$\begin{cases} e \geqslant 0, \\ \mathbb{1}_n e \to max, \\ Be \leqslant \mathbb{1}_n. \end{cases} \qquad \begin{cases} v \geqslant 0, \\ \mathbb{1}_m v \to min \\ B^\intercal v \geqslant \mathbb{1}_m. \end{cases}$$

$\square$

Sample Maximal Matching



6 candidates applied for 6 jobs, first candidate applied for A, B, second candidate for D, E etc. How to hire maximum number of candidates?

Scheduling

Say we have n activities, each activity starts at time p_i, it finishes at time q_i and it brings profit c_i when completed. How to pick non–overlapping activities with maximal profit? Consider the following problem

$$c^\mathsf{T} x \to max,$$

$$x_i + x_j \leqslant 1, \quad \text{for each overlapping activities } i, j,$$

$$x \in \{0, 1\}^n.$$

It has the same optimal solutions as the problem

$$c^\mathsf{T} x \to max,$$

$$x_i + x_j \leqslant 1, \quad \text{for each overlapping activities } i, j,$$

$$x \geqslant 0,$$

as the matrix is an incidence matrix of a bipartite graph (activities i, j are joined by an edge if they overlap) hence totally unimodular.

Fourier–Motzkin Elimination

Theorem

Let $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the subspace spanned by the first $n-1$ standard unit vectors, i.e.

$$P(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}).$$

Let $X \subset \mathbb{R}^n$ be a convex polyhedron. Then $P(X) \subset \mathbb{R}^{n-1}$ is a convex polyhedron.

Fourier–Motzkin Elimination

Theorem

Let $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the subspace spanned by the first $n-1$ standard unit vectors, i.e.

$$P(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}).$$

Let $X \subset \mathbb{R}^n$ be a convex polyhedron. Then $P(X) \subset \mathbb{R}^{n-1}$ is a convex polyhedron.

Proof.

Assume $X \neq \mathbb{R}^n$ is given by the system of inequalities

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leqslant b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leqslant b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leqslant b_m \end{cases}$$

Fourier–Motzkin Elimination (continued)

Proof.

Let $N_<, N_0, N_>$ be a partition of the set $\{1, \ldots, m\}$ given by the conditions

$$N_< = \{1 \leqslant i \leqslant m \mid a_{in} < 0\}, N_0 = \{1 \leqslant i \leqslant m \mid a_{in} = 0\},$$

$$N_> = \{1 \leqslant i \leqslant m \mid a_{in} > 0\}.$$

Any $(x_1, \ldots, x_{n-1}) \in P(X)$ satisfies inequality $a_i^\mathsf{T} x \leqslant b_i$ for $i \in N_0$ and a linear combination (with non–negative coefficients) of inequalities $i \in N_<, j \in N_>$

$$a_{jn} \left(\sum_{k=1}^{n} a_{ik} x_k \right) - a_{in} \left(\sum_{k=1}^{n} a_{jk} x_k \right) \leqslant a_{jn} b_i - a_{in} b_j,$$

where x_n is eliminated, i.e.,

$$a_{jn} \left(\sum_{k=1}^{n-1} a_{ik} x_k \right) - a_{in} \left(\sum_{k=1}^{n-1} a_{jk} x_k \right) \leqslant a_{jn} b_i - a_{in} b_j.$$

Fourier–Motzkin Elimination (continued)

Proof.

After dividing by $-1/a_{in}a_{jn}$ this can be rewritten as

$$-\frac{1}{a_{in}}\left(\sum_{k=1}^{n-1} a_{ik}x_k\right) + \frac{1}{a_{jn}}\left(\sum_{k=1}^{n-1} a_{jk}x_k\right) \leqslant -\frac{1}{a_{in}}b_i + \frac{1}{a_{jn}}b_j,$$

that is

$$-\frac{1}{a_{in}}\left(\sum_{k=1}^{n-1} a_{ik}x_k - b_i\right) \leqslant -\frac{1}{a_{jn}}\left(\sum_{k=1}^{n-1} a_{jk}x_k - b_j\right).$$

This implies that

$$\max_{i\in N_<} -\frac{1}{a_{in}}\left(\sum_{k=1}^{n-1} a_{ik}x_k - b_i\right) \leqslant \min_{j\in N_>} -\frac{1}{a_{jn}}\left(\sum_{k=1}^{n-1} a_{jk}x_k - b_j\right).$$

Choosing x_n between those numbers one can see that $(x_1, \ldots, x_n) \in X$. $\square$

Gale's Theorem

Theorem

Let $A \in M(m \times m; \mathbb{R})$, $b \in M(m \times 1; \mathbb{R})$. Then the following conditions are equivalent

i) *the inequality $Ax \leqslant b$ has no solutions,*

ii) *there exists $y \in \mathbb{R}^m$, $y \geqslant 0$ such that $A^\intercal y = 0$, $b^\intercal y < 0$.*

Proof.

Use Fourier–Motzkin elimination to project convex polyhedron X give by $Ax \leqslant b$ onto $0-$dimensional subspace. The image of projection is non–empty is and only if X is non–empty. Each projection amount to multiplying the inequality $Ax \leqslant b$ by some matrix $y \in M(r \times m; \mathbb{R})$, $y \geqslant 0$. The product of such $y's$ gives inequality $y^\intercal A0 \leqslant y^\intercal b$. If X is empty one of the inequalities is $0 \leqslant c$ where $c < 0$. $\square$

Farkas' Lemma Revisited

Corollary (Farkas' Lemma)

For $A \in M(m \times n; \mathbb{R})$, $b \in M(n \times 1; \mathbb{R})$ exactly one of the following sentences is true

i) *there exists $x \in \mathbb{R}^n$ such that $Ax = b$, $x \geqslant 0$,*

ii) *there exists $y \in \mathbb{R}^m$ such that $A^\intercal y \leqslant 0$ and $y^\intercal b > 0$.*

Proof.

As in the previous proof, both conditions cannot be satisfied. If $Ax = b, x \geqslant 0$ has a solution, then $\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leqslant \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$ has a solution.

Farkas' Lemma Revisited (continued)

Proof.

By Gale's Theorem for all $\overline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geqslant 0$

$$A^\mathsf{T} y_1 - A^\mathsf{T} y_2 - y_3 \neq 0, \quad \text{or} \quad b^\mathsf{T} y_1 - b^\mathsf{T} y_2 \geqslant 0.$$

With $y = y_2 - y_1$ this can be rewritten as

$$A^\mathsf{T} y \neq -y_3, \quad \text{or} \quad b^\mathsf{T} y \leqslant 0,$$

for all $y_3 \geqslant 0$, i.e., for any $y \in \mathbb{R}^m$

$$A^\mathsf{T} y \nleqslant 0, \quad \text{or} \quad y^\mathsf{T} b \leqslant 0,$$

which is exactly the opposite of the condition ii) of Farkas' Lemma.

The converse can be proven in a similar way (exercise). $\square$

Certificate of Infeasibility

Remark

To prove that the problem $Ax = b, x \geqslant 0$ *is infeasible it is enough to find* $y \in \mathbb{R}^m$ *such that* $A^\intercal y \leqslant 0$ *and* $y^\intercal b > 0$. *Therefore any such* y *is called a* **certificate of infesibility**.

Extremal Set Theory

Let S be a finite set and let $\mathcal{A} \subset P(S)$ be a family of subsets of the set S. Let A be a matrix which rows are indicator vectors of subsets in $\mathcal{A}$. Then optimal solutions of the first problems correspond to subsets of $X \subset S$ of maximal cardinality such that $|X \cap A| \leqslant 1$ and the and optimal solutions of the second problem to a subfamilies $\mathcal{Y} \subset \mathcal{A}$ of minimal cardinality such that $\bigcup \mathcal{Y} = S$.

$$\begin{cases} x \in \mathbb{Z}, \\ x \geqslant 0, \\ \mathbb{1}^\intercal x \to max, \\ Ax \leqslant \mathbb{1}. \end{cases} \qquad \begin{cases} y \in \mathbb{Z}, \\ y \geqslant 0, \\ \mathbb{1}^\intercal v \to min \\ A^\intercal y \geqslant \mathbb{1}. \end{cases}$$

Extremal Set Theory (continued)

Optimal solutions of the first problems correspond to subsets of $X \subset S$ of minimal cardinality such that $|X \cap A| \geqslant 1$ (that is X intersects all subsets in the family $\mathcal{A}$) and the and optimal solutions of the second problem to a subfamilies $\mathcal{Y} \subset \mathcal{A}$ of maximal cardinality, containing pairwise disjoint sets.

$$\begin{cases} x \in \mathbb{Z}, \\ x \geqslant 0, \\ \mathbb{1}^\mathsf{T} x \to min, \\ Ax \geqslant \mathbb{1}. \end{cases} \qquad \begin{cases} y \in \mathbb{Z}, \\ y \geqslant 0, \\ \mathbb{1}^\mathsf{T} v \to max \\ A^\mathsf{T} y \leqslant \mathbb{1}. \end{cases}$$

However, for some families $\mathcal{A}$ optimal values of these **integral linear programming problems** may differ. For example let $\mathcal{A} = \{\{1,2\},\{1,3\},\{2,3\}$ and $S = \{1,2,3\}$.

Modeling in Linear Programming[4]

Sometimes it is desirable to impose additional constraints on the optimal solution. This can be achieved by introducing auxiliary variables $t, y_1, \dots, y_n$ (or $t \in \mathbb{R}$ if needed)

$$t \geqslant \max\{x_1, \dots, x_n\} \iff t \geqslant x_i \text{ for } i = 1, \dots, n,$$

$$t \leqslant \min\{x_1, \dots, x_n\} \iff t \leqslant x_i \text{ for } i = 1, \dots, n,$$

$$t \geqslant \max\{a_i^\top x + b_i \mid i = 1, \dots, m\} \iff t \geqslant a_i^\top x + b_i \text{ for } i = 1, \dots, m,$$

in particular

$$t \geqslant |x_i| \iff -t \leqslant x_i \leqslant t,$$

as $|x| = \max\{-x, x\}$.

[4] based on https://docs.mosek.com/modeling-cookbook/index.html

Modeling in Linear Programming (continued)

$$|x_1| + \ldots + |x_n| \leqslant t \iff |x_i| \geqslant y_i \text{ for } i = 1, \ldots, n, \ \sum_{i=1}^{n} y_i = t \iff$$

$$\iff -y_i \leqslant x_i \leqslant y_i \text{ for } i = 1, \ldots, n, \ \sum_{i=1}^{n} y_i = t.$$

The above observation may be used to look (by a heuristic rule) for a sparse solution of the system $Ax = b$ by solving a linear programming problem

$$y_1 + \ldots + y_n \to min,$$

with constraints

$$Ax = b, \quad y \geqslant 0, \quad -y_i \leqslant x_i \leqslant y_i, \quad i = 1, \ldots, n.$$

Sparse Solution of a System of System of Linear Equations – Example

Remark

The previous method quintuples the number of variables. Note that in the generic case an overdetermined system of linear equations $Ax = b$ will have rank A*- sparse solution. Note that in practice large matrices will have low rank which is computationally expensive to estimate.*

Consider the sytem $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

It does not have 1-sparse solution (why?), i.e. with at most one non–zero entry. To find a solution with the least 1-norm we should add variables $y_1, y_2, y_3, y_4 \geqslant 0$ with inequalities $-y_i \leqslant x_i \leqslant y_i$ for $i = 1, \ldots, 4$ and minimze $y_1 + \ldots + y_4$.

Sparse Solution of a System of System of Linear Equations – Example

The standard form of the above problem is

$$A' = \left[\begin{array}{r|r|c|c|r} A & -A & 0 & 0 & 0 \\ \hline I & -I & I & 0 & -I \\ \hline -I & I & 0 & I & -I \end{array}\right],$$

with $b' = (b, 0, \ldots, 0)$ and $c' = (0, \ldots, 0, 1, 1, 1, 1)$. The first 8 columns of A' correspond to $x_i = x_i^+ - x_i^-$. The second line corresponds to inequalites $x_i + x_i' \leqslant y_i$ i.e., $x_i^+ - x_i^- + x_i' - y_i = 0$. The third line corresponds to inequalites $-y_i \leqslant x_i$ i.e., $-x_i^+ + x_i^- + x_i'' - y_i = 0$. Moreover $x_i^+, x_i^-, x_i', x_i'', y_i \geqslant 0$ for $i = 1, \ldots, 4$.

Sparse Solution of a System of System of Linear Equations – Example

The problem has an optimal solution

$$\bar{x} =$$

$$= \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 0 & 0 & 3 & 0 & \frac{1}{2} \end{bmatrix}^T$$

which leads to a 2-sparse solution $x = \left(0, 3, 0, -\frac{1}{2}\right)$.

Modeling in Linear Programming – Sum of m Maximal Components

Proposition

Let $X \subset\in \mathbb{R}^n$ be a section of an n-dimensional cube with a hyperplane $\sum_{i=1}^{n} = m$ where $m \in \{0, 1, \ldots, n\}$, i.e.,

$$X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leqslant x_i \leqslant 1,\ x_1 + \ldots + x_n = m\}.$$

Then vertices of polytope X are of the form

$$(x_1, \ldots, x_n) \quad \textit{where} \quad x_i \in \{0, 1\}, x_1 + \ldots + x_n = m,$$

i.e., sums of m different vectors of the standard basis of $\mathbb{R}^n$.

Modeling in Linear Programming – Sum of m Maximal Components (continued)

Proof.

The constrainst can be rewritten as
$\sum x_i \leqslant m, -\sum x_i \leqslant -m, x_1 \leqslant 1, -x_1 \leqslant 0, \ldots, x_n \leqslant 1, -x_n \leqslant 0$ It is enough to consider submatrices of matrix,

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & \cdots & -1 & -1 & -1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 \end{bmatrix}$$

consisting of rows corresponding to active inequalities of rank A. The unique solution is exactly of the required form. Both first rows are always active. $\square$

Modeling in Linear Programming – Sum of m Maximal Components (continued)

Corollary

A solution of the linear programming program $c^\intercal x \to \max$ over X is the sum of m maximal components of vector c.

Modeling in Linear Programming – Sum of m Maximal Components (continued)

If you want to optimize the sum of m maximal components of a point in polyhedron the objective function becomes quadratic. This can by avoided by passing to a dual problem und using the strong duality.

$$d^\mathsf{T} x \to min, Ax = b, x \geqslant 0 \iff b^\mathsf{T} y \to max, A^\mathsf{T} y \leqslant d.$$

Take

$$b = \begin{bmatrix} -m \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \quad A = \left[\begin{array}{c|c} -1 & \\ \vdots & -I \\ -1 & \\ \hline 0 & -I \end{array}\right], \quad d = \begin{bmatrix} -c_1 \\ \vdots \\ -c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Modeling in Linear Programming – Sum of m Maximal Components (continued)

The dual problem becomes

$$mt + \sum_{i=1}^{n} y_i \to min,$$

under the constraints

$$y_i + t \geqslant c_i,$$

$$y_i \geqslant 0,$$

for $i = 1, \ldots, n$.