Linear Algebra

Lecture 11 - Affine Space $\mathbb{R}^n$

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9 January 2025

Affine Space

Definition

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$$+ \colon E \times V \to E,$$

satisfying the following conditions

i) $p + 0 = p$ for any $p \in E$,

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ii) $(p + v) + w = p + (v + w)$ for any $p \in E, v, w \in V$ (associativity),
iii) for any $p, q \in E$ there exits a unique vector $\overrightarrow{pq} \in V$ such that $p + \overrightarrow{pq} = q$.

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ii) $p + (\overrightarrow{pq} + \overrightarrow{qr}) = (p + \overrightarrow{pq}) + \overrightarrow{qr} = q + \overrightarrow{qr} = r$,

iii) follows form i) and ii) for $r = p$.

Affine Space (continued)

Remark

Elements of the set E are called **points** *and elements of vector space V are called* **vectors**.

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Remark

For any $p \in E$ the map

$$V \ni v \mapsto p + v \in E,$$

is a bijection.

Affine Space (continued)

Proof.

It is injective

$$(p + v = p + w = q) \Rightarrow (v = w = \overrightarrow{pq}),$$

and surjective

$$q = p + \overrightarrow{pq}.$$

$\square$

Translation

Definition

For any $v \in V$ the **translation** by v is the map

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Proof.

It is injective

$$(p + v = q + v = r) \Rightarrow (v = \overrightarrow{pr} = \overrightarrow{qr}) \Rightarrow (p = r + \overrightarrow{rp} = r + \overrightarrow{rq} = q),$$

and surjective

$$t_v(q - v) = q.$$

$\square$

Affine Subspace

Definition

Let E be an affine space over V. For any $p \in E$ and any subspace W of the vector space V the set

$$F = p + W = \{p + w \in E \mid w \in W\},$$

is called an **affine subspace** of E. The subspace W is called the **direction** of F and it is denoted by $\overrightarrow{F} = W$.

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Remark

The 0-dimensional affine subspaces are called points, the 1-dimensional affine subspaces are called lines, the 2-dimensional affine subspaces are planes.

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Remark

The 0*-dimensional affine subspaces are called points, the* 1*-dimensional affine subspaces are called lines, the* 2*-dimensional affine subspaces are planes.*

Remark

The affine space $F = p + W$ is invariant under translations t_w for any $w \in W$, i.e.

$$t_w(F) = F.$$

Affine Subspace (continued)

Proposition

Let $F = p + W$ be an affine subspace of E. Then for any $q \in F$

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Let $F = p + W$ be an affine subspace of E. Then for any $q \in F$

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Proof.

Since $q \in F$ then $q = p + w$ for some $w \in W$, i.e. $\overrightarrow{pq} = w$. Therefore

$$q + W = (p + w) + W = p + W.$$

$\square$

Affine Subspace (continued)

Proposition

For any $q, r \in F = p + W$

$$\overrightarrow{qr} \in W,$$

i.e. any vector joining two points of an affine subspace F *belongs to its direction* $\overrightarrow{F} = W$.

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Proposition

For any $q, r \in F = p + W$

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i.e. any vector joining two points of an affine subspace F belongs to its direction $\overrightarrow{F} = W$.

Proof.

Since $q = p + \overrightarrow{pq}, r = p + \overrightarrow{pr}$, both $\overrightarrow{pq}, \overrightarrow{pr} \in W$ and

$$\overrightarrow{qr} = \overrightarrow{qp} + \overrightarrow{pr} \in W.$$

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Remark

Note that any affine subspace F *is an affine space over* $W = \overrightarrow{F}$ *with the operation* $+$ *restricted to* $F \times W$.

Affine Combination

Let E be an affine space over V.

Definition

Let $p_0, \ldots, p_k \in E$ be points. For any $a_i \in \mathbb{R}$ such that $\sum_{i=0}^{k} a_i = 1$ and any point $p \in E$ the point

$$\sum_{i=0}^{k} a_i p_i = p + \sum_{i=0}^{k} a_i \overrightarrow{pp_i}$$

is called the **affine combination** of $p_0, \ldots, p_k$.

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Proposition

For any $p, q \in E$

$$p + \sum_{i=0}^{k} a_i \overrightarrow{pp_i} = q + \sum_{i=0}^{k} a_i \overrightarrow{qp_i}.$$

Affine Combination (continued)

Proof.

$$q + \sum_{i=0}^{k} a_i \overrightarrow{qp_i} = q + \sum_{i=0}^{k} a_i (\overrightarrow{qp} + \overrightarrow{pp_i}) = p + \sum_{i=0}^{k} a_i \overrightarrow{pp_i}.$$

$\square$

Corollary

The affine combination of $p_0, \ldots, p_k$ does not depend on the point $p \in E$.

Affine Combination (continued)

Corollary

Let $F = p + W$ be an affine subspace. Then any affine combination of $p_0, \ldots, p_k \in F$ belongs to F, i.e. any affine subspace is closed under taking affine combinations.

Proof.

For any $\sum_{i=0}^{k} a_i = 1$

$$\sum_{i=0}^{k} a_i p_i = p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i} \in F,$$

because $\overrightarrow{p_0 p_i} \in W$ for $i = 0, \ldots, k$. $\square$

The Main Example of Affine Space

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Any vector space V is an affine space over itself with the operation $+$ being the vector addition from V and

$$\overrightarrow{pq} = q - p.$$

Remark

Any affine space can be obtained in this way.

Affine Space $\mathbb{R}^n$

Remark

From now on we will be dealing only with the affine space $\mathbb{R}^n$ (as a vector space over itself) and its affine subspaces of the form

$$E = p + V,$$

where $V \subset \mathbb{R}^n$ is a subspace.

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Example

Let $p = (1,1,1), q = (1,2,3)$. Then $\overrightarrow{pq} = q - p = (0,1,2)$.

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Example

Let $p = (1, -1)$ and $V = \text{lin}((2, 3)) \subset \mathbb{R}^2$. Then

$$E = p + V = \{(1 + 2t, -1 + 3t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

Affine Span

Definition

Let $p_0, \ldots, p_k \in \mathbb{R}^n$. The **affine span** (or the **affine hull**) of $p_0, \ldots, p_k$ is the set of all affine combinations of $p_0, \ldots, p_k$, i.e.

$$\operatorname{aff}(p_0, \ldots, p_k) = \left\{ \sum_{i=0}^{k} a_i p_i \in \mathbb{R}^n \mid \sum_{i=0}^{k} a_i = 1 \right\}.$$

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Proposition

Let $p_0, \ldots, p_k \in \mathbb{R}^n$. *Then*

$$\operatorname{aff}(p_0, \ldots, p_k) = p_0 + \operatorname{lin}(\overrightarrow{p_0 p_1}, \ldots, \overrightarrow{p_0 p_k}).$$

Affine Span (continued)

Proof.

Let $\sum_{i=0}^{k} a_i = 1$. Then

$$\sum_{i=0}^{k} a_i p_i = p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i} \in p_0 + \mathrm{lin}(\overrightarrow{p_0 p_1}, \ldots, \overrightarrow{p_0 p_k}).$$

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Assume $p = p_0 + \sum_{i=1}^{k} a_i \overrightarrow{p_0 p_k} \in p_0 + \mathrm{lin}(\overrightarrow{p_0 p_1}, \ldots, \overrightarrow{p_0 p_k})$ for some $a_1, \ldots, a_k \in \mathbb{R}$. Then

$$p = (1 - \sum_{i=1}^{k} a_i) p_0 + \sum_{i=1}^{k} a_i p_k.$$

$\square$

Affine Span (continued)

Proof.

Let $\sum_{i=0}^{k} a_i = 1$. Then

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Corollary

The affine subspace $\operatorname{aff}(p_0, \dots, p_k)$ *is the smallest affine subspace of* $\mathbb{R}^n$ *containing points* $p_0, \dots, p_k$.

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$$\mathrm{aff}((1,1,1),(1,2,3),(3,2,1)) = (1,1,1) + \mathrm{lin}((0,1,2),(2,1,0))).$$

Parametrization

Definition

Let $E = p + \mathrm{lin}(v_1, \dots, v_k) \subset \mathbb{R}^n$ where vectors $v_1, \dots, v_k$ are linearly independent (i.e. $v_1, \dots, v_k$ is a basis of $\overrightarrow{E}$). Then any point $q \in E$ can be uniquely written as

$$q = p + \sum_{i=1}^{k} t_i v_i.$$

Any such presentation of E is called a **parametrization**.

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Example

$$E = (1, 1, 1) + \text{lin}((0, 1, 2), (2, 1, 0)) =$$

$$= (1, 2, 3) + \text{lin}((0, 1, 2), (1, 1, 1))$$

that is $(1 + 2t_2, 1 + t_1 + t_2, 1 + 2t_1)$, $t_1, t_2 \in \mathbb{R}$ and $(1 + t_2, 2 + t_1 + t_2, 3 + 2t_1 + t_2)$, $t_1, t_2 \in \mathbb{R}$ are two different parametrizations of E.

Parallel Affine Subspaces

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Any affine subspace E of $\mathbb{R}^n$ is equal to a set of solutions of a (possibly non–homogeneous) system of linear equations in n variables.

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Proposition

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Proof.

There exists a homogeneous system of linear equations describing the vector subspace $\overrightarrow{E}$

$$\overrightarrow{E}: \begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = 0 \end{cases}$$

Proof.

Let $E = p + \overrightarrow{E}$. If $p = (y_1, \dots, y_n)$ set

$$\begin{array}{ccccccccc}
b_1 & = & a_{11}y_1 & + & a_{12}y_2 & + & \dots & + & a_{1n}y_n \\
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Then the affine subspace E is described by

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□

The constants $b_1, \ldots, b_m$ do not depend on the point $p \in E$ since any two points in E differ by a vector from $\overrightarrow{E}$.

Examples

Example

Describe by a system of linear equations an affine subspace E parallel to $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ passing through $p = (2, 3, 4)$.

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Example

Describe by a system of linear equations the affine subspace $E = p + V$ in $\mathbb{R}^4$ where

$$p = (1, 1, 2, 1),\ \ V = \mathrm{lin}((1, 1, 3, 0), (1, 0, 1, 0), (0, 1, 2, 0)).$$

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Example

Describe by a system of linear equations the affine subspace $E = p + V$ in $\mathbb{R}^4$ where

$$p = (1, 1, 2, 1), \;\; V = \operatorname{lin}((1, 1, 3, 0), (1, 0, 1, 0), (0, 1, 2, 0)).$$

Vectors $(1, 0, 1, 0), (0, 1, 2, 0)$ form a basis of V. Therefore V is described by the system of equations

$$V\colon \begin{cases} x_1 + 2x_2 - x_3 & = 0 \\ x_4 & = 0 \end{cases}$$

Examples (continued)

Example

Recall $E = (1,1,2,1) + V$. Therefore

$$E: \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_4 = 1 \end{cases}$$

Definition

For any $p, q \in \mathbb{R}^n$ the **distance** between p and q is $\|\overrightarrow{pq}\|$. It is denoted $d(p,q)$.

Examples (continued)

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Definition

For any $p, q \in \mathbb{R}^n$ the **distance** between p and q is $\|\overrightarrow{pq}\|$. It is denoted $d(p,q)$.

It has the following properties:

i) $d(p,q) \geqslant 0$ and $(d(p,q) = 0 \iff p = q)$,

ii) $d(p,q) = d(q,p)$ (symmetry),

iii) $d(p,r) \leqslant d(p,q) + d(q,r)$ (triangle inequality).

Examples (continued)

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$$E\colon \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_4 = 1 \end{cases}$$

Definition

For any $p, q \in \mathbb{R}^n$ the **distance** between p and q is $\|\overrightarrow{pq}\|$. It is denoted $d(p,q)$.

It has the following properties:

i) $d(p,q) \geqslant 0$ and $(d(p,q) = 0 \iff p = q)$,

ii) $d(p,q) = d(q,p)$ (symmetry),

iii) $d(p,r) \leqslant d(p,q) + d(q,r)$ (triangle inequality).

The affine space $\mathbb{R}^n$ equipped with a function satisfying above properties (called metric) becomes a **metric space**.

Affine Transformation

Definition

Let $E, H \subset \mathbb{R}^n$ be two affine subspaces. We say that E, H are orthogonal if $v \perp w$ for every $v \in \overrightarrow{E}, w \in \overrightarrow{H}$.

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Let $E \subset \mathbb{R}^n, H \subset \mathbb{R}^m$ be two affine subspaces. A function $f : E \longrightarrow H$ satisfying the condition

$$f(p + \alpha) = f(p) + f'(\alpha),$$

$$\left(\text{or equivalently } \overrightarrow{f(p)f(p+\alpha)} = f'(\alpha)\right),$$

for some $p \in E$, some linear transformation $f' : \overrightarrow{E} \longrightarrow \overrightarrow{H}$ and any $\alpha \in \overrightarrow{E}$ is called an **affine transformation**.

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If $q \in E$ then $f(q+\alpha) = f(p + \overrightarrow{pq} + \alpha) = f(p) + f'(\overrightarrow{pq}) + f'(\alpha) =$ $= f(q) + f'(\alpha)$ therefore the condition in the definition holds for any $p \in E$.

Properties of Affine Transformation

Proposition

Let E, H be two affine subspaces. Then $f : E \longrightarrow H$ is an affine transformation if and only if

$$f\left(\sum_{i=0}^{k} a_i p_i\right) = \sum_{i=0}^{k} a_i f(p_i),$$

for any $p_i \in E$ and $a_i \in \mathbb{R}$ such that $\sum_{i=0}^{k} a_i = 1$.

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Proof.

($\Rightarrow$) Assume that f is an affine transformation. Then

$$f\left(\sum_{i=0}^{k} a_i p_i\right) = f\left(p_0 + \sum_{i=0}^{k} a_i \overrightarrow{p_0 p_i}\right) = f(p_0) + \sum_{i=0}^{k} a_i f'(\overrightarrow{p_0 p_i}) =$$

$$= f(p_0) + \sum_{i=0}^{k} a_i \left(\overrightarrow{f(p_0) f(p_i)}\right) = \sum_{i=0}^{k} a_i f(p_i).$$

Properties of Affine Transformation (continued)

Proof.

($\Leftarrow$) Assume that function f satisfies the condition of the Proposition for $k = 1$. Let $p_0, p_1 \in E$ be any points and $a \in \mathbb{R}$, then

$$f((1-a)p_0 + ap_1) = f(p_0 + a\overrightarrow{p_0p_1}) = (1-a)f(p_0) + af(p_1) =$$

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Formula of an Affine Transformation

Remark

Any affine transformation $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is given by a formula*

$$f((x_1, x_2, \ldots, x_n)) = (a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1, \ldots,$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + b_m),$$

where $a_{ij}, b_k \in \mathbb{R}$*. The linear transformation* f' *has matrix*

$$M(f')^{st}_{st} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

in standard bases (and it is equal to the total derivative of f *at any point* $p \in \mathbb{R}^n$*).*

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Proof.

Choose $p = (0, \ldots, 0), \alpha = (x_1, \ldots, x_n)$ so
$f((x_1, \ldots, x_n)) = f((0, \ldots, 0)) + f'((x_1, \ldots, x_n))$.

Affine Orthogonal Projection and Reflection

Definition

Let $E \subset \mathbb{R}^n$ be an affine subspace and let $p_0 \in E$. The affine transformation $\pi_E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\pi_E(p) = \pi_E(p_0 + \overrightarrow{p_0 p}) = p_0 + P_{\overrightarrow{E}}(\overrightarrow{p_0 p}),$$

where $P_{\overrightarrow{E}}$ is the (linear) orthogonal projection on $\overrightarrow{E}$, is called an **(affine) orthogonal projection** on E.

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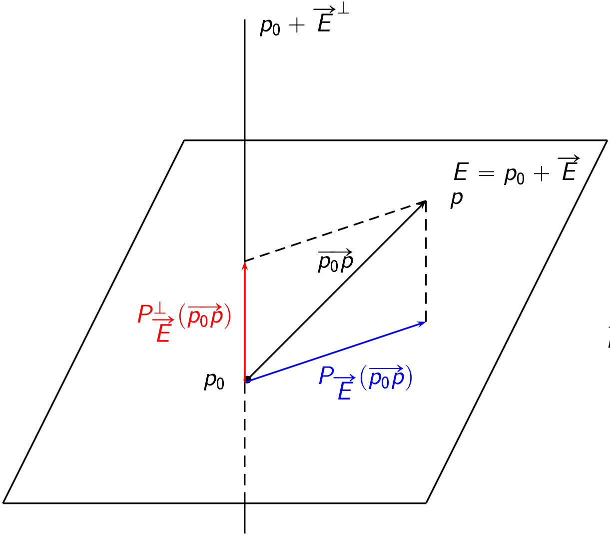
The transformation $\sigma_E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\sigma_E(p) = \sigma_E(p_0 + \overrightarrow{p_0 p}) = p_0 + S_{\overrightarrow{E}}(\overrightarrow{p_0 p}),$$

where $S_{\overrightarrow{E}}$ is the (linear) orthogonal reflection about $\overrightarrow{E}$, is called an **(affine) orthogonal reflection** about E.

Orthogonal Projection



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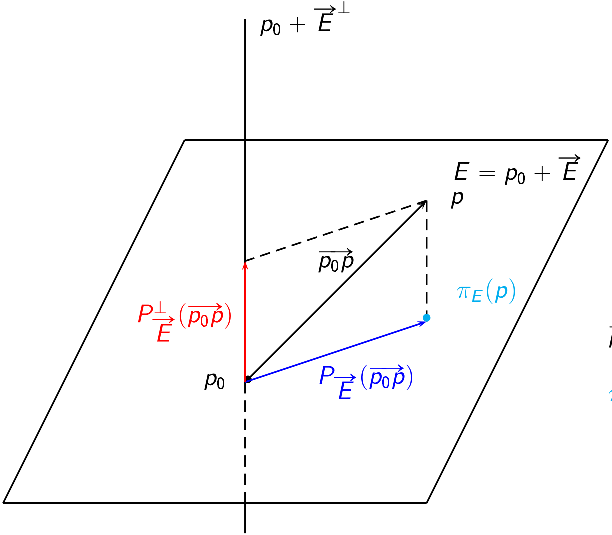

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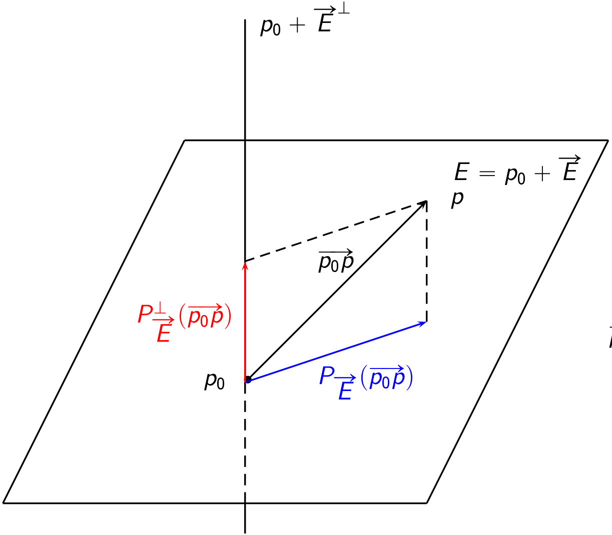

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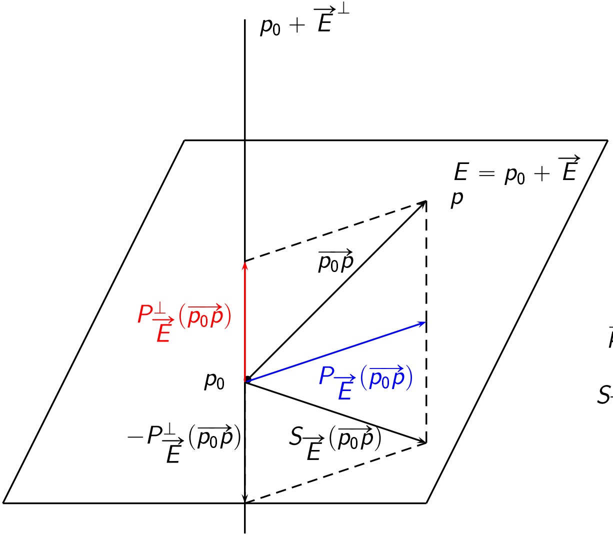

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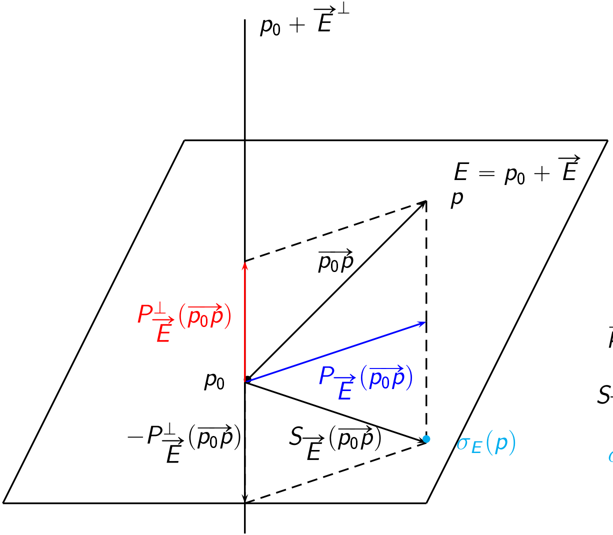

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$$\sigma_E(p) = p_0 + S_{\overrightarrow{E}}(\overrightarrow{p_0p})$$

Example

Let $p_0 = (1,1,1), p_1 = (1,2,3)$. Let $E = \mathrm{aff}(p_0, p_1)$ be an affine line. Compute orthogonal projection of $p = (2,0,1)$ on E.

$$\overrightarrow{p_0 p} = (2,0,1) - (1,1,1) = (1,-1,0), \quad \overrightarrow{E} = \mathrm{lin}((0,1,2)),$$

The linear projection of $\overrightarrow{p_0 p}$ on $\overrightarrow{E}$ is

$$P_{\overrightarrow{E}}(\overrightarrow{p_0 p}) = \frac{(1,-1,0)\cdot(0,1,2)}{0^2+1^2+2^2}(0,1,2) = -\frac{1}{5}(0,1,2).$$

Therefore $\pi_E(p) = (1,1,1) - \frac{1}{5}(0,1,2) = \frac{1}{5}(5,4,3)$.

Intersection of Affine Subspaces

Proposition

Let $E = p + V, H = q + W \subset \mathbb{R}^n$ *be two affine subspaces. Then either* $E \cap H = \varnothing$ *or* $p_0 \in E \cap H$ *and*

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$$\overrightarrow{pq} = v + w.$$

Intersection of Affine Subspaces (continued)

Proof.

Assume $\overrightarrow{pq} = v + w$ as above. Then $q - w \in H$ and $q - w = p + \overrightarrow{pq} - w = p + v \in E$.

Intersection of Affine Subspaces (continued)

Proof.

Assume $\overrightarrow{pq} = v + w$ as above. Then $q - w \in H$ and $q - w = p + \overrightarrow{pq} - w = p + v \in E$. Assume that $p_0 \in E \cap H$. Then $\overrightarrow{pq} = \overrightarrow{pp_0} + \overrightarrow{p_0 q}$ where $\overrightarrow{pp_0} \in V$ and $\overrightarrow{p_0 q} \in W$. $\square$

Projection as Intersection

Proposition

Let $V \subset \mathbb{R}^n$ be a vector subspace. For any $p, q \in \mathbb{R}^n$ the affine subspaces $p + V$ and $q + V^\perp$ intersect in exactly one point.

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By the previous lecture $\overrightarrow{pq} = P_V(\overrightarrow{pq}) + P_{V^\perp}(\overrightarrow{pq})$ and $V \cap V^\perp = \{0\}$. $\square$

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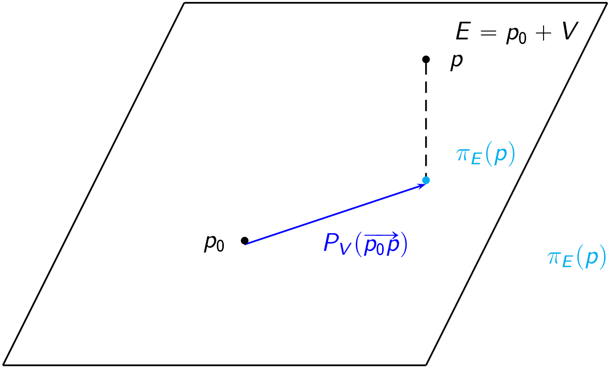
Proof.

We know $\overrightarrow{p_0 p} = P_V(\overrightarrow{p_0 p}) + P_{V^\perp}(\overrightarrow{p_0 p})$. As in the previous proof the only point of the intersection is equal to $p_0 + P_V(\overrightarrow{p_0 p})$. This is equal to $\pi_E(p)$ by definition. $\square$

Orthogonal Projection (again)



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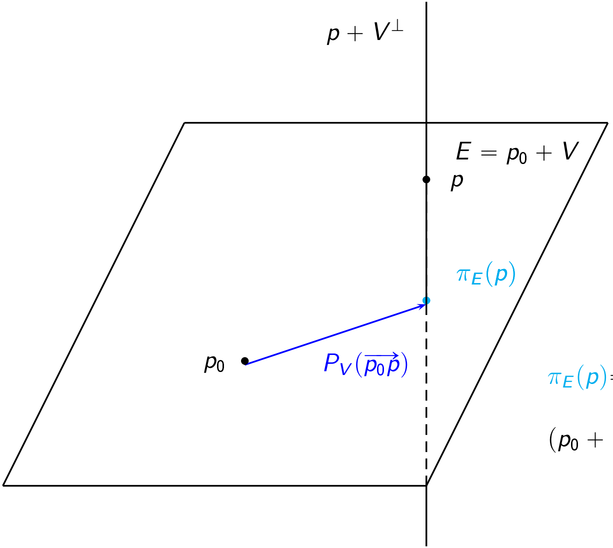

$$\pi_E(p) = p_0 + P_V(\overrightarrow{p_0 p})$$

$$(p_0 + V) \cap (p + V^{\perp}) = \pi_E(p)$$

Example

Let $p_0 = (1, 1, 1), p_1 = (1, 2, 3)$. Let $E = \text{aff}(p_0, p_1)$ be an affine line. Compute orthogonal projection of $p = (2, 0, 1)$ on E.

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$$E = \{(1, 1, 1) + t(0, 1, 2) \mid t \in \mathbb{R}\}.$$

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The orthogonal complement to $\overrightarrow{E}$ is two-dimensional hence given by a single equation $x_2 + 2x_3 = 0$. The point p satisfies the equation, therefore $p + \overrightarrow{E}^{\perp}$ is described by $x_2 + 2x_3 = 2$. By substituting the parametrization to the equation we get

$$(1+t) + 2(1+2t) = 2 \Longrightarrow t = -\frac{1}{5}.$$

Hence $\pi_E(2,0,1) = (1,1,1) - \frac{1}{5}(0,1,2) = \frac{1}{5}(5,4,3)$.

Example

Find a formula of an orthogonal projection onto the affine subspace $E = \mathrm{aff}((1,1,1,1),(1,0,1,0),(1,1,0,0)) \subset \mathbb{R}^4$. The subspace E can be written as $E = (1,1,1,1) + \mathrm{lin}((0,1,0,1),(0,0,1,1))$. We need to find an orthogonal basis of $\overrightarrow{E}$. Set $v_1 = (0,1,0,1), v_2 = (0,0,1,1)$. Then

$$w_1 = v_1 = (0,1,0,1),$$

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$$\pi_E(x_1,x_2,x_3,x_4) = (1,1,1,1) + P_{\overrightarrow{E}}(x_1-1,x_2-1,x_3-1,x_4-1) =$$

$$= (1,1,1,1) + \frac{x_2+x_4-2}{2}(0,1,0,1) + \frac{-x_2+2x_3+x_4-2}{6}(0,-1,2,1) =$$

Example (continued)

$$\pi_E(x_1, x_2, x_3, x_4) = (1,1,1,1) + P_{\overrightarrow{E}}(x_1 - 1, x_2 - 1, x_3 - 1, x_4 - 1) =$$

$$= (1,1,1,1) + \frac{x_2 + x_4 - 2}{2}(0,1,0,1) + \frac{-x_2 + 2x_3 + x_4 - 2}{6}(0,-1,2,1) =$$

$$= \left(1, \frac{2x_2 - x_3 + x_4 + 1}{3}, \frac{-x_2 + 2x_3 + x_4 + 1}{3}, \frac{x_2 + x_3 + 2x_4 - 1}{3}\right).$$

Example (continued)

Alternatively, by the definition $\pi'_E = P_{\overrightarrow{E}}$, therefore if $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, the linear part of the affine projection π_E is given by

$$M(P_{\overrightarrow{E}})^{st}_{st} = A(A^\intercal A)^{-1}A^\intercal = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

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It follows that

$$\pi_E(x_1, x_2, x_3, x_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix},$$

because $\pi_E(1, 1, 1, 1) = (1, 1, 1, 1)$.

Distance from an Affine Hyperplane

Proposition

Let $E \subset \mathbb{R}^n$ *be an affine hyperplane given by the equation*

$$E\colon a_1x_1 + \ldots + a_nx_n = b,$$

equivalently

$$E\colon a^{\mathsf{T}}x = b,$$

where $a = (a_1, \ldots, a_n), x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$*. Then the signed distance (positive in the direction of vector* $a \in \mathbb{R}^n$ *and negative otherwise) of the point* $p \in \mathbb{R}^n$ *from the affine hyperplane* E *is equal to*

$$d_s(p, E) = \frac{a^{\mathsf{T}}p - b}{\|a\|}.$$

Distance from an Affine Hyperplane (continued)

Proof.

The signed distance $d = d_s(p, E)$ is given by a system of equations

$$\left\{ \begin{array}{rcll} q & = & p - d\frac{a}{\|a\|}, & \text{i.e., } \|\overrightarrow{qp}\| = \left\| d\frac{a}{\|a\|} \right\| = |d| \\ a^{\mathsf{T}} q & = & b, & \text{i.e., } q \text{ belongs to } E \end{array} \right. ,$$

where $q \in E$ is the image of point p under the affine orthogonal projection onto E. The first equation multiplied by a^{T} on the left gives

$$b = a^{\mathsf{T}} q = a^{\mathsf{T}} p - d\|a\|.$$

$\square$

Distance from an Affine Hyperplane (continued)

Example

The signed distance of the point $p = (1, 2, 3, 4) \in \mathbb{R}^4$ from the affine hyperplane

$$E\colon x_1 - x_2 + 2x_3 - x_4 = 5,$$

is equal to

$$d_s(p, E) = \frac{1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 2 + 4 \cdot (-1) - 5}{\sqrt{1^2 + (-1)^2 + 2^2 + (-1)^2}} = -\frac{4}{\sqrt{7}}.$$

Distance from an Affine Subspace

Corollary

Let $E \subset \mathbb{R}^n$ be an affine subspace of $\mathbb{R}^n$ given by the system of linear equations

$$\begin{cases} a_1^\top x = b_1 \\ \quad\vdots \\ a_m^\top x = b_m \end{cases},$$

where $a_1, \dots, a_m \in \mathbb{R}^n$ are pairwise orthogonal, i.e.,

$$a_i \cdot a_j = a_i^\top a_j = 0 \quad \textit{for} \quad i \neq j.$$

The distance of point $p \in \mathbb{R}^n$ from the subspace E is equal to

$$d(p, E) = \sqrt{\sum_{i=1}^{m} \left(\frac{a_i^\top p - b_i}{\|a_i\|} \right)^2}.$$

Distance of Parallel Affine Hyperplanes

Corollary

Let $E, H \subset \mathbb{R}^n$ be two parallel affine hyperplanes given by the equations

$$E\colon a_1x_1 + \ldots + a_nx_n = b,$$

$$E'\colon a_1x_1 + \ldots + a_nx_n = b',$$

equivalently

$$E\colon a^{\mathsf{T}}x = b,$$

$$E'\colon a^{\mathsf{T}}x = b',$$

where $a = (a_1, \ldots, a_n),\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $b, b' \in \mathbb{R}$. Then distance between E and E' is equal to

$$d(E, E') = \frac{|b - b'|}{\|a\|}.$$

Two Lines in $\mathbb{R}^n$

Let $L_1, L_2 \subset \mathbb{R}^n$ be two lines in $\mathbb{R}^n$. Then either

i) the lines intersect, i.e.

$$L_1 \cap L_2 \neq \emptyset$$

  a) $\overrightarrow{L}_1 \neq \overrightarrow{L}_2$ (the lines intersect in exactly one point),
  b) $\overrightarrow{L}_1 = \overrightarrow{L}_2$ (the lines coincide).

ii) the lines are disjoint, i.e.

$$L_1 \cap L_2 = \emptyset$$

  a) $\overrightarrow{L}_1 \neq \overrightarrow{L}_2$ (the lines are skew),
  b) $\overrightarrow{L}_1 = \overrightarrow{L}_2$ (the lines are parallel).

Distance of Two Skew Lines in $\mathbb{R}^3$

Proposition

Let

$$L_1 = p_1 + \mathrm{lin}(v_1),$$

$$L_2 = p_2 + \mathrm{lin}(v_2),$$

be two skew lines in $\mathbb{R}^3$*, that is* $p_i \in \mathbb{R}^3$ *and* $v_i \in \mathbb{R}^3$ *for* $i = 1, 2$*. Then the distance between line* L_1 *and line* L_2 *is equal to*

$$d(L_1, L_2) = \frac{|v_3^{\mathsf{T}}(p_1 - p_2)|}{\|v_3\|},$$

where

$$\mathrm{lin}(v_3) = \mathrm{lin}(v_1, v_2)^{\perp}.$$

Distance of Two Skew Lines in $\mathbb{R}^3$

Proposition

Let

$$L_1 = p_1 + \mathrm{lin}(v_1),$$
$$L_2 = p_2 + \mathrm{lin}(v_2),$$

be two skew lines in $\mathbb{R}^3$, that is $p_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ for $i = 1, 2$. Then the distance between line L_1 and line L_2 is equal to

$$d(L_1, L_2) = \frac{|v_3^{\mathsf{T}}(p_1 - p_2)|}{\|v_3\|},$$

where

$$\mathrm{lin}(v_3) = \mathrm{lin}(v_1, v_2)^{\perp}.$$

Proof.

Use the formula for distance between two parallel planes containing respectively L_1 and L_2. Alternatively, the distance is equal to length of the image of the orthogonal projection of $\overrightarrow{p_1 p_2}$ onto the subspace $\mathrm{lin}(v_3)$. $\square$

Distance Between Two Affine Subspaces in $\mathbb{R}^n$

Proposition

Let $E\colon Ax = b$, and $H\colon Cx = d$, be two affine subspaces of $\mathbb{R}^n$, where $A \in M(s \times n, \mathbb{R})$ and $C \in M(t \times n, \mathbb{R})$. Assume that

i) *the rows of matrix A are orthonormal,*

ii) *the rows of matrix C are linearly independent,*

iii) *the columns of matrix $\begin{bmatrix} A \\ C \end{bmatrix}$ are linearly independent.*

Then the equation

$$\begin{bmatrix} A^\intercal A & C^\intercal \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A^\intercal b \\ d \end{bmatrix}$$

has a unique solution $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and the distance between E and H is equal to

$$d(E, H) = \|Ax_0 - b\|.$$

Distance Between Two Affine Subspaces in $\mathbb{R}^n$ (continued)

Proof.

Let $a_1, \ldots, a_s \in \mathbb{R}^n$ be the rows of matrix A. Then

$$d(p, E) = \min_{x \in H} \sqrt{\sum_{i=1}^{s} (a_i^{\mathsf{T}} x - b_i)^2},$$

that is, we need to solve the following constrained least squares problem:
minimize

$$\|Ax - b\|^2,$$

under the constraints

$$Cx = d.$$

I follow closely the proof which can be found in L. Vanderberghe's lecture[1].

[1] see `http://www.seas.ucla.edu/~vandenbe/133A/lectures/cls.pdf`, slide 11.4

Distance Between Two Affine Subspaces in $\mathbb{R}^n$ (continued)

Proof.

Assume that $Cx = d$. Then x_0 is optimal since

$$\|Ax - b\| = \|A(x - x_0) + (Ax_0 - b)\|^2 =$$

$$= \|A(x - x_0)\|^2 + \|Ax_0 - b\|^2 + 2(x - x_0)^\intercal A^\intercal (Ax_0 - b) =$$

$$(\text{as } A^\intercal A x_0 + C^\intercal y_0 = A^\intercal b)$$

$$= \|A(x - x_0)\|^2 + \|Ax_0 - b\|^2 - 2(x - x_0)^\intercal C^\intercal y_0 =$$

$$(\text{as } Cx = Cx_0 = d, \text{ i.e., } x, x_0 \in H)$$

$$= \|A(x - x_0)\|^2 + \|Ax_0 - b\|^2 \geqslant \|Ax_0 - b\|^2.$$

Distance Between Two Affine Subspaces in $\mathbb{R}^n$ (continued)

Proof.

Moreover, if $x_0, x_0' \in H \subset \mathbb{R}^n$ are optimal then $C(x_0 - x_0') = 0$, and by the first part of the proof, $A(x - x_0) = 0$, which by the condition *iii*), gives $x_0 - x_0' = 0$. It can be also checked that

$$\begin{bmatrix} A^\mathsf{T} A & C^\mathsf{T} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies that

$$x^\mathsf{T}(A^\mathsf{T} A x + C^\mathsf{T} y) = 0, \quad Cx = 0,$$

$$Ax = Cx = 0,$$

that is $x = 0$ by the condition *iii*). This implies that $C^\mathsf{T} y = 0$, which, by the condition *ii*) implies that $y = 0$. Therefore, the above matrix is non–singular. $\square$

Distance Between Two Affine Subspaces in $\mathbb{R}^n$ (continued)

Remark

The condition $iii)$ *guarantees that the affine subspaces* E, H *are either disjoint or they intersect in an exactly one point.*

Distance Between Two Affine Subspaces in $\mathbb{R}^n$ (continued)

Remark

The condition $iii)$ guarantees that the affine subspaces E, H are either disjoint or they intersect in an exactly one point.

In constrained least squares problem, that is:
minimize

$$\|Ax - b\|^2,$$

under the constraints

$$Cx = d,$$

we need to assume only $ii)$ and $iii)$. Condition $i)$ is needed to use the formula for the distance between a point and an affine plane.

Linear Isometries

Definition

Linear transformation $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is called a linear isometry if

$$\|\varphi(v)\| = \|v\|,$$

for any $v \in \mathbb{R}^n$.

Linear Isometries (continued)

Proposition

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. The following conditions are equivalent

i) φ *is an isometry,*

ii) *for any* $v, w \in \mathbb{R}^n$

$$\varphi(v) \cdot \varphi(w) = v \cdot w,$$

iii) *for any (or some) orthonormal basis* $\mathcal{A}$ *of* $\mathbb{R}^n$ *if* $A = M(\varphi)_{\mathcal{A}}^{\mathcal{A}}$ *then*

$$A^{\mathsf{T}} A = I,$$

i.e. the matrix A *is orthogonal.*

Proof.

Exercise. $\square$

Orthogonal Group

Definition

The group

$$O(n) = \{\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n \mid \varphi \text{ is a linear isometry}\},$$

is called the **orthogonal group**.

Example

Any orthogonal linear symmetry is a linear isometry.

Affine Isometries

Definition

Affine transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called a linear isometry if

$$d(f(p), f(q)) = d(p, q),$$

for any $p, q \in \mathbb{R}^n$.

Proposition

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation. Then it is equal to an linear isometry followed by a translation. In particular $\overrightarrow{f} \in O(n)$.

Proof.

Let $f(0) = q$. Let

$$\widetilde{f}(q) = f(q) + \overrightarrow{p0}.$$

Then $\widetilde{f}$ is a linear isometry, hence

$$f = t_{-\overrightarrow{p0}} \circ \widetilde{f}.$$

□

Affine Orthogonal Group

Definition

The group

$$AO(n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \mid f \text{ is an affine isometry}\},$$

is called the **affine orthogonal group**. The group

$$T(n) = \{t_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n \mid v \in \mathbb{R}^n\},$$

is called the **translation group**.

Affine Orthogonal Group (continued)

Proposition

For any affine isometry $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and any vector* $v \in \mathbb{R}^n$

$$f \circ t_v \circ f^{-1} = t_{f(v)}.$$

Proof.

Exercise. $\square$

Affine Orthogonal Group

Corollary

The affine orthogonal group is a semidirect product of groups $T(n)$ and $O(n)$, i.e.

$$AO(n) = T(n) \ltimes O(n),$$

in particular

i) $O(n)T(n) = AO(n), \quad O(n) \cap T(n) = \{id\}, \quad T(n) \lhd AO(n),$

ii) *for any $f \in AO(n)$ there exist unique $\varphi \in O(n)$, $v \in \mathbb{R}^n$ such that $f = \varphi \circ t_v$,*

iii) *for any $f \in AO(n)$ there exist unique $\varphi \in O(n)$, $v \in \mathbb{R}^n$ such that $f = t_v \circ \varphi$,*

iv) *the sequence*

$$1 \to T(n) \to AO(n) \to O(n) \to 1,$$

is exact.

Center of Mass

Let $p_1, \ldots, p_k \in \mathbb{R}^n$ be a points of mass $m_1, \ldots, m_k \in \mathbb{R}$ such that $M = \sum_{i=1}^{k} m_i \neq 0$ (negative mass is allowed).

Definition

The **center of mass** of points $p_1, \ldots, p_k$ is the affine combination

$$\overline{p} = \frac{1}{M} \sum_{i=1}^{k} m_i p_i .$$

Proposition

When $M > 0$ (resp. $M < 0$) the center of mass minimizes (resp. maximizes) the weighted sum of squared distances to points $p_1, \ldots, p_k$, i.e.

$$\overline{p} = \operatorname*{argmin}_{p \in \mathbb{R}^n} \sum_{i=1}^{k} m_i \|p - p_i\|^2 .$$

Center of Mass (continued)

Proof.

Assume $M > 0$. Let

$$f(p) = Mp^\intercal p - 2\sum_{i=1}^{k} p^\intercal p_i.$$

We need to show that

$$\overline{p} = \operatorname*{argmin}_{p \in \mathbb{R}^n} f(p).$$

Note that

$$\nabla f(p) = 2Mp - 2\sum_{i=1}^{k} m_i p_i,$$

therefore

$$\nabla f(\overline{p}) = 0.$$

Center of Mass (continued)

Proof.

Moreover $D^2 f = I$, and by the multivariate Taylor's formula

$$f(\overline{p} + h) = f(\overline{p}) + 2M\frac{1}{2!}h^{\mathsf{T}}h,$$

which proves that at $\overline{p} \in \mathbb{R}^n$ the function f attains its global minimum. $\square$

Affine Independence

Proposition

Points $p_0, \ldots, p_k \in \mathbb{R}^n$ are affine dependent if and only if there exist $a_0, \ldots, a_k \in \mathbb{R}$ not all equal to 0 such that

$$\sum_{i=0}^{k} a_i p_i = 0, \quad \sum_{i=0}^{k} a_i = 0.$$

Proof.

Easy exercise. If say $a_0 \neq 0$, dividing by a_0 we see that p_0 is an affine combination of $p_1, \ldots, p_k$. The converse is proven in a similar way. $\square$

Corollary

Points $p_0, \ldots, p_k \in \mathbb{R}^n$ are affinely dependent if and only if vectors $(p_0, 1), \ldots, (p_k, 1) \in \mathbb{R}^{n+1}$ are linearly dependent.

Affine Independence (continued)

Example

Points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ are colinear if and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

Points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) \in \mathbb{R}^3$ are coplanar if and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$