A series whose sum range is an arbitrary finite set

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Abstract

In finitely-dimensional spaces the sum range of a series has to be an affine subspace. It is long known that this is not the case in infinitely dimensional Banach spaces. In particular in 1984 M. I. Kadets and K. Wożniakowski obtained an example of a series the sum range of which consisted of two points, and asked whether it is possible to obtain more than two, but finitely many points. This paper answers the question positively, by showing how to obtain an arbitrary finite set as the sum range of a series in any infinitely dimensional Banach space.

1 Introduction

For a finitely-dimensional linear space $X$ the well-known Steinitz theorem states that for any conditionally convergent series the set of all possible limits of the series (called the sum range) is a affine subspace of $X$. In the "Scottish Book" S. Banach posed the problem whether the same holds for infinitely dimensional Banach spaces. The problem was solved negatively in the same book by J. Marcinkiewicz. In his example the sum range is the set $M$ of all integer-valued functions in $L_2[0,1]$. The next example, due to M. I. Ostrovskii, showed that the sum range does not have to be a closed set - the sum range of Ostrovskii's series was of the form $M + \sqrt{2}M$. Finally M. I. Kadets constructed an example in which the sum range consisted of two points, disproving, in particular, H. Hadwiger's conjecture that the sum range has to be the coset of some additive subgroup of $X$. The justification of the example was obtained independently by K. Woźniakowski and P. A. Kornilov in 1986.
It is still unknown what sets can be the sum ranges of series. In this paper it is shown that any finite subset of $X$ can be the sum range of a conditionally convergent series, which solves the problem posed by M. I. Kadets along with his two-point example (the problem is stated in [S91] in the general case, and in [U02] for $X = C(\Delta)$ and $n = 3$). The example is an extension of the 2-point example of M. I. Kadets as given in [S91]. As far as possible I shall try to keep the notation consistent with the notation given there, although the lack of suitable letters in the latin alphabet will force me to abandon the notation in a few places.

Everywhere all spaces are considered with the $L_1$ norm, i.e. $||f||_X = \int_X |f(x)| \, dx$. Frequently it is assumed it is obvious on which space the norm is taken, and only $||f||$ is written.

2 The results of K. Woźniakowski

The work in this paper is strongly inspired by the 2-point example of M. I. Kadets and the proof by K. Woźniakowski. In this paper not only the final result of Woźniakowski’s work will be used, but also multiple technical facts than can be found in the proof. Rather than force the reader to search for those in the original paper, I shall reiterate here Woźniakowski’s work, at times formulating the results in a way that will make them easier to use in the subsequent sections. The whole content of this section is based on [S91], and a reader familiar with this work may probably skip to the next section.

Let $Q = [0, 1]^\omega$ be the infinite dimensional cube, i.e. the product of a countable number of unit segments, equipped with the standard product topology and measure. By $x = (x_1, x_2, \ldots)$ we shall denote the variable on $Q$. Suppose we have two sequences of functions on the cube: $a^n_m$ and $b^n_{m,j}$, where $n \in \mathbb{N}$ and for a given $n$ both $m$ and $j$ belong to some finite sets $M_n$ and $J_n = M_{n+1}$. By $A_n$ we shall denote the set $\{a^n_m : m \in M_n\}$, and by $B_n$ the set $\{b^n_{m,j} : m \in M_n, j \in J_n\}$. For convenience if $X$ is a set of functions, by $\tilde{X}$ we shall denote the sum of the functions from $X$. We shall assume the following properties of the functions $a^n_m$ and $b^n_{m,j}$:

\begin{align*}
\tilde{A}_n(x) &= 1 \quad \forall n \in \mathbb{N} \forall x \in Q \quad (1) \\
||a^n_m|| &= \frac{1}{|M_n|} \quad (2) \\
\lim_{n \to \infty} |M_n| &= \infty \quad (3) \\
\text{The function } a^n_m \text{ depends only on the variable } x_n \quad (4) \\
\text{The functions } a^n_m \text{ assume only values 0 and 1} \quad (5) \\
b^n_{m,j} &= -a^n_m \cdot a^{n+1}_j \quad (6)
\end{align*}
We shall this collection of properties the **Kadets properties** on the cube $Q$. These properties mean that for each $n$ the interval $[0,1]$ is divided into $|M_n|$ sets $V^n_m$ of equal measure, and $a^n_m(x_1, x_2, \ldots) = 1$ iff $x_n \in V_m$. The functions $b^n_{m,j}$ are negative, and are supported on rectangles $(x_n, x_{n+1}) \in V^n_m \times V^{n+1}_j$.

From the Kadets properties we can easily deduce another few properties, mainly about the behaviour of $b^n_{m,j}$ based on properties 1 and 6:

\begin{align}
    a^n_m &= - \sum_{j \in J_n} b^n_{m,j}, \quad (7) \\
    a^{n+1}_j &= - \sum_{m \in M_n} b^n_{m,j} \quad (8) \\
    \delta B_n(x) &= -1 \forall n \in \mathbb{N} \quad (9) \\
    ||b^n_{m,j}|| &= \frac{1}{|M_n \times J_n|} \quad (10)
\end{align}

The function $b^n_{m,j}$ depends only on the variables $x_n$ and $x_{n+1}$ (12)

The functions $b^n_{m,j}$ assume only values 0 and -1 (13)

$a^n_m$ and $a^{n'}_{m'}$ have almost disjoint supports for $m \neq m'$ (14)

These properties follow easily from the Kadets properties. In property 14 by almost disjoint supports we mean that the intersection of two supports is of measure zero, we can obviously modify $a^n_m$ so that the Kadets properties still hold and the sets $\{x : a^n_m(x) > 0\}$ are disjoint for any constant $n$ and any two different values of $m$.

Let $c_k, k \in \mathbb{N}$ be any ordering of all the functions $a^n_m$ and $b^n_{m,j}$. Following Woźniakowski we shall investigate the convergence of any reordering $c_{\sigma(k)}$ of $c_k$.

**Proposition 2.1.** For any family of functions $c_k$ having the Kadets properties there exist such two permutations $\sigma$ and $\tau$ of $\mathbb{N}$ that $\sum c_{\sigma(k)} \to 0$ and $\sum c_{\tau(k)} \to 1$.

**Proof.** For $\sigma$ it is enough to order the functions $a^n_m$ lexicographically, i.e. $a^n_m$ appears before $a^{n'}_{m'}$ iff $n < n'$ or $n = n'$ and $m < m'$, and then immediately after each $a^n_m$ to put the whole set $\{b^n_{m,j} : j \in J_n\}$. Then the sum of each block consisting of a single function $a^n_m$ and the functions $b^n_{m,j}$ following it sums up to zero due to property 7, so the norm of each partial sum is the norm of the currently open block, which converges to zero due to properties 2, 10 and 3.

To get $\tau$ we order the functions $a^n_m$ in the same way, but we follow each function $a^n_m$ for $n > 1$ by the set $\{b^{n-1}_{l,m} : l \in M_{n-1}\}$, the functions $a^1_m$ are not followed by anything (as there
are no functions \( b_{m,j}^0 \). Then the functions \( a_m^1 \) sum up to the constant function 1 due to property 1. The following blocks again sum up to zero, this time due to property 8, so the norm of the difference between 1 and a particular partial sum is equal to the norm of the currently open block, which again converges to zero due to properties 2, 10 and 3.

**Remark 1.** The series of functions from Proposition 2.1 converge not only in the \( L_1 \) norm, but also in any \( L_p \) norm for any \( p < \infty \).

**Proof.** Again it is only a question of investigating the norm of any given block, as the sum of the previous blocks is zero. Functions \( a_m^n \) assume only values 0 and 1 and have disjoint supports for a set \( n \) from properties 5 and 14. Functions \( b_{m,j}^n \) for a given \( n \) have disjoint supports (this follows from properties 6 and 14) and assume values 0 and \(-1\) (from 13). Thus for any \( f \) being a sum of any set of functions \( a_m^n \) and \( b_{m,j}^n \) for a fixed \( n \) (or \( a_m^n \) and \( b_{m,j}^{n-1} \) for a fixed \( n \) in the case of \( \tau \) we have \( \|f\|_\infty \leq 1 \). This implies for any \( 1 \leq p < \infty \)

\[
\|f\|_p = \left( \int |f|^p \right)^{1/p} = \left( \int |f| \cdot |f|^{p-1} \right)^{1/p} \leq (\|f\|_1 \cdot \|f|^{p-1}\|_\infty)^{1/p} \leq \|f\|_1^{1/p} \cdot 1 = \|f\|_1^{1/p}.
\]

Thus if the sum of the series tended to zero in the \( L_1 \) norm with \( n \) tending to infinity, it also tends to zero in any \( L_p \) norm for \( p < \infty \).

**Proposition 2.2.** If a reordering \( c_{\sigma(k)} \) of a family \( c_k \) having the Kadets properties converges, it converges to a constant integer function.

**Proof.** Due to properties 4 and 12 and the finiteness of the sets \( M_n \) and \( J_n \) only finitely many of the functions \( c_{\sigma(k)} \) depend on a given variable \( x_l \) - precisely the functions belonging to \( A_l, B_l \) and \( B_{l-1} \). Moreover the sum of all these functions equals to the constant function \(-1\) due to properties 1 and 9. Thus for some integer \( K_0 \) the function \( \sum_{k=1}^{K} c_{\sigma(k)} \) is constant with regard to \( x_l \) for \( K \geq K_0 \), and thus the limit of the series also has to be constant with regard to \( x_l \). As this applied to an arbitrary \( l \), the limit simply has to be constant.

As the functions \( c_k \) are integer-valued (properties 5 and 13), their sums also have to be integer-valued. Thus all the partial sums of the series are integer-valued, and so the limit is also integer-valued, which ends the proof.

The next step will be to show that 0 and 1 are the only possible limits of a rearrangement of a family of functions with the Kadets property. We shall set a fixed rearrangement \( c_{\sigma(k)} \) of a given Kadets family, and we shall assume that the sum \( \sum_k c_{\sigma(k)} \) converges to some constant integer \( C \neq 1 \) (we know \( C = 1 \) can be achieved, it remains to prove that under these assumptions \( C = 0 \)).
Take an arbitrary $\delta > 0$ and fix $K_0 = K_0(\delta)$ such that for any $K > K_0$,

$$\| C - \sum_{k=1}^{K} c_{\sigma(k)} \| \leq \delta$$

(15)

and for any $m > l > K_0$ the Cauchy condition holds, i.e.

$$\| \sum_{k=l}^{m} c_{\sigma(k)} \| \leq \delta. \quad (16)$$

In addition to the sets $A_n$ and $B_n$ introduced earlier we shall also consider $V_n = \bigcup_{k=1}^{n} (A_k \cup B_k)$. Let $M$ be any integer such that

$$c_{\sigma(k)} \in V_M \cup A_{M+1} \quad \text{for any } k \leq K. \quad (17)$$

Let $c_k = c_{\sigma(k)}$ if $c_{\sigma(k)} \in V_M \cup A_{M+1}$, 0 otherwise. Similarly let $\hat{c}_k = c_{\sigma(k)}$ if $c_{\sigma(k)} \in B_{M+1}$, 0 otherwise. By $c^*$ we shall denote $\sum_{k=K_0+1}^{\infty} c_k^*$, while by $c$ we shall denote $\sum_{k=1}^{K_0} c_{\sigma(k)}$. The sum $c + c^*$ is equal to $\hat{V}_M + \tilde{A}_{M+1} = 0 + 1 = 1$. Hence $\| c^* \| = \| 1 - c^* \| \geq \| 1 - C \| - \| C - c \| \geq 1 - \delta$. Let $k_0 = K_0$ and

$$k_{j+1} = \min \left\{ k : \frac{1}{4} - \frac{5\delta}{4} \leq \| \sum_{i=k_{j+1}}^{k} c^*_k \| \leq \frac{1}{4} - \frac{\delta}{4} \right\}. \quad (18)$$

The indices $k_j$ are well defined for $j$ from 1 to 4 because the total norm of the sum $c^*$ is at least $1 - \delta$ and each single $c^*_k$ has norm $\leq \delta$ due to the Cauchy condition (16). For $j = 0, 1, 2, 3$ define the following functions:

$$c^*_j = \sum_{k=k_j+1}^{k_{j+1}} c^*_k, \quad \hat{c}_j = \sum_{k=k_j+1}^{k_{j+1}} \hat{c}_k, \quad \tilde{c}_j = \sum_{k=k_j+1}^{k_{j+1}} \tilde{c}_k,$$

and for $j = 1, 2, 3, 4$ set $r_j = \hat{c}_j - \tilde{c}_j - c^*_j$.

In plain words this means that we divide the functions $c_k$ for $k_j < k \leq k_{j+1}$ into three sets - those from $A_n$ for $n \leq M + 1$ or $B_n$ for $n \leq M$ (these add up to $c^*_j$), those from $B_{M+1}$ (these add up to $\tilde{c}_j$) and the rest (these add up to $r_j$). We will show that the functions from $B_{M+1}$ are placed in $c_k$ in similar proportions as the functions from $V_M \cup A_{M+1}$ — if, say, about a half of the functions from $V_M \cup A_{M+1}$ appeared in $c_k$ (that happens at $k_2$) then about a half of the functions from $B_{M+1}$ must have appeared, too.

We shall need to estimate the norm of two sums, which we would like to be negligible: $\| r_j \|$ and $\| \sum_{k=k_{j+1}}^{\infty} c^*_k \|$. We know that the sum of all $c_k$ up to $k_j$ is negligible, thus if the high-$n$ functions ($r_j$) are negligible, the functions from $V_M \cup A_{M+1}$ and $B_{M+1}$ have to approximately cancel each other out. This motivates the following proposition:
Proposition 2.3. For a Kadets family of functions $c_k$, its rearrangement $c_{\sigma(k)}$ converging to some $C \neq 1$, an arbitrary $\delta$ and an arbitrary $M > K_0(\delta)$ as above, with the notation as above we have $\sum_{j=1}^{4} ||r_j|| \leq 18\delta$.

Proof. As $c_j^{**}$ is integer-valued (being a sum of some functions from a Kadets family), the condition $||c_j^{**}|| \leq \frac{1}{3}$ implies $|\text{supp}c_j^{**}| \leq \frac{1}{3}$. Thus we can use lemma 1 (from the section "Auxiliary lemmas"

\[
||c_j^{**} + r_j|| \geq ||c_j^{**}|| + (1 - 2|\text{supp}c_j^{**}||)||r_j|| = ||c_j^{**}|| + \frac{1}{2}r_j.
\]

Of course $||\hat{c}_j|| \leq \delta$ from the Cauchy condition (16). We thus have

\[
1 \geq \sum_{j=1}^{4} ||\hat{c}_j|| = \sum_{j=1}^{4} ||\hat{c}_j - c_j^{**} - r_j|| \geq \sum_{j=1}^{4} ||c_j^{**} + r_j|| - \sum_{j=1}^{4} ||\hat{c}_j|| \geq
\]

\[
\geq \sum_{j=1}^{4} (||c_j^{**}|| + \frac{1}{2} ||r_j||) - 4\delta = 1 - 5\delta + \frac{1}{2} \sum_{j=1}^{4} ||r_j|| - 4\delta,
\]

which gives us the sought estimate upon $||r_j||$, namely $\sum_{j=1}^{4} ||r_j|| \leq 18\delta$. In particular, of course, each $||r_j||$ is bounded by $18\delta$.

\[\blacksquare\]

Corollary 2.4. With the notation and assumptions as above, $||\bar{c}_j + c_j^{**}|| \leq 19\delta$

Proof. $||\bar{c}_j + c_j^{**}|| = ||\bar{c}_j - r_j|| \leq ||\hat{c}_j|| + ||r_j|| \leq \delta + 18\delta = 19\delta$.

\[\blacksquare\]

Proposition 2.5. For a Kadets family of functions $c_k$, its rearrangement $c_{\sigma(k)}$ converging to some $C \neq 1$, an arbitrary $\delta$ and an arbitrary $M > K_0(\delta)$ as above, with the notation as above we have $||\sum_{k=k_4+1}^{\infty} c_k^*|| \leq 11\delta$.

Proof. We have

\[
||\bar{c}_j|| = ||\hat{c}_j - c_j^{**} - r_j|| \geq ||c_j^{**} + r_j|| - ||\hat{c}_j|| \geq ||c_j^{**}|| + \frac{1}{2} ||r_j|| - ||\hat{c}_j|| \geq ||c_j^{**}|| - \delta \geq \frac{1}{4} - \frac{9\delta}{4}.
\]

Take any index $k' > k_4$. If the norm $||\sum_{k=k_4+1}^{k'} c_k^*||$ were greater then $11\delta$, then there would exist some $k_5 \in (k_4, k')$ such that $12\delta \geq ||\sum_{k=k_4+1}^{k_5} c_k^*|| > 11\delta$. Then by a similar argument ($||\bar{c}_5|| \geq ||c_5^*|| + (1 - 24\delta)||r_5|| - ||\hat{c}_5|| \geq 11\delta - \delta$) the norm of $\sum_{k=k_4+1}^{k_5} \bar{c}_k$ would be larger then $10\delta$ — but all the functions $\bar{c}_k$ are negative, so $||\sum \bar{c}_k|| = \sum ||\bar{c}_k||$, which in this case gives $1 \geq ||\sum_{k=k_4+1}^{k_5} \bar{c}_k|| = \sum_{j=1}^{4} ||\bar{c}_j|| + ||\sum_{k=k_4+1}^{k_5} \bar{c}_k|| > 1 - 9\delta + 10\delta$, a contradiction. Thus the norm $||\sum_{k=k_4+1}^{\infty} c_k^*||$ has to be no greater than $11\delta$ (the sum is convergent, as it is in fact the sum of a finite number of functions, all coming from $V_{M+1}$). Let us denote this sum by $c_5^*$. 

\[\blacksquare\]
Now we can prove the main theorem of Woźniakowski’s work:

**Theorem 2.6.** For a Kadets family of functions $c_k$ and some rearrangement $c_{\sigma(k)}$ converging to $C \neq 1$ we have $|C - \frac{1}{2}| \leq \frac{1}{2}$, which (due to lemma 2.2) implies $C = 0$.

**Proof.** Consider any $\delta$, and the partial sum $S = \sum_{k=1}^{k_4} c_{\sigma(k)}$ with the notation as above. As $k_4 > K_0$, from assumption 15 we know that $||S - C|| \leq \delta$, so it will suffice to estimate $||S - \frac{1}{2}||$. We have

$$||S - \frac{1}{2}|| = ||c + \sum_{j=1}^{4} c_j^* + \sum_{j=1}^{4} \bar{c}_j + \sum_{j=1}^{4} r_j + c_5^* - c_5^* - \frac{1}{2}|| =$$

$$= ||c - \frac{1}{2} + \sum_{j=1}^{4} \bar{c}_j + \sum_{j=1}^{4} r_j - c_5^*|| \leq \left|\left|\frac{1}{2} + \sum_{j=1}^{4} \bar{c}_j\right| + \left|\sum_{j=1}^{4} r_j\right| + \left|c_5^*\right|\right|.$$

The function $\sum_{j=1}^{4} \bar{c}_j$ is a sum of functions from $B_{M+1}$, which means assumes only the values 0 and 1, thus $|\frac{1}{2} + \sum_{j=1}^{4} \bar{c}_j|$ is always equal to $\frac{1}{2}$. Inserting this and the bounds upon $r_j$ and $c_5^*$ we get

$$||S - \frac{1}{2}|| \leq \frac{1}{2} + 18\delta + 11\delta = \frac{1}{2} + 29\delta.$$

As $||S - C|| \leq \delta$ we get $||C - \frac{1}{2}|| \leq \frac{1}{2} + 30\delta$. As $\delta$ was chosen arbitrarily, we get the thesis. \qed

**Corollary 2.7.** The sum range of any Kadets family consists of two points, the constant functions 0 and 1, in any $L_p$ norm for $1 \leq p < \infty$

**Proof.** From Proposition 2.1 and Remark 1 we know that the two constant functions belong to the sum range. From the Proposition 2.2 we know that all functions in the sum range in the $L_1$ norm are constant integer functions, and from Theorem 2.6 we know that only the two functions 0 and 1 are eligible. If any permutation of the series converged to some function $g$ in some $L_p$ norm, then $||S_n - g||_p$ would tend to zero. But from the Hölder inequality we know that $||S_n - g||_p \geq ||S_n - g||_1$ (as the measure of the whole space is 1), which would imply that the series $S_n$ converges also in the $L_1$ norm, contradicting Theorem 2.6. \qed

### 3 The 3-point series

Denote by $Q_i = [0, 1]^ω$, $i = 1, 2, 3$ the infinite dimensional cube, i.e., the product of a countable number of unit segments equipped with the standard product probability measure. The example will be constructed in $L_1(Q_1 \cup Q_2 \cup Q_3)$. In the whole paper $t = (t_1, t_2, \ldots)$ will
denote the variable on $Q_1$, $u = (u_1, u_2, \ldots)$ will denote the variable on $Q_2$ and $v = (v_1, v_2, \ldots)$ will denote the variable on $Q_3$.

Our series will consist of functions of three kinds. The functions of the first kind are defined as follows:

$$f^n_m(t) = \begin{cases} 1 & \text{if } \frac{m-1}{n} < t < \frac{m}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$f^n_m(u) = f^n_m(v) = 0$$

for $n \in \mathbb{N}, m \in \{1, 2, \ldots, n\}$.

The second kind of functions is defined on all three cubes:

$$g^n_{m,j}(t) = \begin{cases} -1 & \text{if } \frac{m-1}{n} < t < \frac{m}{n} \text{ and } \frac{j-1}{n+1} < t_{n+1} < \frac{j}{n+1} \\ 0 & \text{otherwise} \end{cases}$$

$$g^n_{m,j}(u) = \begin{cases} \frac{1}{n+1} & \text{if } \frac{m-1}{n} < u < \frac{m}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$g^n_{m,j}(v) = \begin{cases} 1 & \text{if } \frac{(m-1)(n+1)+j-1}{n(n+1)} < v < \frac{(m-1)(n+1)+j}{n(n+1)} \\ 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}, m \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, n+1\}$.

The functions of the third kind are defined on $Q_2$ and $Q_3$:

$$h^n_{m,j,k}(t) = 0$$

$$h^n_{m,j,k}(u) = \begin{cases} -\frac{1}{(n+1)^2(n+2)} & \text{if } \frac{m-1}{n} < u < \frac{m}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$h^n_{m,j,k}(v) = \begin{cases} -1 & \text{if } \frac{(m-1)(n+1)+j-1}{n(n+1)} < v < \frac{(m-1)(n+1)+j}{n(n+1)} \text{ and } \frac{k-1}{(n+1)(n+2)} < v_{n+1} < \frac{k}{(n+1)(n+2)} \\ 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}, m \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, n+1\}, k \in \{1, 2, \ldots, (n+1)(n+2)\}$.

These functions have properties we want to generalize. Suppose we have three families of indices: $M_n, J_n$ and $K_n$, with $J_n = M_{n+1}$ and $K_n = M_{n+1} \times J_{n+1}$ (here $M_n = \{1, 2, \ldots, n\}$ and the mapping between $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n+1\}$ and $\{1, 2, \ldots, n(n+1)\}$ is given by $(m, j) \mapsto (m-1)(n+1) + j$). We have three families of functions: the first kind $\{f^n_m : n \in \mathbb{N}, m \in M_n\}$, the second kind $\{g^n_{m,j} : n \in \mathbb{N}, m \in M_n, j \in J_n\}$ and the third kind $\{h^n_{m,j,k} : n \in \mathbb{N}, m \in M_n, j \in J_n, k \in K_n\}$ defined on the union $Q_1 \cup Q_2 \cup Q_3$ of Hilbert cubes.
The families \( f \) and \( g \) form a Kadets family on \( Q_1 \), while the functions \( h \) disappear on \( Q_1 \). On \( Q_3 \) the functions \( g \) and \( h \) form a Kadets family (with \( M_n \times J_n \) being the first index set and \( K_n \) the second), while functions \( f \) disappear. The properties of the functions on \( Q_2 \) are different, as follows:

\[
\sum_{m \in M_n} \sum_{j \in J_n} g_{m,j}^n = 1
\]

\[
\sum_{m \in M_n} \sum_{j \in J_n} \sum_{k \in K_n} h_{m,j,k}^n = -1,
\]

\[
g_{m,j}^n = -\sum_{k \in K_n} h_{m,j,k}^n,
\]

\[
\sum_{m' \in M_{n+1}} g_{m',j}'^{n+1} = -\sum_{m \in M_n} \sum_{j \in J_n} \sum_{m' \in M_{n+1}} h_{m,j,(m',j)'}^n.
\]

\[
\sum_{j \in J_n} g_{m,j}^n \text{ assumes only values 0 and 1}
\]

\[
\int_{Q_2} g_{m,j}^n = \int_{Q_3} g_{m,j}^n
\]

\[
\int_{Q_2} h_{m,j,k}^n = \int_{Q_3} h_{m,j,k}^n
\]

\[
||g_{m,j}^n|| = \frac{1}{|M_n \times J_n|}
\]

\[
||h_{m,j,k}^n|| = \frac{1}{|M_n \times J_n \times K_n|}
\]

\[
\text{The functions } g_{m,j}^n \text{ and } h_{m,j,k}^n \text{ on } Q_2 \text{ depend only on } u_n
\]

Such a family of functions will be called a 3-Kadets family. It is easy (although maybe a bit tedious) to check that the family defined at the beginning of the section is a 3-Kadets family.

We shall denote by \( F_n \) the set \( \{m \times (m \in M_n)\} \), by \( G_n \) the set \( \{g_{m,j}^n : m \in M_n ; j \in J_n\} \) and by \( H_n \) the set \( \{h_{m,j,k}^n : m \in M_n , j \in J_n ; k \in K_n\} \). Also, by \( V_M \) we shall denote \( \bigcup_{k=1}^M F_k \cup G_k \cup H_k \). Denote by \( d_n \) any set enumeration of the whole 3-Kadets family. We are investigating the possible limits of \( \sum_{n=1}^{\infty} d_{\sigma(n)} \) for all permutations \( \sigma \) of \( \mathbb{N} \).

If a given rearrangement \( d_{\sigma(n)} \) of a 3-Kadets family converges, it converges on each of the cubes separately. On \( Q_1 \) and \( Q_3 \) we have Kadets families of functions, so the series on each of these cubes converges either to 0 or to 1 due to theorem 2.6. The new part is the
behaviour on \( Q_2 \). Same as in the first part of Proposition 2.2 only finitely many functions depend on a given variable \( u_n \) – the functions \( g_{m,j}^n \) and \( h_{m,j,k}^n \) – and their sum is constant, equal to zero due to property (21) applied to each \( j \) separately. Thus again the limit of the series \( \sum d_{\sigma(n)} \) on \( Q_2 \) has to be a constant function.

As \( \int_{Q_2} d_n = \int_{Q_3} d_n \) for any \( d_n \) (it is 0 for functions of the first kind and follows from properties 24 and 25 for the second and third kind), we get \( \int_{Q_2} \sum_{n=1}^{N} d_{\sigma(n)} = \int_{Q_3} \sum_{n=1}^{N} d_{\sigma(n)} \). As the integral is a continuous functional on \( L_1(Q_2) \) and \( L_1(Q_3) \) we get that the integrals of the limits have to be equal – but we know that the limit of \( \sum d_{\sigma(n)} \) on both \( Q_2 \) and \( Q_3 \) is a constant function, so the equality of integrals implies the equality of the limits. Thus the limit of the whole series is described by a pair of integers - the value on \( Q_1 \) and the value on \( Q_3 \). Let us denote the limit function by \( d_\infty \).

We are to show that it is possible to obtain exactly three different sums – precisely we can obtain \((0,0), (1,0) \) and \((1,1)\). To obtain any of these limits we first arrange the functions \( f \) and \( g \) as by Proposition 2.1 for a Kadets family on \( Q_1 \), and then after each \( g \) we put the \( h \) functions as by Proposition 2.1 for the cube \( Q_3 \). It remains to be seen if we get convergence on \( Q_2 \).

In the case of \((0,0)\) after a given \( f_m^n \) there appear the all functions \( g_{m,j}^n \) and \( h_{m,j,k}^n \) with the same \( m \) and \( n \). The sum of all these functions on \( Q_2 \) is equal to \( 0 \) due to property (21) for each \( j \) separately. Thus the norm of the partial sum on \( Q_2 \) is equal to the norm of the functions appearing after the last \( f \), and this tends to zero due to properties 26, 27 and 3 (all the functions have the same index \( m \), so the sum of their norms is equal to \( \frac{2}{\| \sigma \|} \)).

In the case of \((1,0)\) after a given \( f_m^n \) we get the functions \( g_{l,m}^{n-1} \) and \( h_{l,m,k}^{n-1} \). The sum of all these functions on \( Q_2 \) is again \( 0 \) due to property 21, this time applied to each \( l \) separately. Again the norm of the difference between the partial sum and \((1,0)\) is the norm of the part after the last \( f \), and that again tends to \( 0 \).

In the case of \((1,1)\) after a given \( f_m^n \) we get the functions \( g_{l,m}^{n-1} \) and \( h_{l,m,k}^{n-2} \). Their sum is \( 0 \) due to property 22 applied to them all. Again the norm of the difference between the partial sum and \( 1 \) tends to \( 0 \).

Again it is easy to check that the convergence occurs not only in the \( L_1 \) norm, but also in any \( L_p \) norm for \( p < \infty \) in the same way as in Remark 1 — on each of the cubes the \( L_\infty \) norm of the partial sums is bounded by \( 1 \).

One may wonder why the same arguments will not imply the convergence of the series arranged by rows in \( G_n \) and columns in \( H_{n-1} \) to \((0,1)\). The answer is we lack the equivalent of property 22 for this arrangement. To illustrate this let us look at the 3-Kadets family given at the beginning of the section arranged in this natural way. The sum \( \sum_{j=1}^{n+1} g_{m,j}^n \) on \( Q_2 \) is equal to \( 1 \) on \( \frac{m-1}{n} < u_n < \frac{m}{n} \), while the sum of the appropriate column of \( H_{n-1} \), \( \sum_{j=1}^{n+1} \sum_{m'=1}^{n-1} \sum_{j'=1}^{m-1} h_{m,j',(m-1)(n+1)+j} \) is equal to \( -\frac{1}{n} \) on the whole cube \( Q_2 \). Thus the partial
sums before each function of the first kind do not disappear as they did in the previous three cases, and when half of these functions from a given $F_n$ have appeared, the norm of the partial sum on $Q_2$ is $\frac{1}{2}$ regardless of $n$ – thus this particular series does not converge. Of course we still have to prove this is true for any rearrangement – but this example shows the nature of the reason why only three and not four possible limits exist.

\section{Auxiliary lemmas}

Before we begin the main part of this paper – i.e. the proof that our series cannot converge to $(0, 1)$ – we shall need three auxiliary lemmas:

\textbf{Lemma 1.} (Lemma given without proof in [O89]) Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces with probability measures. Let $f(x, y)$ and $g(x, y)$ be functions in $L_1(X \times Y)$, each of which depends on only one variable: $f(x, y) = \tilde{f}(x), g(x, y) = \tilde{g}(y)$. Then

$$||f + g|| \geq ||f|| + ||g||(1 - 2\mu(\text{supp} \tilde{f})).$$

\textbf{Proof.} $||f + g|| = \int_{X \times Y} |f + g| = \int_{\text{supp} \tilde{f} \times Y} |f + g| + \int_{(X \setminus \text{supp} \tilde{f}) \times Y} |g| \geq \int_{\text{supp} (f) \times Y} |f| - \int_{\text{supp}(\tilde{f}) \times Y} |g| + (1 - \mu(\text{supp} \tilde{f}))/||g|| = ||f|| - \mu(\text{supp} \tilde{f})||g|| + (1 - \mu(\text{supp} \tilde{f}))/||g|| = ||f|| + ||g||(1 - 2\mu(\text{supp} \tilde{f})).$ \hfill \Box

\textbf{Lemma 2.} Let $A, B, C$ be arbitrary spaces equipped with probabilistic measures and let $X = A \times B \times C$ be equipped with the standard product measure. Suppose $f, g$ are bounded functions defined on $X$ of the form $f(a, b, c) = \tilde{f}(a, b) = \sum_{k=1}^{N} s_k \chi_{A_k \times B_k}$ and $g(a, b, c) = \tilde{g}(b, c) = \sum_{k=1}^{N} t_k \chi_{B_k \times C_k}$, and $||f - g|| \leq \varepsilon$. Then there exists a function $h(a, b, c) = \tilde{h}(b)$ such that $||h - g|| \leq 2\varepsilon$ and $||h - f|| \leq 2\varepsilon$. Moreover if $f$ is integer-valued then $h$ can also be chosen to be integer-valued, and if for a family of sets $B_\alpha$ we have $\forall_{a, b_1, b_2 \in B_\alpha} \forall_{a \in A} f(a, b_1, c) = f(a, b_2, c)$, then we can choose a function $h$ constant on any set $B_\alpha$.

\textbf{Proof.} For any given $b \in B$ we take $\tilde{h}(b)$ such that

$$\int_A |\tilde{f}(a, b) - \tilde{h}(b)| da = \inf_{x \in \mathbb{R}} \int_A |\tilde{f}(a, b) - x| da.$$ 

This is well defined, as $f$ is bounded, and thus in fact the inf is taken over a bounded, and thus compact set. For such an $h$ we have

$$||h - f|| = \int_X |f(a, b, c) - \tilde{h}(b)| = \int_C \int_B \int_A |\tilde{f}(a, b) - \tilde{h}(b)| = \int_C \int_B \inf_x \{ \int_A |\tilde{f}(a, b) - x(b)| \} \leq$$
\[ \leq \int_C \int_B \int_A |\tilde{f}(a,b) - \tilde{g}(b, c)| \leq \int_C \int_B \int_A |f(a, b, c) - g(a, b, c)| = \|f - g\| \leq \varepsilon. \]

As \(\|h - f\| \leq \varepsilon\) and \(\|f - g\| \leq \varepsilon\), we immediately have \(\|g - h\| \leq 2\varepsilon\). As for the additional assumptions, if \(f\) and \(g\) are integer-valued, we can take the inf in the definition of \(h\) to be taken only over integers, with the same result. Regardless of that which option we choose, if \(f\) is constant with regard to \(b\) on any \(B_a\), then from the definition \(h\) also can be chosen to be constant on that set.

\textbf{Lemma 3.} Let \(A, B\) be arbitrary spaces equipped with probabilistic measures and \(X = A \times B\) equipped with the standard product measure. Suppose \(f, g, h\) are integer-valued functions defined on \(X\) fulfilling \(f(a, b) = \tilde{f}(a)\) and \(h(a, b) = \tilde{h}(b)\) for some \(\tilde{f}, \tilde{h}\). Suppose too that the function \(g\) assumes only two adjacent values (i.e. \(k\) and \(k+1\) for some \(k\)). Finally suppose that \(\|f + g + h\| < \delta < \frac{1}{9}\). Then either \(f\) or \(h\) is a constant function equal some integer \(c\) on a set of measure \(\geq 1 - 2\sqrt{\delta}\). Furthermore the function satisfies \(\|\tilde{f} - c\| < 3\sqrt{\delta}\) (or \(\|h - c\| < 3\sqrt{\delta}\), respectively).

\textbf{Proof.} The sets \(F_n = \tilde{f}^{-1}((-\infty, n])\) and \(H_n = \tilde{h}^{-1}((-\infty, n])\) form two increasing families, the sum of each is the whole space \(X\) and the intersection of each is empty. The measures \(|F_n|\) thus form an ascending sequence with elements arbitrarily close to \(0\) when \(n \to -\infty\) and arbitrarily close to \(1\) when \(n \to \infty\). As \(F_n \setminus F_{n-1} = \tilde{f}^{-1}(n)\), if \(\tilde{f}\) is not constant on any set of measure \(\geq 1 - 2\sqrt{\delta}\), then at least one element of the sequence \(|F_n|\), say \(F_{n_f}\), has to fall into the interval \([\sqrt{\delta}, 1 - \sqrt{\delta}]\). Similarly if \(\tilde{h}\) is constant on no set of measure \(\geq 1 - 2\sqrt{\delta}\), then for some \(n_h\) we have \(\sqrt{\delta} \geq |H_{n_h}| \geq 1 - \sqrt{\delta}\). Then on the set \(X_1 = F_{n_f} \times H_{n_h}\) we have \(f(a, b) + h(a, b) \leq n_h + n_f\), while on \(X_2 = (A \setminus F_{n_f}) \times (B \setminus H_{n_h})\) we have \(f(a, b) + h(a, b) \geq n_h + n_f + 2\). As \(g\) assumes two adjacent values, it is either \(-n_h - n_f + 1\) or \(-n_h - n_f + 1\) on the whole space \(X\). Thus on one of the sets \(X_1, X_2\) we have \(|f + g + h| \geq 1\), call it \(X_i\). As both \(X_1\) and \(X_2\) are products of two sets of measure \(\geq \sqrt{\delta}\), we have \(\|f + g + h\| = \int_X |f(a, b) + g(a, b) + h(a, b)| \geq \int_{X_i} |f(a, b) + g(a, b) + h(a, b)| \geq |X_i| \geq \delta\), which contradicts the assumptions of the lemma.

Thus one of the functions has to be constant on a large set. Without the loss of generality we may assume it is \(h\), and that it is equal to some integer \(c\). Let us examine the function \(f\), taking into account that all the functions are integer-valued, and thus if their sum is non-zero, it is at least one:

\[ \delta > \|f + g + h\| \geq \|f + g + c\|_{A \times h^{-1}(c)} \geq |\{\tilde{f}(a) \not\in \{-k-c, -k-c-1\}\} \times h^{-1}(c)| = |\{\tilde{f}(a) \not\in \{-k-c, -k-c-1\}\}| \cdot (1 - 2\sqrt{\delta}), \]
which implies \( \tilde{f}(a) \in \{-k - c, -k - c - 1\} \) on a set of measure at least \( 1 - \frac{\delta}{1 - 2\sqrt{\delta}} \). Denote this set by \( A' \). Now we return to the function \( h \):

\[
\|h - c\|_X \leq \frac{1}{1 - 2\sqrt{\delta}} \|h - c\|_{A' \times B} = \frac{1}{1 - 2\sqrt{\delta}} \|h - c\|_{A' \times (B \setminus h^{-1}(c))}.
\]

On the set \( A' \) the function \( f + g + c \) assumes values of absolute value \( \leq 1 \), so by substituting \( f + g \) for \( -c \) we shall decrease the norm at most by

\[
1 \cdot |A' \times (B \setminus h^{-1}(c))| \leq (1 - \frac{\delta}{1 - 2\sqrt{\delta}})(2\sqrt{\delta}) \leq 2\sqrt{\delta},
\]

thus giving the inequality

\[
\|h - c\|_X \leq \frac{1}{1 - 2\sqrt{\delta}} \|h + f + g\|_{A' \times (B \setminus h^{-1}(c))} + 2\sqrt{\delta} \leq \frac{1}{1 - 2\sqrt{\delta}} \|f + g + h\|_X + 2\sqrt{\delta} \leq \frac{\delta}{1 - 2\sqrt{\delta}} + 2\sqrt{\delta}.
\]

As \( \delta \leq \frac{1}{9} \), we have \( \frac{\delta}{1 - 2\sqrt{\delta}} \leq \sqrt{\delta} \), and thus \( \|h - c\| \leq 3\sqrt{\delta} \).

\[\square\]

5 The fourth point

Now we can begin to prove the main theorem of the paper:

**Theorem 5.1.** The function \( d_\infty = (0, 1) \) does not belong to the sum range of any 3-Kadets family series.

**Proof.** Suppose we have a rearrangement of some 3-Kadets family \( d_{\sigma(n)} \) the sum of which converges to \( d_\infty \). Again, take an arbitrarily small \( \delta > 0 \) (we shall need \( 927\sqrt{\delta} < \frac{1}{4} \), i.e. \( \delta < \frac{1}{13479264} \)) and an integer \( K \) satisfying inequalities (15) and (16), i.e. the tails and Cauchy sums are smaller than \( \delta \) for \( N > K \). Then, again, we take any \( M \) satisfying (17), i.e. such that \( V_M \) contains the first \( K \) elements of our series. Then we take an \( N_0 \) such that

\[
V_M \subset \{d_{\sigma(1)}, d_{\sigma(2)}, \ldots, d_{\sigma(N_0)}\}. \tag{29}
\]

Consider any fixed \( N > N_0 \). We will prove that

\[
\int_{Q^3} \sum_{n=1}^{N} d_{\sigma(n)} < \frac{1}{4}.
\]
Of course this suffices to prove that our series does not converge to 1 on $Q_3$, which contradicts the assumption the rearrangement converged to $(0, 1)$.

Denote for any $L, k \in \mathbb{Z}$ by $D_k$ the set $\{d_{\sigma(1)}, \ldots, d_{\sigma(k)}\}$, and by $F_L^k, G_L^k, H_L^k$ and $V_L^k$ the intersections of sets $F_L, G_L, H_L$ or $V_L$, respectively, with the set $D_k$. First we shall prove the following lemma:

**Lemma 4.** If functions $f_m^n, g_{m,j}^n$ and $h_{m,j,k}^n$ are a 3-Kadets family on the cubes $Q_1, Q_2$ and $Q_3$ and their set permutation $d_{\sigma(n)}$ tends to 0 on $Q_1$ and 1 on $Q_2$ and $Q_3$, and for a given $L$ we have $\int_{Q_1} \tilde{G}_L^N \geq \frac{1}{2} + 38\delta$, where $N > N_0$ as above, then there exists a $P \subset [0, 1]$ such that $|P| = \frac{1}{2}$ and $[(\tilde{H}_L^N)^{-1}(0)] \cap \{v : v_L \in P\} \subset Q_3$ has measure $\leq 450\delta$.

**Remark 2.** What this lemma really tells us is: if up to the $N$th element of the series at least half plus something (38$\delta$) of the $G_L$ functions have appeared, then at least half minus something (450$\delta$) of the $H_L$ functions had to appear. Moreover, the $H_L$ functions do not appear in a haphazard fashion - we know that at least half minus something rows had to appear (a row is the set of the functions $h_{m,j,k}^L$ with fixed $m$ and $j$ and varying $k$).

**Proof.** If $L \leq M$ then our thesis is automatically fulfilled - all functions from $H_L$ belong to the set $D_N$, thus we can take any set of measure $\frac{1}{2}$ for $P$ and the set $(\tilde{H}_L^N)^{-1}(0)$ will be empty, so $P$ will satisfy the required conditions.

Now consider the case $L > M$. The numbers $K$ and $L - 1$ satisfy the conditions (15), (16) and (17) (as $L > M$ and $M$ satisfied (17)). Thus we know there exist numbers $n_i$ satisfying (18). We shall prove that $N \geq n_2$.

We know that $\int_{Q_1} \tilde{G}_L^N = -\int_{Q_1} \tilde{G}_L^N$ (as all $g_{m,j}^n$ are of the same constant sign on each cube, the absolute value of the integral is equal to the norm, and the norms on each cube are equal). If $N < n_2$, then

$$\|\tilde{G}_L^N\|_{Q_1} \leq \|\tilde{G}_L^{n_2}\|_{Q_1} = \|\tilde{d}_1 + \tilde{d}_2\| \leq \|d_1^*\| + 19\delta + \|d_2^*\| + 19\delta < \frac{1}{2} + 38\delta,$$

which contradicts our assumption (the first inequality follows from the fact, that $g_{m,j}^n$ are all non-negative functions on $Q_1$, the second inequality from corollary 2.4).

Thus $N > n_2$. Consider $\tilde{V}_{L-1}^{n_2} + \tilde{F}_L^{n_2}$ on $Q_1$. This function is dependent on variables $t_1, t_2, \ldots, t_L$, while $\tilde{G}_L^{n_2} = \tilde{d}_1 + \tilde{d}_2$ on $Q_1$ depends on $t_L$ and $t_{L+1}$. From property (15) and Corollary 2.4 we get

$$\|\tilde{V}_{L-1}^{n_2} + \tilde{F}_L^{n_2} + \tilde{G}_L^{n_2}\| \leq \|\tilde{D}_k\| + \|d_1^* + \tilde{d}_1\| + \|d_2^* + \tilde{d}_2\| \leq \delta + 19\delta + 19\delta = 39\delta.$$

We can thus use lemma 2 for functions $-\tilde{V}_{L-1}^{n_2} - \tilde{F}_L^{n_2}$ and $\tilde{G}_L^{n_2}$ to get that on $Q_1$ both these functions are closer than $39\delta$ to some integer-valued function $\tilde{A}$ depending only on $t_L$. 14
Each function \( f^n_m \) depends only on \( t_n \) and assumes values 0 and 1 only (properties 5 and 4), so it is in fact the characteristic function of a set \( \{ t : t_n \in S^n_m \} \) for some \( S^n_m \subset [0, 1] \). As the \( f^n_m \) functions have disjoint support for a fixed \( n \), they are all constant on any given \( S^n_m \).

The \( g \) functions are also constant with regard to \( t_n \) on the \( S^n_m \) due to property 6, and all the other functions are constant with regard to \( t_n \) on the whole interval. Thus the functions \(-V_{L-1}^{n_2} - \tilde{F}_L^{n_2} \) and \( G_L^{n_2} \) are constant with respect to \( t_L \) on sets \( \{ t_L \in S^L_m \} \) we can choose \( \bar{A} \) to be constant on those sets. Thus \( \bar{A} \) coincides on \( Q_1 \) with the sum of some of the rows of \( G_L \), i.e. \( \bar{A} \) corresponds to some subset \( A \) of \( G_L \) such that for a fixed \( m \) either all or none of the functions \( g_{m,j}^{n_2} \) belong to \( A \). Define \( \bar{A} \) on \( Q_2 \) and \( Q_3 \) as the sum of all the elements of \( A \) as well, which agrees with our notation that \( \bar{U} \) is the sum of all the elements of \( U \) for an arbitrary set of functions.

We know from (18) and Proposition 2.5 that \( \| \sum_{n=n_2+1}^{\infty} d_n^* \|_{Q_1} \leq \frac{1-\delta}{4} + \frac{1-\delta}{4} + 11\delta \leq \frac{1}{2} + 11\delta \). Remark that \( (V_{L-1}^{n_2} + F_L^{n_2} + \sum_{n=n_2+1}^{\infty} d_n^*)|_{Q_1} = (V_{L-1} + F_L)|_{Q_1} = 1|_{Q_1} \), so \( \| V_{L-1}^{n_2} + F_L^{n_2} \|_{Q_1} \geq \frac{1}{2} - 11\delta \). On the other hand \( \| V_{L-1}^{n_2} + F_L^{n_2} \|_{Q_1} = \| \bar{D}_K + d_1^{n_2} + d_2^{n_2} \|_{Q_1} \leq \delta + \frac{1-\delta}{4} \leq \frac{1}{2} + \delta \). As \( \| V_{L-1}^{n_2} + F_L^{n_2} - \bar{A} \|_{Q_1} \leq 39\delta \), taking into account the equality \( \| \bar{A} \|_{Q_1} = \| \bar{A} \|_{Q_2} \) we can estimate that

\[
\frac{1}{2} - 50\delta \leq \| \bar{A} \|_{Q_2} \leq \frac{1}{2} + 40\delta.
\]

(30)

Distinct functions from \( G_L \) have disjoint supports on \( Q_1 \) (this follows from the properties 14 and 6 of Kadets families), and each has the same norm \( \psi = \frac{1}{|M_L|} \). Thus if the distance between two functions corresponding to two subsets of \( G_L \) on \( Q_1 \) is smaller than \( n\psi \), then at most \( n \) functions belong to the symmetric difference of those two subsets. If at most \( n \) functions belong to the symmetric difference, then the distance between the two functions on \( Q_2 \) is at most \( n\psi \) (as on \( Q_2 \) the norm of a single function is also equal \( \psi \) by property 26). Thus, in general, if \( B, C \subset G_L \), then \( \| \bar{B} - \bar{C} \|_{Q_1} \geq \| \bar{B} - \bar{C} \|_{Q_2} \). In particular \( G_L^{n_2} \) is at most \( 39\delta \) distant from \( \bar{A} \) on \( Q_2 \).

Now consider what happens on \( Q_2 \). From (23) the restriction of \( \bar{A} \) to \( Q_2 \) is equal to 1 on some set (on intervals \( t_L \in [\frac{m_1}{2}, \frac{m_2}{2}] \) for \( m \) such that \( g_{m,j}^{n_2} \in A \)) and 0 on the rest. From (15), as \( n_2 \geq K \), we have \( \| \bar{D}_n - 1 \|_{Q_2} \leq \delta \). If we substitute \( \bar{A} \) for \( G_L^{n_2} \), we will be at most \( 40\delta \) distant from zero, precisely

\[
\| \bar{D}_n - 1 - G_L^{n_2} + \bar{A} \|_{Q_2} \leq 40\delta.
\]

However as only \( G_L \) and \( H_L \) depend on \( u_L \), this sum is composed of two parts - the part \( \bar{A} + H_L^{n_2} \) dependent on \( u_L \) and the whole rest (i.e. \( \bar{D}_n - (G_L^{n_2} + H_L^{n_2}) \)) dependent on other variables. Thus we can apply a simplified version of lemma 2, with \( f = \bar{A} + H_L^{n_2} \), \( g = -(\bar{D}_n - V_L^{n_2}) \), and a trivial one-point space as \( B \). We learn that both our functions are
within 80δ from a function c dependent on b – but as B was a one-point space, c is a constant function. As A assumes values 0 and 1, and H^{n_2}_L \in [-1,0], their sum is non-negative on supp(A) and non-positive on the remainder of Q_2.

From (30) we know that |supp(A)| \geq \frac{1}{2} - 50δ, thus A + H^{n_2}_L is non-negative on a set of measure \geq \frac{1}{2} - 50δ. If c is positive, then (as \delta < \frac{1}{200})

\[ 80\delta \geq \|A + H^{n_2}_L - c\| \geq c\left(\frac{1}{2} - 50\delta\right) \geq \frac{c}{4}, \]

which implies c \leq 320\delta. Similarly if c is negative, we know from (30) that |Q_2 \setminus supp(A)| \geq \frac{1}{2} - 40\delta, yielding again c > -\frac{300}{80}\delta. Thus |c| < 320\delta, so \|A + H^{n_2}_L\| \leq \|A + H^{n_2}_L - c\| + |c| \leq 80\delta + 320\delta = 400\delta.

Thus H^{n_2}_L is within 400\delta of a function with values 0 and -1 on Q_2 – the function \bar{A}. Remark, that \bar{A} = -\bar{A}' on Q_2 for a subset A' of H_L with the property that for a given m either all of the functions h_{m,j,k}^L belong to A', or none of the functions belongs to A' (if a given g_{m,j}^L belongs to A, then all h_{m,j,k}^L belong to A'). If A', where A' \subset H_L, is a function assuming only values 0 and 1 on Q_2 and B \subset H_L, then

\[
\|\bar{A}' - \bar{B}\|_{Q_2} = \|\bar{A}' - \bar{B}\|_{supp\bar{A}'} + \|\bar{A}' - \bar{B}\|_{Q_2 \setminus supp\bar{A}'}
= \frac{1}{|M_L \times J_L \times K_L|} \left|\{h : h \in A' \land h \notin B\}\right| + \frac{1}{|M_L \times J_L \times K_L|} \left|\{h : h \notin A' \land h \in B\}\right|
= \frac{1}{|M_L \times J_L \times K_L|} |A \triangle B| = \|\bar{A}' - \bar{B}\|_{Q_3}.
\]

Let us take any subset A'' of H_L depending only on m and j with exactly half of the elements of H_L and containing A' or contained in A'. If B \subset C \subset H_L or C \subset B \subset H_L, then \|\bar{C} - \bar{B}\| = \|\bar{C}\| - \|\bar{B}\|, because all the the functions in H_L are non-positive. As A' = \bar{A} on Q_2 and from (30) \|\bar{A}\|_{Q_2} - \frac{1}{2} \leq 50\delta, we get \|\bar{A}' - \bar{A}'\|_{Q_2} \leq 50\delta, and thus \|H^{n_2}_L - \bar{A}'\|_{Q_3} = \|H^{n_2}_L - \bar{A}'\|_{Q_2} \leq 450\delta.

Now consider what happens on Q_3. As H^{n_2}_L and \bar{A}' are both integer-valued on Q_3, this means they differ on a set of measure at most 450\delta, and thus their difference can be positive on a set of measure at most 450\delta. When we increase n from n_2 to N the set where the difference is positive can only decrease. Thus \|\{}H^{N}_L - \bar{A}'\|_{Q_3}\| \leq 450\delta. Now for P we take supp\bar{A}''. The set \{(H^{N}_L)^{-1}(0) \cap \{v : v_L \in P\}\} is the set where H^{N}_L is equal to zero and \bar{A}'' is negative — thus their difference is positive, so the set has to have measure smaller than 450\delta, which is what we had to prove.

Now the main proof. Assume d_{\infty} = (0, 1), i.e. our series converges to 1 on Q_2 and Q_3 and to 0 on Q_1. We shall prove by induction upon L that \int_{Q_3} \bar{V}_L^N \leq \frac{1}{4}. As \sum_{n=1}^{N} d_{\sigma(n)} is finite,
its elements are contained in some $V_L$, thus if the thesis is true, we get $\int_{Q_3} \sum_{n=1}^N d_{\sigma(n)} \leq \frac{1}{4}$, which is what we had to prove. For $L < M$ we have $V_L \subset D_N$ and from property (7) $\int_{Q_3} V_L \leq 0 \leq \frac{1}{4}$. Now suppose we have the thesis for $L - 1$ and attempt to prove it for $L$. Denote by $P_1$ the function $(\tilde{V}^N_{L-1} + \tilde{G}^N_L)|_{Q_3}$ and by $P_2$ the function $\sum_{n>L} \tilde{G}_n^N + \tilde{H}_n^N|_{Q_3}$. Consider the function $H^N_L|_{Q_3}$. It depends on variables $v_L$ and $v_{L+1}$. The function $P_1$ depends on $v_1, \ldots, v_L$, while $P_2$ depends on $v_{L+1}, \ldots, v_Z$ for some $Z \in \mathbb{Z}$. The function $H^N_L|_{Q_3}$ assumes only values 0 and -1, all three functions $-H^N_L|_{Q_3}$, $P_1$ and $P_2$ are integer-valued, and from (15) their sum is less then $\delta$ distant from 1 on $Q_3$. Thus by taking $P'_1 = P_1 - 1$ we have three functions fulfilling the assumptions of lemma 3. Thus either $P_1$ or $P_2$ is within $3\sqrt{\delta}$ of a constant function. In each of these cases the proof will also depend on whether $\int_{Q_3} \tilde{G}^N_L \leq \frac{1}{2} + 38\delta$ or $\int_{Q_3} \tilde{G}^N_L > \frac{1}{2} + 38\delta$. Thus we have in total four cases to consider.

Suppose first that $P_2$ is within $3\sqrt{\delta}$ of a constant function. As $\|P_1 + P_2 + H^N_L - 1\| \leq \delta$, this means that $P_1 + H^N_L$ is within $3\sqrt{\delta} + \delta \leq 4\sqrt{\delta}$ of a constant function. If $\int_{Q_3} \tilde{G}^N_L \leq \frac{1}{2} + 38\delta$, then $\int_{Q_3} \tilde{V}^N_L = \int_{Q_3} \tilde{V}^N_{L-1} + \tilde{G}^N_L + \tilde{H}^N_L \leq \frac{1}{2} + \frac{1}{2} + 38\delta + 0 = \frac{3}{4} + 38\delta$. But this function is equal $P_1 + \tilde{H}^N_L$, and so is within $4\sqrt{\delta}$ of some constant integer $c$ and its integral also has to be within $4\sqrt{\delta}$ of $c$. As $4\sqrt{\delta} + 38\delta < \frac{1}{4}$, we get $c \leq 0$, thus $\int_{Q_3} \tilde{V}^N_L \leq c + 4\sqrt{\delta} \leq \frac{1}{4}$.

If $P_2$ is within $3\sqrt{\delta}$ of a constant function, and $\int_{Q_3} \tilde{G}^N_L > \frac{1}{2} + 38\delta$, then again $P_1 + \tilde{H}^N_L$ is within $4\sqrt{\delta}$ from a constant integer $c$. From lemma 4 we have in particular that $\int_{Q_3} \tilde{H}^N_L \leq -\frac{1}{2} + 450\delta$. Obviously $\int_{Q_3} \tilde{G}^N_L \leq 1$, thus $\int_{Q_3} \tilde{V}^N_L = \int_{Q_3} \tilde{V}^N_{L-1} + \tilde{G}^N_L + \tilde{H}^N_L \leq \frac{1}{4} - \frac{1}{2} + 450\delta = \frac{3}{4} + 450\delta$. As this is supposed again to within $4\sqrt{\delta}$ of $c$, we have $c \leq 0$ as $450\delta + 4\sqrt{\delta} \leq \frac{1}{4}$. Again thus $\int_{Q_3} \tilde{V}^N_L \leq c + 4\sqrt{\delta} \leq \frac{1}{4}$.

In the third case we suppose that $P'_1$, and thus also $P_1$ is within $3\sqrt{\delta}$ of a constant function and $\int_{Q_3} \tilde{G}^N_L \leq \frac{1}{2} + 38\delta$. As $\int_{Q_3} \tilde{V}^N_{L-1} \leq \frac{1}{4}$ from the inductive assumption, we have $\int_{Q_3} P_1 \leq \frac{3}{4} + 38\delta$. As $P_1$ is supposed to be within $3\sqrt{\delta}$ of some constant integer $c$, its integral also has to be within $3\sqrt{\delta}$ of $c$, which again implies $c \leq 0$ and $\int_{Q_3} P_1 \leq 3\sqrt{\delta}$. As $\tilde{V}^N_L = P_1 + \tilde{H}^N_L$ and $\tilde{H}^N_L \leq 0$, we get $\int_{Q_3} \tilde{V}^N_L \leq 3\sqrt{\delta} \leq \frac{1}{4}$.

The last case is when $P_1$ is within $3\sqrt{\delta}$ of a constant integer $c$ and $\int_{Q_3} \tilde{G}^N_L > \frac{1}{2} + 38\delta$. In this case from lemma 4 we know there exists a set $P' \subset Q_3$ dependent only on $v_L$ such that $|P'| = \frac{1}{2}$ and $\int_{P'} \tilde{H}^N_L \leq -\frac{1}{2} + 450\delta$. If $P_1$ is within $3\sqrt{\delta}$ of a constant integer function and $P_1 + P_2 + \tilde{H}^N_L$ is within $\delta$ of 1 (from 15) then $P_2 + \tilde{H}^N_L$ is within $3\sqrt{\delta} + \delta \leq 4\sqrt{\delta}$ of some constant integer function $C$. Taking $P_2' = P_2 - C$ we arrive in the situation of lemma 2: $\tilde{H}^N_L$ depends on $v_L$ and $v_{L+1}$ while $P_2'$ depends on $v_{L+1}, v_{L+2}, \ldots, v_Z$. This means that each of them is within $8\sqrt{\delta}$ of some integer function $P_3$ dependent only on $v_{L+1}$. As $\int_{P'} \tilde{H}^N_L \leq -\frac{1}{2} + 450\delta$ and $\|\tilde{H}^N_L - P_3\| \leq 8\sqrt{\delta}$, we gather that $\int_{P'} P_3 \leq -\frac{1}{2} + 450\delta + 8\sqrt{\delta} \leq -\frac{1}{4} + 458\sqrt{\delta}$. As $P'$
depends only on $v_L$ and $P_3$ only on $v_{L+1}$ and $|P'| = |Q_3 \setminus P'|$,
\[ \int_{Q_3} P_3 = \int_{P'} P_3 + \int_{Q_3 \setminus P'} P_3 = 2 \int_{P'} P_3 \leq -1 + 916\sqrt{\delta}. \]

Returning to $\tilde{H}_L^N$ we get $\int_{Q_3} \tilde{H}_L^N \leq \int_{Q_3} P_3 + 8\sqrt{\delta} \leq -1 + 924\sqrt{\delta}$.

As $\int_{Q_3} \tilde{G}_L^N \leq 1$ and $\int_{Q_3} V_{L-1} \leq \frac{1}{4}$ we get $\int_{Q_3} P_1 \leq \frac{5}{16}$. As before, $\int_{Q_3} P_1$ has to be within $3\sqrt{\delta}$ of the integer $c$, implying $c \leq 1$ and $\int_{Q_3} P_1 \leq 1 + 3\sqrt{\delta}$. We have $\int_{Q_3} V_L^N = \int_{Q_3} P_1 + \tilde{H}_L^N \leq 1 + 3\sqrt{\delta} - 1 + 924\sqrt{\delta} \leq 927\sqrt{\delta} \leq \frac{1}{4}$.

Thus in all four cases we have completed the induction step, which proves in a finite number of steps that $\int_{Q_3} \tilde{D}_N \leq \frac{1}{4}$. This holds for an arbitrary $N > N_0$, and would thus have to hold for the limit function, $\int_{Q_3} d_\infty \leq \frac{1}{4}$, which obviously contradicts the assumption that $d_{\infty}|_{Q_3} = 1$.

**Corollary 5.2.** A 3-Kadets series has a 3-point sum range, consisting of the functions $(0, 0)$, $(1, 0)$ and $(1, 1)$. As previously, this holds for any $L_p$ with $1 \leq p < \infty$.

### 6 More points

From the previous section we know how to make 3 points out of 2. The same mechanism can be applied to make $r + 1$ points out of $r$.

**Theorem 6.1.** For any $r > 1$ there exist a family $d_k$ of functions defined on a union of cubes $Q_1, \ldots, Q_N$ with an $r$-point sum range. Additionally we can distinguish two disjoint subsets $\mathcal{F}$ and $\mathcal{G}$ of $\{d_k : k \in \mathbb{N}\}$ which form a Kadets family on $Q_N$, while all other functions $d_k$ disappear on $Q_N$. Moreover one function in the sum range of $d_k$ is equal to 1 on $Q_N$ and all the other functions from the sum range disappear on $Q_N$. Finally there exist rearrangements convergent to any point of the sum range in which the sets $\mathcal{F}$ and $\mathcal{G}$ are arranged as in Proposition 2.1.

**Proof.** We shall prove the thesis by induction upon $r$. For $r = 2$ the original Kadets example with $N = 1$ satisfies the given conditions.

Suppose we have an appropriate family for $r - 1$. We add two cubes to the domain of $d_k$: $Q_{N+1}$ and $Q_{N+2}$. Denote by $x = (x_1, x_2, \ldots)$ the variable on $Q_{N+1}$ and by $y = (y_1, y_2, \ldots)$ the variable on $Q_{N+2}$. All the functions except $\mathcal{G}$ will disappear on these cubes. For each $n$ we divide the unit interval $[0, 1]$ into $|M_n|$ sets $S_m^n, m \in M_n$ of measure $\frac{1}{|M_n|}$ each. We define $g_{m,j}^n$ to be equal $\frac{1}{|M_n|}$ if $x_n \in S_m^n$, 0 otherwise. Next we define $K_n = M_{n+1} \times J_{n+1}$ and divide
the unit interval \([0,1]\) into \(|K_n|\) sets \(T^n_k\) of equal measure, and on \(Q_{N+2}\) define \(g^n_{m,j}\) to be equal to 1 if \(y_n \in T^n_{(m,j)}\), 0 otherwise. Finally to the functions \(d_k\) we add a set of functions \(\mathcal{H} = \{h^n_{m,j,k}\}\) which disappear on the cubes \(Q_1\) to \(Q_N\), and satisfy 
\[h^n_{m,j,k} = -\frac{1}{|K_n|}g^n_{m,j}\]
on \(Q_{N+1}\) and 
\[h^n_{m,j,k} = -g^n_{m,j} \cdot q^n_{k+1}\] on \(Q_{M+2}\).

It is again easy, although tedious, to check that \(\mathcal{F}, \mathcal{G}\) and the new functions \(\mathcal{H}\) form a 3-Kadets family on \(Q_N, Q_{N+1}, Q_{N+2}\). We claim that the set \(\{d_k\} \cup \mathcal{H}\) satisfies the conditions given in the theorem. The sets \(\mathcal{G}\) and \(\mathcal{H}\) form a Kadets family on \(Q_{N+2}\), all other functions disappear on \(Q_{N+2}\). We have to check the sum ranges. Let us fix any convergent rearrangement \(\epsilon_k\) of \(\{d_k\} \cup \mathcal{H}\). From the properties of 3-Kadets families given in section 3 we know that the limit on \(Q_{N+1}\) and \(Q_{N+2}\) is going to be the same, and equal either 0 or 1. From theorem 5.1 we know that if the series converges to 0 on \(Q_M\), it has to converge to 0 on \(Q_{N+1}\) and \(Q_{N+2}\). Thus at most \(r+1\) limits can be achieved - the functions with 0 on \(Q_N\) generate one each (by the 0-extension onto \(Q_{N+1} \cup Q_{N+2}\)), while the single function with 1 on \(Q_N\) can be extended by either 0 or 1 to \(Q_{N+1} \cup Q_{N+2}\). This also satisfies the condition that only one of the points in the sum range is 1 on \(Q_{N+2}\), while the other points disappear on \(Q_2\).

We can of course attain all the desired points in the sum range with \(\mathcal{G}\) and \(\mathcal{H}\) ordered as in Proposition 2.1 by taking the rearrangements with \(\mathcal{F}\) and \(\mathcal{G}\) ordered as in the proposition and inserting \(\mathcal{H}\) as in section 3.

Thus it is possible to attain an affine-independent finite set of any size \(r\) as a sum range of a conditionally convergent series. Again, this works for any \(L_p, 1 \leq p < \infty\).

To attain full generality on \(L_p\) we would attain arbitrary sum ranges, and not only the affine-independent sum range given above. We will do that according to the scheme from [K90], as follows:

**Lemma 5.** Let \(\Omega\) be an arbitrary probability space, \(c_n \in \mathbb{R}, c_n \to 0\) and let \(f_n \in L_2(\Omega)\) be a sequence of integer-valued functions. Then the series \(\sum_{n=1}^{\infty}(f_n + c_n)\) converges if and only if both \(\sum_{n=1}^{\infty}f_n\) and \(\sum_{n=1}^{\infty}c_n\) converge.

**Proof.** The “if” part is obvious. For the “only if” part it is enough to prove that if \(\sum c_n\) diverges, then \(\sum(f_n + c_n)\) has to diverge as well. In fact if \(\sum c_n\) diverges then there exists an \(\varepsilon \in (0, 1/4)\) such that for any \(N \in \mathbb{N}\) we have a large Cauchy sum above \(N\), i.e. for some \(l > k > N\) we have \(\sum_{n=k}^{l} c_n > \varepsilon\). As \(c_n \to 0\) we can take \(N\) large enough to ensure \(|c_j| < \varepsilon\) for \(j > N\). Thus we can select \(l = l(k)\) such that \(\varepsilon < \sum_{n=k}^{l(k)} c_n < 2\varepsilon < \frac{1}{2}\). But then 
\[\|\sum_{n=k}^{l} (f_n + c_n)\| \geq \varepsilon\] as a sum of an integer-valued function and a constant \(c \in (\varepsilon, 1/2)\), which ensures the divergence of \(\sum(f_n + c_n)\). \(\square\)
Now let us apply this lemma to our example from Theorem 6.1. We have a series \( d_k \) with an \( r + 2 \)-point sum range \( D \) defined on \( \Omega = \bigcup_{i=1}^{2r+1} Q_i \) of cubes. We consider it as a series defined on \( L_2(\Omega) \). Let us denote \( X = \text{lin}\{\chi_{Q_1}, \chi_{Q_2}, \ldots, \chi_{Q_{2r+1}}\} \), i.e. the subspace of the piece-wise constant functions on \( \Omega \). Let \( P : L_2(\Omega) \to X \) be the orthogonal projection onto \( X \). Denote by \( Y \) the subspace of \( X \) consisting of those piecewise constant functions \( (f_i)_{i=1}^{2r+1} \), where \( f_i \) is the value of \( f \) on \( Q_i \), that \( f_{2j} = f_{2j+1} \) for \( j = 1, 2, \ldots, r \). Thus for any \( d_k \) we have \( P(d_k) \in Y \), and thus \( P(D) \) is in fact a subset of \( Y \). Recall also that for odd indices \( j \) the functions \( d_k \) are integer-valued. Let \( T : Y \to Y \) be an arbitrary linear operator. Put \( d'_k = d_k + TP(d_k) \).

**Theorem 6.2.** The sum range \( D' \) of the series \( \sum d'_k \) equal \( (I + T)(D) \).

**Proof.** The inclusion \( (I + T)(D) \subset D' \) is evident. To prove the inverse inclusion consider an arbitrary arrangement \( (b'_k) \) of \( (d'_k) \) and the corresponding rearrangement \( (b_k) \) of \( (d_k) \). If \( (b'_k) \) converges to some point \( b' \in D' \), then its restrictions to \( Q_j \) for odd indices \( j \) satisfy the conditions of the lemma. Thus the restrictions to \( Q_j \) for odd \( j \) of \( TP(b_k) \) converge. Now the restrictions of \( TP(b_k) \) to \( Q_{j-1} \) are equal to the corresponding restrictions to \( Q_j \), so the whole series \( TP(b_k) \) converges. Then \( \sum b_k = \sum (b'_k - TP(b_k)) \) also has to converge. The sum of this series \( b \) belongs to \( D \), hence \( b' = b + TP(b) \) belongs to \( (I + T)(D) \). \( \square \)

This example can be transferred to any infinite-dimensional Banach space \( Y \) using the results of V.M. Kadets. In [S91], Theorem 7.2.2 states: Let \( X \) and \( Y \) be Banach spaces, \( X \not\rightarrow Y \). Suppose that \( X \) has a basis \( \{e_k\}_{k=1}^\infty \) and let \( \sum_{k=1}^\infty x_k \) be a series in \( X \) such that \( \text{SR}(\sum_{k=1}^\infty x_k) \) is not a linear set. Then for any monotone sequence of positive numbers \( \{a_k\}_{k=1}^\infty \) with \( a_k \to \infty, k \to \infty \), there exists a series \( \sum_{k=1}^\infty y_k \) in \( Y \) such that \( \text{SR}(\sum_{k=1}^\infty y_k) \) is not a linear set and \( \|y_k\| \leq a_k \|x_k\| \) for all \( k \in \mathbb{N} \), Corollary 7.2.1 points out that if \( X \) is \( l_2 \) then by Dvoretzky’s theorem \( X \not\rightarrow Y \), and Corollary 7.2.2 states that In any infinite-dimensional Banach space there are series whose sum range consists of two points. This is achieved by applying the two-point example in \( l_2 \) to Corollary 7.2.1 and following the proof of Theorem 7.2.2 to see that no new points appear and all the old ones are transferred to the space \( Y \). We have an \( n \)-point example in \( l_2 \) which can be in the same manner, through obvious modifications in the proof of Theorem 7.2.2 transferred to any Banach space \( Y \). Finally for any finite-dimensional subspaces \( H_1, H_2 \) of an infinitely dimensional Banach space \( Y \) and any isomorphism \( f : H_1 \to H_2 \) there exists an isomorphism \( \tilde{f} : Y \to Y \) extending \( f \). Thus having any \( n \) points satisfying some linear equations as a sum range of \( y_k \) in \( Y \) we can take an \( f \) transferring them to any other \( n \) points satisfying the same linear equations and then transfer the whole series by \( \tilde{f} \).
References


