Locally Finite Constraint Satisfaction Problems

joint work with Bartek Klin, Eryk Kopczyński
and Szymon Toruńczyk

Joanna Ochremiak

University of Warsaw

Verification Seminar, Oxford
2nd June 2015

Outline

1. Constraint Satisfaction Problems
   - Examples
   - Definition
   - Locally finite templates
2. Decidability of locally finite CSPs
   - Invariant solutions
   - Monotone-invariant solutions
3. Complexity
4. Descriptive complexity

Atoms

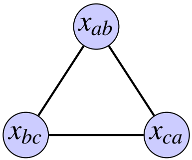
$\mathbb{A} = \{a, b, c, \ldots\}$ - countably infinite set of *atoms*

Graph colorability

G - an **infinite**, undirected graph:

- vertices indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- edges: x_{ab} — x_{bc}, where a and c are distinct

Subgraph of G:



Question: Is the infinite graph G three-colorable?

Systems of linear equations over $\mathbb{Z}_2$

E - an **infinite** system of linear equations over $\mathbb{Z}_2$

- variables indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- equations:

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinc}$$
$$x_{ab} + x_{bc} + x_{ca} = 0, \text{where } a, b \text{ and } c \text{ are distinct}$$

Question: Does system E have a solution?

Systems of linear equations over $\mathbb{Z}_2$

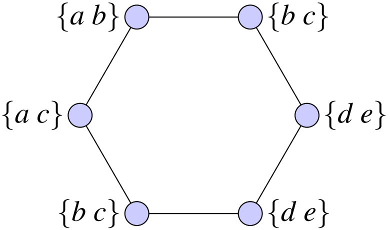
$$
\begin{array}{ccccccccccccccccccccccccccc}
ab & + & ba & & & & & & & & & & & & & & & & & & & & & & & = & 1 \\
ab & & & + & bc & + & ca & & & & & & & & & & & & & & & & & & & = & 0 \\
 & & ba & & & & & + & ac & + & cb & & & & & & & & & & & & & & & = & 0 \\
 & & & & bc & & & & & & & + & cd & + & db & & & & & & & & & & & = & 0 \\
 & & & & & & ca & & & & & & & & & + & ae & & & + & ec & & & & & = & 0 \\
 & & & & & & & & ac & & & + & cd & & & & & & & & & & & + & da & = & 0 \\
 & & & & & & & & & & cb & & & & & & & + & be & + & ec & & & & & = & 0 \\
 & & & & & & & & & & & & & & db & & & + & be & & & + & ed & & & = & 0 \\
 & & & & & & & & & & & & & & & & ae & & & & & + & ed & + & da & = & 0 \\
\hline
0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & = & 1
\end{array}
$$

Generalized graph colorability

G - a **finite**, undirected graph

We treat atoms as colors.

To each vertex we assign a set of n possible colors.



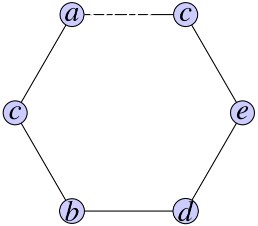
Question: Can this graph be colored with atoms such that no two adjacent vertices share the same color?

Generalized graph colorability

G - a **finite**, undirected graph

We treat atoms as colors.

To each vertex we assign a set of n possible colors.



Question: Can this graph be colored with atoms such that no two adjacent vertices share the same color?

Constraint Satisfaction Problem

A CSP *instance* $\mathbb{I} = (V, T, \mathcal{C})$:

- a set of variables: $V = \{x, y, \ldots\}$
- a set of their possible values: T
- a set of constraints: $\mathcal{C}$

A *constraint* is ...

A *solution* is an assignment which satisfies all the constraints.

Graph colorability as a CSP instance

G - an infinite, undirected graph:

- vertices indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- edges: x_{ab} — x_{bc}, where a and c are distinct

Question: Is this graph three-colorable?

$\mathbb{I}_G$ - a CSP instance:

- variables: vertices $V = \{x_{ab} \mid a, b \in \mathbb{A} \text{ distinct}\}$
- values: possible colors $T = \{1, 2, 3\}$
- constraints: $\mathcal{C} = \{\big((x_{ab}, x_{bc}), R\big) \mid a, b, c \in \mathbb{A} \text{ distinct}\}$

For each edge x_{ab} — x_{bc} there is a constraint: $\big((x_{ab}, x_{bc}), R\big)$, where $R = \big((1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\big)$.

Question: Is there a solution?

Constraint Satisfaction Problem

A CSP *instance* $\mathbb{I} = (V, T, \mathcal{C})$:

- a set of variables: $V = \{x, y, \ldots\}$
- a set of their possible values: T
- a set of constraints: $\mathcal{C}$

A *constraint* is a pair $(\bar{x}, R)$, where $\bar{x}$ is an n-tuple of variables and R is an n-ary relation over T.

A *solution* is an assignment which satisfies all the constraints.

Template

Let $\mathbb{T} = (T, R_1, R_2, \ldots)$. An instance $\mathbb{I}$ is over the *template* $\mathbb{T}$ if all relations in all constraints are from $\mathbb{T}$.

Example

3-colorability is over $\mathbb{T} = (\{1, 2, 3\}, R)$, where $R = \big((1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\big)$.

Constraint Satisfaction Problem

Problem: CSP($\mathbb{T}$)
Instance: CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

What kind of instances and templates do we consider?

Definable sets

Example

Set of variables in $\mathbb{I}_G$:

$\{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}$.

Example

Set of constraints in $\mathbb{I}_G$:

$\mathcal{C} = \{\big((x_{ab}, x_{bc}), R\big) \mid a, b, c \in \mathbb{A}, a \neq b, a \neq c, b \neq c\}$.

Definable templates and instances

Templates and instances are definable.

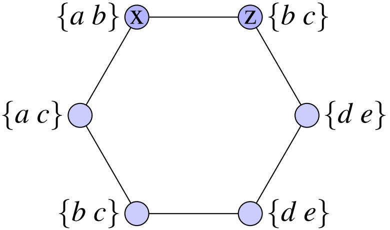
$$\Downarrow$$

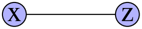
They can be treated as an input for algorithms.

Locally finite templates

A template $\mathbb{T} = \{T, R_1, R_2, \ldots\}$ is *locally finite* is every relation of $\mathbb{T}$ is finite.

Generalized graph colorability

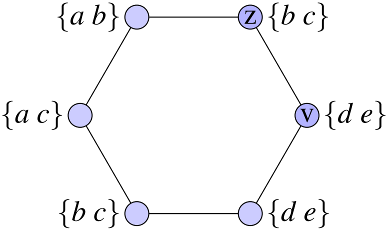


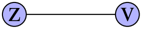
$c_1 = \Big($ 

$, R_{\{a,b\}\{b,c\}}\Big)$

$$R_{\{a,b\}\{b,c\}} = \{(a,b), (a,c), (b,c)\}$$

Generalized graph colorability

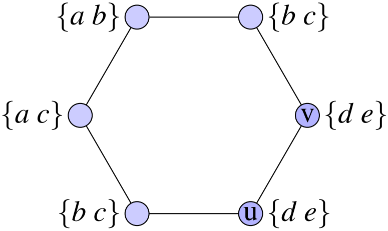


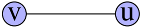
$c_2 = \Big($ 

$, R_{\{b,c\}\{d,e\}} \Big)$

$$R_{\{b,c\}\{d,e\}} = \{(b,d), (b,e), (c,d), (c,e)\}$$

Generalized graph colorability



$c_3 = ($ 

$, R_{\{d,e\}\{d,e\}})$

$R_{\{d,e\}\{d,e\}} = \{(d,e),(e,d)\}$

Locally Finite Constraint Satisfaction Problem

$\mathbb{T}$ - a definable, locally finite template

Problem: CSP($\mathbb{T}$)
Instance: a definable CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

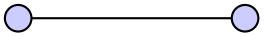
Linearly $\mathcal{P}$-patched structure

Fix a finite graph $\mathcal{P}$.

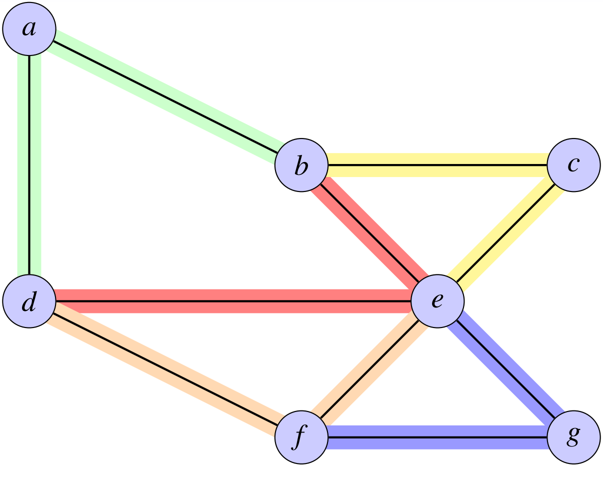
A linearly $\mathcal{P}$-patched structure is a graph:

- Covered by subgraphs (vertexwise and edgewise) called *patches*.
- Each patch is isomorphic to $\mathcal{P}$.
- There is a linear order on patches.

Example

If $\mathcal{P} =$  then linearly $\mathcal{P}$-patched structures are graphs with a linear order on the set of edges.

Linearly $\bigwedge$-patched structure



$\bigwedge < \bigwedge < \bigwedge < \bigwedge < \bigwedge$

Isomorphism of linearly $\mathcal{P}$-patched structures

Fix a finite graph $\mathcal{P}$.

Problem: Isomorphism of linearly $\mathcal{P}$-patched structures
Instance: $\mathbb{A}$, $\mathbb{B}$ - linearly $\mathcal{P}$-patched structures (vertices - atoms)
Decide: Are $\mathbb{A}$ and $\mathbb{B}$ isomorphic?

Problem: Isomorphism of linearly $\mathcal{P}$-patched structures
Instance: CSP instance $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$
Decide: Does $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$ have a solution?

All instances $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$ are over a locally finite template $T_{\mathcal{P}}$.

Isomorphism of linearly $\mathcal{P}$-patched structures

$\mathbb{A}, \quad \mathcal{P}_1^{\mathbb{A}} < \mathcal{P}_2^{\mathbb{A}} < \ldots < \mathcal{P}_n^{\mathbb{A}}$
$\mathbb{B}, \quad \mathcal{P}_1^{\mathbb{B}} < \mathcal{P}_2^{\mathbb{B}} < \ldots < \mathcal{P}_n^{\mathbb{B}}$

CSP instance $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$:

- variables: $v_i = (\mathcal{P}_i^{\mathbb{A}}, \mathcal{P}_i^{\mathbb{B}})$ - pairs of corresponding patches,
- values: isomorphisms $f_i : \mathcal{P}_i^{\mathbb{A}} \to \mathcal{P}_i^{\mathbb{B}}$,
- constraints: for a pair (v_i, v_j) a constraint $((v_i, v_j), R)$.

R says that (f_i, f_j) is an isomorphism from $\mathcal{P}_i^{\mathbb{A}} \cup \mathcal{P}_j^{\mathbb{A}}$ to $\mathcal{P}_i^{\mathbb{B}} \cup \mathcal{P}_j^{\mathbb{B}}$

Atom permutations

Example

$\mathrm{Aut}(\mathbb{A},=)$ acts on set of variables in $\mathbb{I}_G$:
$\{x_{ab} \mid a,b \in \mathbb{A}, a \neq b\}$.

π - a permutation of atoms
$\pi(x_{ab}) = x_{\pi(a)\pi(b)}$

Example

$\mathrm{Aut}(\mathbb{A},=)$ acts on set of constraints in $\mathbb{I}_G$:
$\mathcal{C} = \{\big((x_{ab}, x_{bc}), R\big) \mid a,b,c \in \mathbb{A}, a \neq b, a \neq c, b \neq c\}$.

π - a permutation of atoms
$\pi\big((x_{ab}, x_{bc}), R\big) = \big((x_{\pi(a)\pi(b)}, x_{\pi(b)\pi(c)}), R\big)$

Invariant assignments

$\mathbb{I} = (V, T, \mathcal{C})$

The group $\mathrm{Aut}(\mathbb{A}, =)$ (atom permutations) acts on the set of assignments $f : V \to T$.

$$\begin{array}{ll} f & x \mapsto t \\ \pi \cdot f & \pi(x) \mapsto \pi(t) \end{array}$$

$$\text{fixpoint} \ \leftrightarrow \ \text{invariant assignment}$$

An assignment $f : V \to T$ is *invariant* if $\pi \cdot f = f$ for every permutation π of atoms.

Invariant assignments

Example

An infinite system of linear equations over $\mathbb{Z}_2$

- variables indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- equations:

$$x_{ab} + x_{ba} = 0, \text{ where } a \text{ and } b \text{ are distinct}$$

$f(x_{ab}) = 0$ for all pairs ab $\rightarrow$ invariant solution

$f(x_{ab}) = 0$ for all pairs except for cd and dc
$f(x_{cd}) = f(x_{dc}) = 1$ $\rightarrow$ not invariant solution

Invariant solutions

Fact

- *There are finitely many invariant assignments* $f : V \to T$.
- *Invariant assignments* $f : V \to T$ *can be represented in a finite way (by first order formulas using* $=$*).*

Fact

For any definable, locally finite template $\mathbb{T}$*, it is decidable whether a given definable instance* $\mathbb{I}$ *over* $\mathbb{T}$ *has an invariant solution.*

Sadly, an instance can have a solution but no invariant one.

Invariant solutions

Example

An infinite system of linear equations over $\mathbb{Z}_2$

- variables indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- equations:

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

Solution:
for every set $\{a, b\}$ of distinct atoms $f(x_{ab}) = 1$ and $f(x_{ba}) = 0$

There is no invariant solution.

Monotone-invariant assignments

Fix a linear order on atoms isomorphic to $(\mathbb{Q}, \leq)$.
The group $\mathrm{Aut}(\mathbb{A}, \leq)$ (monotone permutations) acts on the set of assignments $f : V \to T$. (the same way)

fixpoint $\leftrightarrow$ monotone-invariant assignment

An assignment $f : V \to T$ is *monotone-invariant* if $\pi \cdot f = f$ for every monotone permutation π of atoms.

Monotone-invariant solutions

Fact

- *There are finitely many monotone-invariant assignments* $f : V \to T$.
- *Monotone-invariant assignments* $f : V \to T$ *can be represented in a finite way (by first order formulas using* $\leq$*).*

Fact

For any definable, locally finite template $\mathbb{T}$*, it is decidable whether a given definable instance* $\mathbb{I}$ *over* $\mathbb{T}$ *has a monotone-invariant solution.*

Locally finite CSPs are decidable

Theorem (Pestov)

Every continuous action of the topological group $\mathrm{Aut}(\mathbb{Q}, \leq)$ *on a compact Hausdorff space has a fixpoint.*

Theorem

An instance $\mathbb{I}$ *has a solution if and only if it has a monotone-invariant solution.*

Proof.

$\mathrm{Sol}(\mathbb{I}, \mathbb{T})$ - the set of solutions (possibly empty)
$\mathrm{Sol}(\mathbb{I}, \mathbb{T})$ is a compact Hausdorff space (because $\mathbb{T}$ is locally finite).

Locally finite CSPs are decidable

Corollary

For any definable, locally finite template $\mathbb{T}$, it is decidable whether a given definable instance $\mathbb{I}$ over $\mathbb{T}$ has a solution.

Locally finite CSPs are decidable

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

x_{ea} x_{ae}

x_{ac}

x_{ca}

x_{ba} x_{ab}

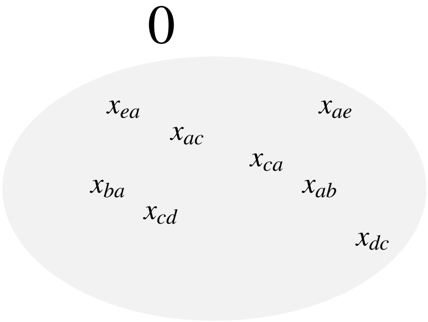
x_{cd}

x_{dc}

Locally finite CSPs are decidable

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

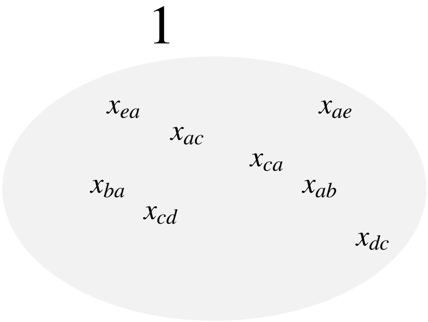
Invariant:



Locally finite CSPs are decidable

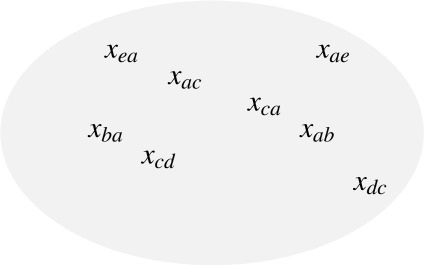
$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

Invariant:



Locally finite CSPs are decidable

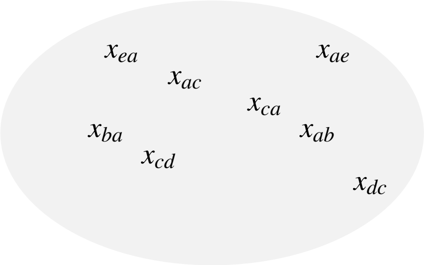
$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$



$\mathrm{Aut}(\mathbb{A}, \leq)$

Locally finite CSPs are decidable

$x_{ab} + x_{ba} = 1,$ where a and b are distinct

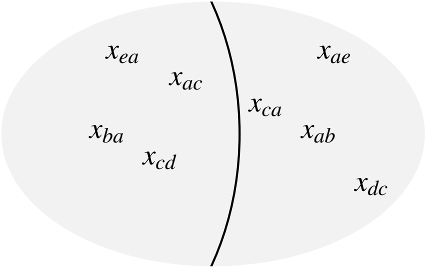


$\mathrm{Aut}(\mathbb{A}, \leq)$

$e < b < a < c < d$

Locally finite CSPs are decidable

$x_{ab} + x_{ba} = 1,$ where a and b are distinct

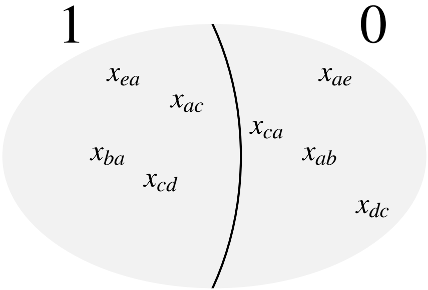


$\mathrm{Aut}(\mathbb{A}, \leq)$ $\quad e < b < a < c < d$

Locally finite CSPs are decidable

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

Monotone-invariant:



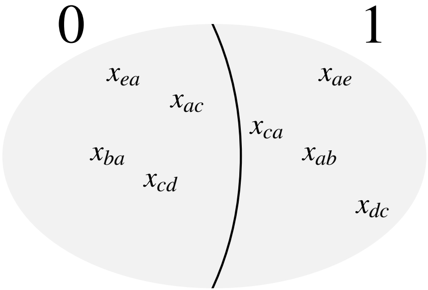
$\mathrm{Aut}(\mathbb{A}, \leq)$

$e < b < a < c < d$

Locally finite CSPs are decidable

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

Monotone-invariant:



$\mathrm{Aut}(\mathbb{A}, \leq)$

$e < b < a < c < d$

Classical Constraint Satisfaction Problem

$\mathbb{T}$ - a finite template

Problem: CSP($\mathbb{T}$)
Instance: a finite CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

Goal: Characterize CSP($\mathbb{T}$) solvable in PTime.

$\mathbb{T}$ - a definable, locally finite template

Problem: $\text{CSP}_{fin}(\mathbb{T})$
Instance: a finite CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

Can those characterizations be generalized to locally finite CSPs?

Bounded width

wide class of templates solvable in PTime

$\downarrow$

templates of *bounded width*

Theorem (Larose; Zádori; Barto; Kozik)

A finite template $\mathbb{T}$ has bounded width (solvable in Datalog) if and only if an instance $\mathbb{I}_{\mathbb{T}}^{bw}$ over $\mathbb{T}$ has a solution.

$\mathbb{I}_{\mathbb{T}}^{bw}$ characterizes some algebraic properties of $\mathbb{T}$ called *p*olymorphisms

Bounded width

Corollary

A locally finite template $\mathbb{T}$ has bounded width (solvable in Datalog) if and only if an instance $\mathbb{I}_{\mathbb{T}}^{bw}$ over $\mathbb{T}$ has a solution.

$\mathbb{I}_{\mathbb{T}}^{bw}$ is a definable instance computable from $\mathbb{T}$

$\Downarrow$

Effective characterization of locally finite templates of bounded width.

Descriptive Complexity Theory

Descriptive complexity theory $\rightarrow$ identify logics which are equiexpressive with known complexity classes, over finite relational structures.

Theorem (Fagin)

Existential second order logic $\exists SO$ captures **NP** *over finite structures.*

- Every property decidable in **NP** can be defined by an ∃SO formula.
- Any ∃SO formula defines an **NP** property.

Central open question: is there a (reasonable) logic which captures PTime.

Least Fixpoint Logic

LFP - extension of first order logic by a fixpoint operator
LFP+C - least fixpoint logic with counting

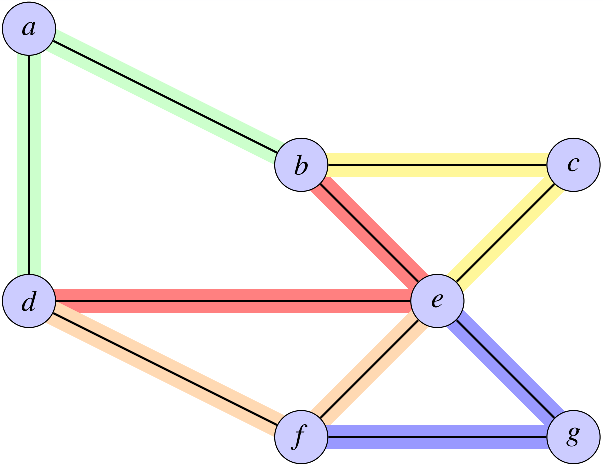
Theorem (Immerman; Vardi)

LFP = PTime over linearly ordered structures.

Theorem (Cai; Fürer; Immerman)

LFP+C ≠ PTime over all structures.

Linearly $\bigwedge$-patched structure



$\bigwedge < \bigwedge < \bigwedge < \bigwedge < \bigwedge$

Isomorphism of linearly $\mathcal{P}$-patched structures

Fix a finite graph $\mathcal{P}$.

Problem: Isomorphism of linearly $\mathcal{P}$-patched structures
Instance: $\mathbb{A}$, $\mathbb{B}$ - linearly $\mathcal{P}$-patched structures (vertices - atoms)
Decide: Are $\mathbb{A}$ and $\mathbb{B}$ isomorphic?

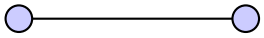
Problem: Isomorphism of linearly $\mathcal{P}$-patched structures
Instance: CSP instance $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$
Decide: Does $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$ have a solution?

All instances $\mathbb{I}_{\mathbb{A}}^{\mathbb{B}}$ are over a locally finite template $T_{\mathcal{P}}$.

Least Fixpoint Logic

LFP - extension of first order logic by a fixpoint operator
LFP+C - least fixpoint logic with counting

LFP+C $=$ LFP over linearly $\mathcal{P}$-patched structures

$\mathcal{P} =$ 

Theorem

LFP $=$ PTime over linearly $\mathcal{P}$-patched structures.

$\mathcal{P} =$ graph with 6 vertices

Theorem

LFP $\neq$ PTime over linearly $\mathcal{P}$-patched structures.

Common generalization

Theorem

The following conditions are equivalent:

- *LFP = PTime over linearly $\mathcal{P}$-patched structures,*
- *$T_{\mathcal{P}}$ has bounded width.*

Moreover, $T_{\mathcal{P}}$ is a locally finite template computable from $\mathcal{P}$ so the second condition is decidable.