Definable Ellipsoid Method, Sums-of-Squares Proofs, and the Isomorphism Problem

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Algorithm analysis through proof complexity

Objective: Identify tractable instances of hard problems.

Algorithms that compute:

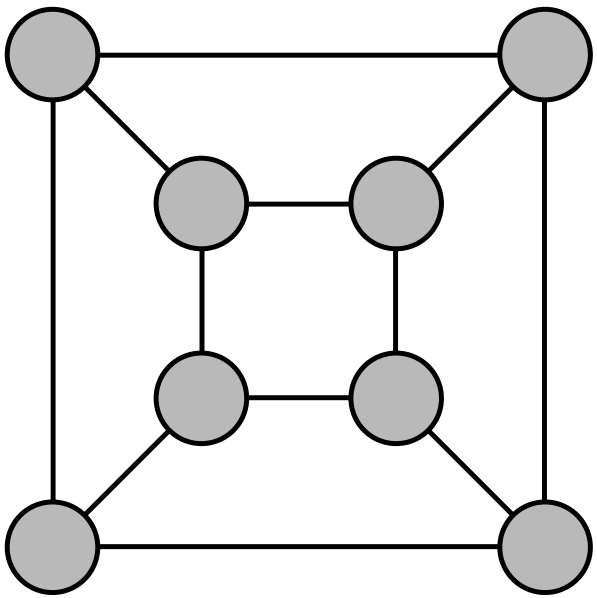
- an answer and
- a certificate/proof that the answer is correct.

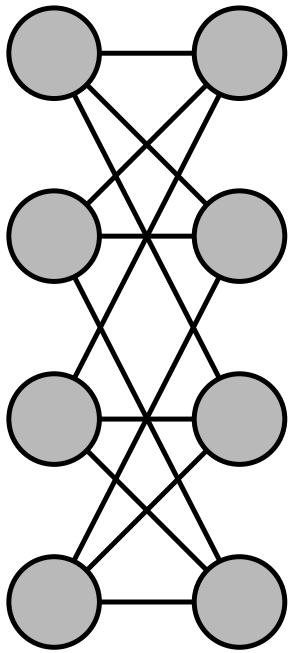
Approach: Study algorithms by analysing proof systems.

The graph isomorphism problem

Input: graphs G and H

Question: are G and H isomorphic?





Algebraic and mathematical-programming techniques

Step 1: encode an instance as a system of equations,

Step 2: solve the system.

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Step 1: encode an instance as a system of equations,

~~**Step 2:** solve the system.~~

Step 2: determine if there EXISTS a solution.

We only want to know if there EXISTS an isomorphism.

Step 1: equations

Input: graphs G and H

Compute: a system of equations $\mathrm{ISO}(G,H)$

$$\begin{cases} x_{vw}^2 - x_{vw} = 0 & \text{for every } v \in V(G), w \in V(H) \\ \sum_{w \in V(H)} x_{vw} - 1 = 0 & \text{for every } v \in V(G) \\ \sum_{v \in V(G)} x_{vw} - 1 = 0 & \text{for every } w \in V(H) \\ x_{vw}x_{v'w'} = 0 & \text{if } (v,v') \in E(G), (w,w') \notin E(H) \\ x_{vw}x_{v'w'} = 0 & \text{if } (v,v') \notin E(G), (w,w') \in E(H) \end{cases}$$

$$\text{SOLUTION} \iff \text{ISOMORPHISM}$$

Solving systems of polynomial equations is intractable.

Algebraic and mathematical-programming techniques

Step 1: encode an instance as a system of equations,

~~Step 2: solve the system.~~

~~Step 2: determine if there exists a solution.~~

Step 2: APPROXIMATELY determine if there exists a solution.

We can approximate using proof systems!

Step 2: computing a proof

Step 2: computing a proof that there is no solution

always correct

Output:

- if the algorithm finds a proof $\rightarrow$ *"no isomorphism"*
- otherwise $\rightarrow$ *"I do not know"*

Which pairs of non-isomorphic graphs the algorithms DISTINGUISH?

output *"no isomorphism"*

Proofs

Step 2: compute a proof that there is no solution

different **type of proof** $\leftrightarrow$ different algorithm

Algorithms:

- linear programming
- Gröbner basis
- semidefinite programming

Semidefinite Proofs

$$\begin{cases} x^2 + y + 2 = 0 \\ x - y^2 + 3 = 0 \end{cases}$$

$$-6\cdot(x^2 + y + 2) + 2\cdot(x - y^2 + 3) + \frac{1}{3} + 2\left(y + \frac{3}{2}\right)^2 + 6\left(x - \frac{1}{6}\right)^2 = -\mathbf{1}$$

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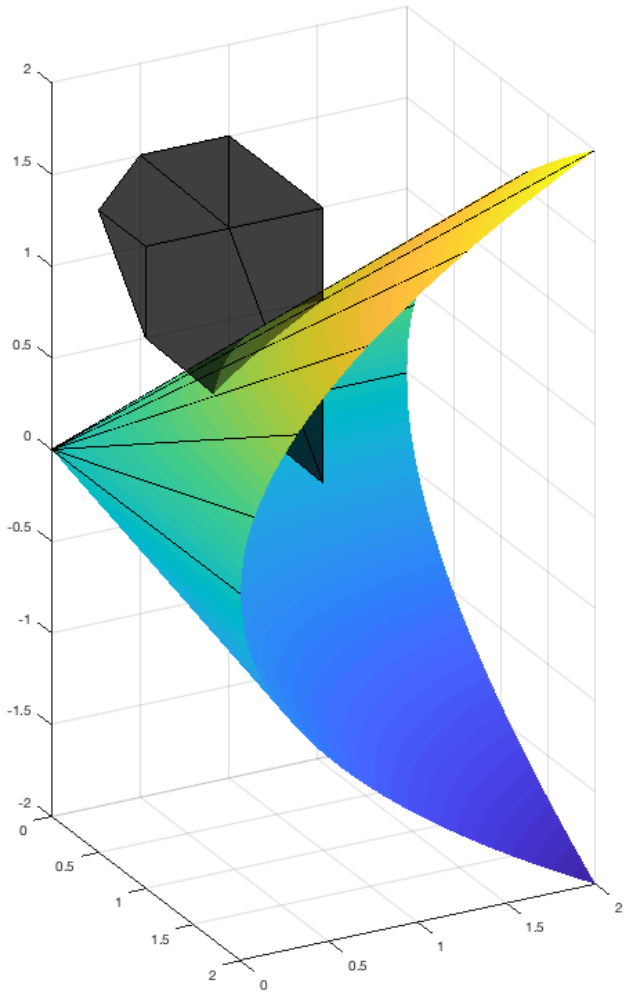
arbitrary polynomials sum of squares of polynomials

degree of the proof $\rightarrow$ max degree of polynomials on the left

Finding Semidefinite Proofs



Proofs

Step 2: compute a proof that there is no solution

restriction of semidefinite programming
finding a proof: linear inequalities

Algorithms:

- linear programming
- Gröbner basis
- semidefinite programming

computing a generating set
in the ideal of polynomials

Proofs

Step 2: compute a proof that there is no solution

Algorithms: Techniques:

- linear programming **hierarchy of algorithms**
- Gröbner basis **hierarchy of algorithms**
- semidefinite programming **hierarchy of algorithms**

degree of polynomials
in the proof

Summary of the setting

Algebraic and mathematical-programming techniques:

Step 1: encode an instance as a system of equations,

Step 2: compute a proof that there is no solution

always correct

Output:

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- otherwise → *"I do not know"*

Which pairs of non-isomorphic graphs the algorithms DISTINGUISH?

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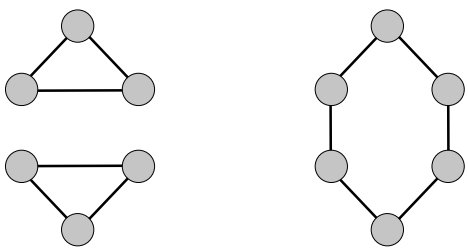
Known before

Theorem [Babai, Kučera'80]. Linear programming degree 2 distinguishes almost all graphs.

outputs *"I do not know"*

It does not distinguish:



Known before

For every pair of non-isomorphic graphs G and H:

Linear programming degree d distinguishes G and H.

$\Downarrow$ **[Berkholz, Grohe'15]**

Gröbner basis degree d distinguishes G and H.

$\Downarrow$ **[Berkholz'18]**

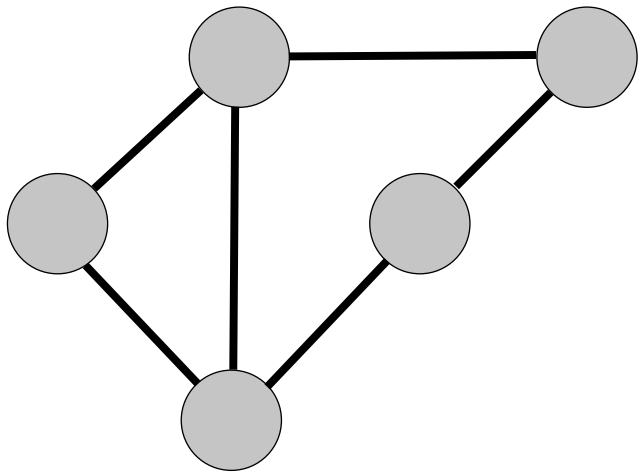
Semidefinite programming degree $2d$ distinguishes G and H.

Does semidefinite programming distinguish more graphs than linear programming?

Hope: yes!

Semidefinite programming much more powerful for many problems.

Example: MAX CUT



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Linear programming: very bad approximation

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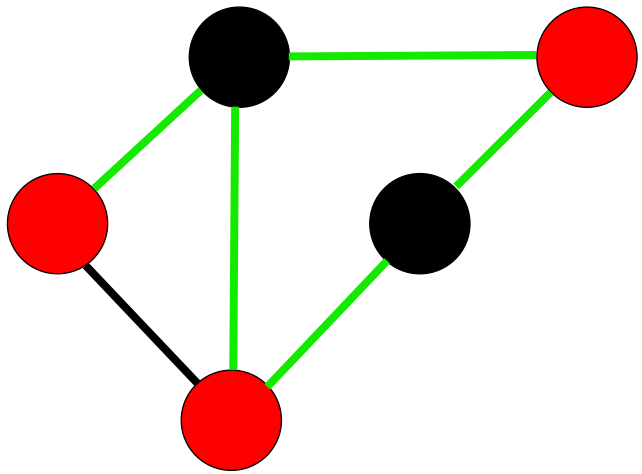

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All algorithms are equally powerful!

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Semidefinite programming degree $2d$ distinguishes G and H.

$\Downarrow$ **this work**

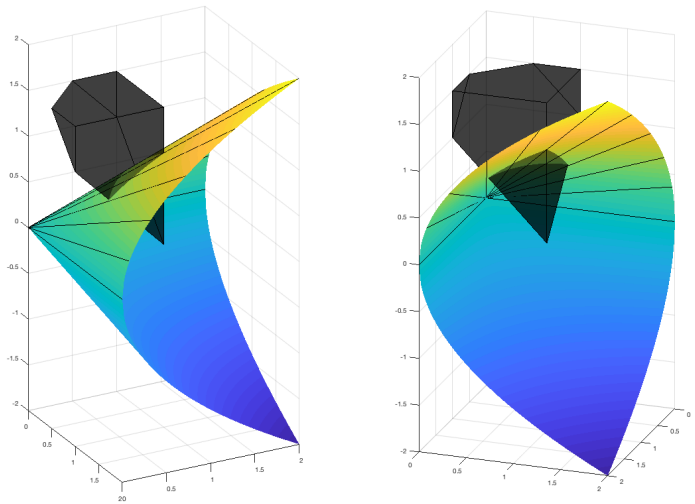
Linear programming degree cd distinguishes G and H.

constant independent from d

All algorithms are equally powerful!

Theorem. For the graph isomorphism problem all three algorithmic techniques are equally powerful, up to a constant factor loss in the degree.

Proof: Finding a semidefinite proof is expressible in counting logic.



Remark: For linear programming and Gröbner basis, showed independently in **[Grädel, Grohe, Pago, Pakusa'18]**.

Collapse

For every pair of non-isomorphic graphs G and H:

Semidefinite programming degree $2d$ distinguishes G and H.

$\Downarrow$

Counting logic $C^{cd}_{\infty\omega}$ distinguishes G and H.

$\Updownarrow$ **[Atserias, Maneva'13] [Malkin'14]**

Linear programming degree cd distinguishes G and H.

Finding semidefinite proofs in $C^{\omega}_{\infty\omega}$

Fact. Existence of semidefinite proofs reduces to feasibility of SDPs.

Is *polytop* $\cap$ *cone of positive semidefinite matrices* non-empty?

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Theorem. Feasibility of SDPs is expressible in $C^{\omega}_{\infty\omega}$.