Deciding FO-definable CSP instances

joint work with Bartek Klin, Eryk Kopczyński and Szymon Toruńczyk

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Atoms

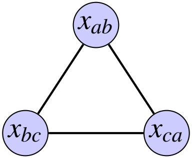
$\mathbb{A} = \{a, b, c, \ldots\}$ - countably infinite set of *atoms*

Graph colorability

G - an **infinite**, undirected graph:

- vertices indexed by ordered pairs of distinct atoms: x_{ab}, x_{ad}, ...
- edges: x_{ab} — x_{bc}, where a and c are distinct

Subgraph of G:



Question: Is the infinite graph G three-colorable?

Systems of linear equations over $\mathbb{Z}_2$

E - an **infinite** system of linear equations over $\mathbb{Z}_2$

- variables indexed by ordered pairs of distinct atoms: $x_{ab}, x_{ad}, \ldots$
- equations:

$$x_{ab} + x_{ba} = 1, \text{ where } a \text{ and } b \text{ are distinct}$$
$$x_{ab} + x_{bc} + x_{ca} = 0, \text{ where } a, b \text{ and } c \text{ are distinct}$$

Question: Does the system E have a solution?

Systems of linear equations over $\mathbb{Z}_2$

$$
\begin{array}{ccccccccccccccccccccccccccc}
ab & + & ba & & & & & & & & & & & & & & & & & & & & & & & = & 1 \\
ab & & & + & bc & + & ca & & & & & & & & & & & & & & & & & & & = & 0 \\
 & & ba & & & & & + & ac & + & cb & & & & & & & & & & & & & & & = & 0 \\
 & & & & bc & & & & & & & + & cd & + & db & & & & & & & & & & & = & 0 \\
 & & & & & & ca & & & & & & & & & + & ae & & & + & ec & & & & & = & 0 \\
 & & & & & & & & ac & & & + & cd & & & & & & & & & & & + & da & = & 0 \\
 & & & & & & & & & & cb & & & & & & & + & be & + & ec & & & & & = & 0 \\
 & & & & & & & & & & & & & & db & & & + & be & & & + & ed & & & = & 0 \\
 & & & & & & & & & & & & & & & & ae & & & & & + & ed & + & da & = & 0 \\
\hline
0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & = & 1
\end{array}
$$

Constraint Satisfaction Problem

A CSP *instance* $\mathbb{I} = (V, T, \mathcal{C})$:

- a set of variables: $V = \{x, y, \ldots\}$
- a set of their possible values: T
- a set of constraints: $\mathcal{C}$

Constraint Satisfaction Problem

G - an infinite, undirected graph:

- vertices indexed by ordered pairs of distinct atoms: $x_{ab}, x_{ad}, \ldots$
- edges: $x_{ab} — x_{bc}$, where a and c are distinct

Question: Is this graph three-colorable?

$\mathbb{I}_G$ - a CSP instance:

- variables: vertices $V = \{x_{ab} \mid a, b \in \mathbb{A} \text{ distinct}\}$
- values: possible colors $T = \{1, 2, 3\}$
- constraints: $\mathcal{C} = \{\big((x_{ab}, x_{bc}), R\big) \mid a, b, c \in \mathbb{A} \text{ distinct}\}$

For each edge $x_{ab} — x_{bc}$ there is a constraint: $\big((x_{ab}, x_{bc}), R\big)$
$R = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$.

Question: Is there a solution?

Classical Constraint Satisfaction Problem

$\mathbb{T} = (T, R_1, R_2, \ldots, R_n)$ - a fixed finite template

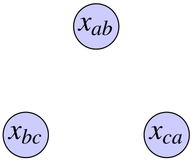
Problem: $\mathrm{CSP}_{\mathit{fin}}(\mathbb{T})$
Input: a finite CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

What kind of instances do we consider?

Definable instances

- variables indexed by tuples of atoms
- constraints defined by a first-order formula over $(\mathbb{A}, =)$

Set of variables in $\mathbb{I}_G$:
$\{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}$.



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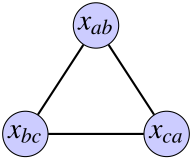
Set of constraints in $\mathbb{I}_G$:
$\{\big((x_{ab}, x_{bc}), R\big) \mid a, b, c \in \mathbb{A}, a \neq b \wedge a \neq c \wedge b \neq c\}$.



Constraint Satisfaction Problem

$\mathbb{T} = (T, R_1, R_2, \ldots, R_n)$ - a fixed finite template

Problem: $\mathrm{CSP}_{inf}(\mathbb{T})$
Input: a definable CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

Complexity

C	**Exp(C)**
P	Exp
NP	NExp
L	PSpace

Theorem. If $\mathrm{CSP}_{fin}(\mathbb{T})$ is C-complete then $\mathrm{CSP}_{inf}(\mathbb{T})$ is Exp(C)-complete.

Complexity

3-colorability of finite graphs – NP-complete

$\Downarrow$

3-colorability of definable graphs – NExp-complete

$\mathrm{CSP}_{inf}(\mathbb{T})$ is decidable

Theorem. It is decidable whether a definable instance $\mathbb{I}$ over a finite template $\mathbb{T}$ has a solution.

Uses Ramsey theorem and topological dynamics.

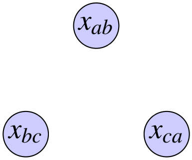
Proof idea: Look for regular solutions.

Atom permutations

$\mathrm{Aut}(\mathbb{A},=)$ acts on set of variables in $\mathbb{I}_G$:
$\{x_{ab} \mid a, b \in \mathbb{A}, a \neq b\}$.

π - a permutation of atoms
$\pi(x_{ab}) = x_{\pi(a)\pi(b)}$

$$a \mapsto b$$
$$b \mapsto c$$
$$c \mapsto a$$



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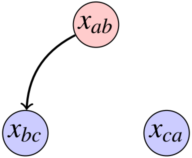

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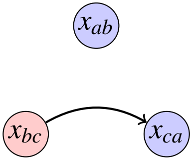

Invariant assignments

$\mathrm{Aut}(\mathbb{A},=)$ acts on the set of assignments $f : V \to T$

$$
\begin{array}{lr}
f & x \mapsto t \\
\pi \cdot f & \pi(x) \mapsto t
\end{array}
$$

fixpoint $\leftrightarrow$ *invariant* assignment

Invariant assignments

$x_{ab} + x_{ba} = 1,$ where a and b are distinct

x_{ea} x_{ae}

x_{ac}

x_{ca}

x_{ba} x_{ab}

x_{cd}

x_{dc}

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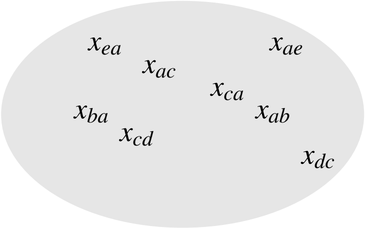

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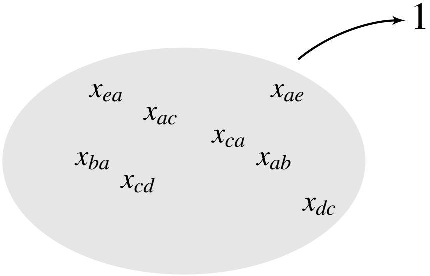

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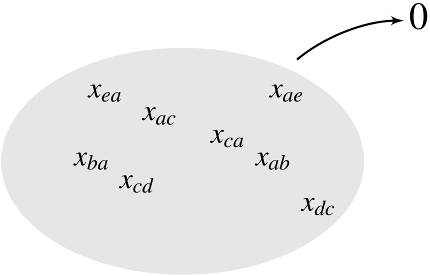

There is no invariant solution.

Monotone-invariant assignments

Fix a linear order on atoms $(\mathbb{A}, \leq)$ isomorphic to $(\mathbb{Q}, \leq)$.

$\mathrm{Aut}(\mathbb{A}, \leq)$ acts on the set of assignments $f : V \to T$

fixpoint $\leftrightarrow$ *monotone-invariant* assignment

Monotone-invariant assignments

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$\mathrm{Aut}(\mathbb{A}, \leq)$ $e < b < a < c < d$

Monotone-invariant assignments

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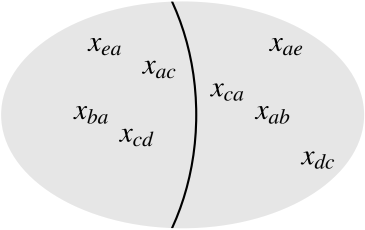

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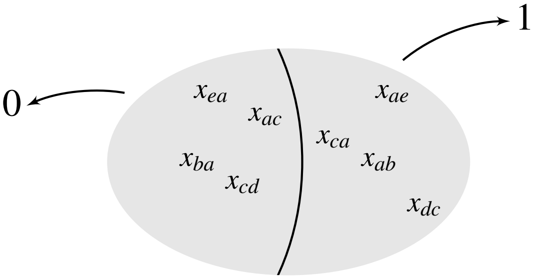

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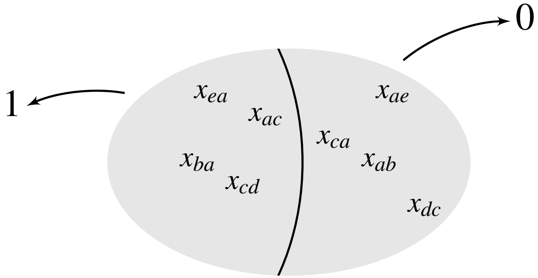

$\mathrm{Aut}(\mathbb{A}, \leq)$

$e < b < a < c < d$

Monotone-invariant assignments

- There are finitely many monotone-invariant assignments $f : V \to T$.
- Monotone-invariant assignments $f : V \to T$ can be represented in a finite way (by first order formulas using $\leq$).

Fact. It is decidable whether a definable instance $\mathbb{I}$ over a finite template $\mathbb{T}$ has a monotone-invariant solution.

$\mathrm{CSP}_{inf}(\mathbb{T})$ is decidable

Theorem. A definable instance $\mathbb{I}$ has a solution if and only if it has a monotone-invariant solution.

Theorem [Pestov]. Every continuous action of the topological group $\mathrm{Aut}(\mathbb{Q}, \leq)$ on a nonempty compact space has a fixpoint.

$\mathrm{CSP}_{inf}(\mathbb{T})$ is decidable

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Proof.
$\mathrm{Sol}(\mathbb{I}, \mathbb{T})$ – the set of solutions (possibly empty)

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$\mathrm{Aut}(\mathbb{A}, \leq)$ acts on $\mathrm{Sol}(\mathbb{I}, \mathbb{T})$ (solutions are mapped to solutions)
$\mathrm{Sol}(\mathbb{I}, \mathbb{T}) \subseteq \mathbb{T}^{\mathbb{I}}$ – a compact space

Theorem [Pestov]. Every continuous action of the topological group $\mathrm{Aut}(\mathbb{Q}, \leq)$ on a nonempty compact space has a fixpoint.

$\mathrm{CSP}_{inf}(\mathbb{T})$ is decidable

Corollary. It is decidable whether a definable instance $\mathbb{I}$ over a finite template $\mathbb{T}$ has a solution.

Complexity

C	**Exp(C)**
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Theorem. If $\mathrm{CSP}_{\mathit{fin}}(\mathbb{T})$ is C-complete then $\mathrm{CSP}_{\mathit{inf}}(\mathbb{T})$ is Exp(C)-complete.

Locally Finite Constraint Satisfaction Problem

A template $\mathbb{T} = \{T, R_1, R_2, \dots\}$ is *locally finite* is every relation of $\mathbb{T}$ is finite.

Locally Finite Constraint Satisfaction Problem

$\mathbb{T} = \{T, R_1, R_2, \ldots\}$ - locally finite, definable template

Problem: $\mathrm{CSP}_{inf}(\mathbb{T})$
Input: a definable CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

Theorem. For any definable, locally finite template $\mathbb{T}$, it is decidable whether a given definable instance $\mathbb{I}$ over $\mathbb{T}$ has a solution.

Open: What about definable instances over arbitrary definable tamplates?

Locally Finite Constraint Satisfaction Problem

$\mathbb{T} = \{T, R_1, R_2, \ldots\}$ - locally finite, definable template

Problem: $\mathrm{CSP}_{\mathit{fin}}(\mathbb{T})$
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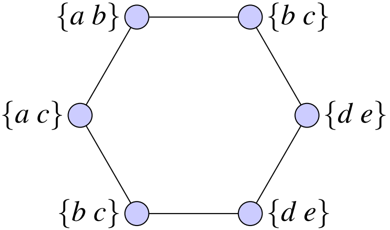
Generalized graph colorability

G - a finite, undirected graph

We treat atoms as colors.

To each vertex we assign a set of n possible colors.



Question: Can this graph be colored with atoms such that no two adjacent vertices share the same color?

Generalized graph colorability

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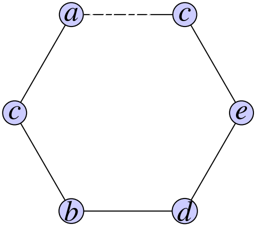

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$\mathbb{T} = \{T, R_1, R_2, \ldots\}$ - locally finite, definable template

Problem: $\mathrm{CSP}_{\mathit{fin}}(\mathbb{T})$
Input: a finite CSP instance $\mathbb{I}$ over $\mathbb{T}$
Decide: Does $\mathbb{I}$ have a solution?

Obviously decidable.

What about the complexity?

Bounded width

Theorem [Larose, Zádori; Barto, Kozik] A finite template $\mathbb{T}$ has bounded width (solvable in Datalog) if and only if an instance $\mathbb{I}_{\mathbb{T}}^{bw}$ over $\mathbb{T}$ has a solution.

$\mathbb{I}_{\mathbb{T}}^{bw}$ has a solution iff $\mathbb{T}$ has certain polymorphisms.

Bounded width

Corollary. A locally finite template $\mathbb{T}$ has bounded width (solvable in Datalog) if and only if an instance $\mathbb{I}_{\mathbb{T}}^{bw}$ over $\mathbb{T}$ has a solution.

$\mathbb{I}_{\mathbb{T}}^{bw}$ is a definable instance computable from $\mathbb{T}$

$\Downarrow$

Effective characterization of locally finite templates of bounded width.

Thank you