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L^p solutions for stochastic evolution equation with nonlinear potential

by

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Abstract. This article deals with the stochastic partial differential equation

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + u^\gamma \xi, \\ u(0,\cdot) = u_0, \end{cases}$$

where ξ is a space/time white noise Gaussian random field, $\gamma \in (1,\infty)$ and the initial condition u_0 is a non-negative measurable mapping, independent of ξ satisfying $u_0 \geq 0$ and additional conditions given in the article. The *space* variable is $x \in \mathbb{S}^1 = [0,1]$ with the identification $0 = 1$. The definition of the stochastic term, taken in the sense of Walsh, will be made clear in the article. The result is that there exists a non-negative solution u such that for all $\alpha \in [0,1)$,

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} u(t,x)^{2\gamma}\,dx\,dt\Big)^{\alpha/2}\Big] \leq K(\alpha)\mathbb{E}\Big[\Big(\int_{\mathbb{S}^1} u_0(x)\,dx\Big)^{\alpha}\Big] < \infty.$$

where the finite constant $K(\alpha)$ is derived from the Burkholder–Davis–Gundy inequality constants. The solution is unique among solutions which satisfy this. Solutions are also shown to satisfy

$$\mathbb{E}\Big[\int_0^T\Big(\int_{\mathbb{S}^1} u(t,x)^p\,dx\Big)^{\alpha/p}\,dt\Big] < \infty \qquad \forall T<\infty,\ 0<p<\infty,\ \alpha \in (0,1/2).$$

1. Introduction. This article shows existence of solutions in suitable function spaces for the equation

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$$\begin{cases} u_t = \frac{1}{2}u_{xx} + u^\gamma \xi, \\ u(0,x) = u_0(x) \geq 0, \\ \lim\limits_{n\to\infty} \mathbb{E}\Big[\Big(\int\limits_{\mathbb{S}^1} (u_0(x) \wedge n)\, dx\Big)^2\Big] = \mathbb{E}\Big[\Big(\int\limits_{\mathbb{S}^1} u_0(x)\, dx\Big)^2\Big] < \infty, \\ \gamma \in (1,\infty), \end{cases} \tag{1}$$

with space variable $x \in \mathbb{S}^1 = [0,1]$, the unit circle with identification $0 = 1$, where u_0 (the initial condition) is a non-negative measurable mapping. The techniques do not accommodate initial conditions which contain Dirac masses (such situations are excluded by the condition on u_0). Here, subscripts denote derivatives: u_t denotes the derivative of $u : \mathbb{R}_+ \times S^1 \times \Omega \to \mathbb{R}$ with respect to the first variable (the *time* variable), while u_{xx} the second derivative with respect to the second variable (the *space* variable). Equation (1) is shorthand for the corresponding stochastic integral equation (SIE) given later (after the machinery to define it has been introduced) as equation (16); the derivatives are understood in this sense. $\xi : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to \mathbb{R}$ is used to denote space/time white noise and the stochastic integral in the SIE is understood in the sense of Walsh [18]. The initial condition u_0 is independent of the white noise field ξ.

Clearly there are no *strong* solutions to (1) in the sense of PDEs; solutions will neither be twice differentiable in the space variables nor once differentiable in the time variable.

1.1. Background. Let W be a standard one-dimensional Wiener process. Consider the stochastic ordinary differential equation

$$u(t) = u_0 + \int_0^t u(s)^\gamma\, dW(s), \qquad u_0 \geq 0, \tag{2}$$

taken in the sense of Itô, for $\gamma > 0$. This has been well studied. Existence and behaviour of solutions can be obtained by comparison with an appropriate Bessel process, in the following way. Let $Y(t) = u(t)^\alpha$. Then for $\alpha \neq 0$, a minor modification of Itô's formula gives

$$Y(t) = u_0^\alpha + \alpha \int_0^t Y(s)^{1+(\gamma-1)/\alpha}\, dW(s) + \frac{\alpha(\alpha-1)}{2}\int_0^t Y(s)^{1+2(\gamma-1)/\alpha}\, ds. \tag{3}$$

Itô's formula may be applied to $f(u(t))$ for functions $f \in C^2(\mathbb{R})$, but for $\alpha < 2$, $\alpha \neq 0$, $f(x) = |x|^\alpha$ is not twice differentiable at 0. The modification involves considering stopping times $\sigma_\epsilon = \inf\{t : u(t) < \epsilon\}$ and applying Itô's formula to $f(u(t \wedge \sigma_\epsilon))$. The comparison with Bessel processes of dimension greater than 2 in (4) below implies that $\lim_{\epsilon\to 0} \sigma_\epsilon = \infty$ almost surely.

For $\alpha = 1-\gamma$, where $\gamma \neq 1$,

$$Y\left(\frac{t}{(\gamma-1)^2}\right) = u_0^{1-\gamma} - (\gamma-1)W\left(\frac{t}{(\gamma-1)^2}\right) + \frac{\gamma}{2(\gamma-1)}\int_0^t \frac{1}{Y\left(\frac{r}{(\gamma-1)^2}\right)}\,dr.$$

Now let $\widetilde{W}(t) = -(\gamma-1)W\left(\frac{t}{(\gamma-1)^2}\right)$, so that $\widetilde{W}$ is a standard Brownian motion, and let $Z(t) = Y\left(\frac{t}{(\gamma-1)^2}\right)$. Then

$$Z(t) = u_0^{1-\gamma} + \widetilde{W}(t) + \frac{\frac{2\gamma-1}{\gamma-1}-1}{2}\int_0^t \frac{1}{Z(s)}\,ds, \tag{4}$$

so that Z is a $\frac{2\gamma-1}{\gamma-1}$-dimensional Bessel process. It follows that for $\gamma \neq 1$, $u(t)^{1-\gamma} = Z((\gamma-1)^2 t)$. A Bessel process of dimension greater than 2 is bounded away from 0 (see Revuz and Yor [14]). Since $\frac{2\gamma-1}{\gamma-1} > 2$ for all $\gamma > 1$, it follows that for initial condition $u_0 > 0$, the solution u is a well defined non-negative local martingale, satisfying $\sup_{0\le t<\infty} u(t) < \infty$. The following asymptotic holds:

$$\frac{u(t)^{2(1-\gamma)}}{(\gamma-1)^2 t} \xrightarrow{(d)} Y$$

where the random variable Y has density function

$$f(y) = \begin{cases} \frac{1}{2^{(2\gamma-1)/(2\gamma-2)}}\,\frac{1}{\Gamma\left(\frac{2\gamma-1}{2\gamma-2}\right)}\,y^{1/(2\gamma-2)}e^{-y/2}, & y \ge 0,\\ 0, & y < 0.\end{cases}$$

This is a straightforward rescaling of the (unnumbered) formulae within Corollary 1.4 towards the bottom of p. 441 of Revuz and Yor [14].

A natural question to ask is the extent to which properties of one-dimensional equations are retained in the presence of mixing. For example, consider an operator A defined on functions over a countable space $\mathcal{X}$ such that $\sum_{y\in\mathcal{X}} A_{x,y} = 0$ for each $x \in \mathcal{X}$, and the system of coupled stochastic differential equations

$$u(t,x) = u_0(x) + \int_0^t \sum_y A_{x,y}u(s,y)\,ds + \int_0^t u(s,x)^\gamma\,dW^{(x)}(s) \tag{5}$$

where $u_0(x) > 0$ for each x and $(W^{(x)})_{x\in\mathcal{X}}$ are independent Wiener processes, each with the same diffusion coefficient. How does the coupling change the nature of the system?

Now consider $\{A^{(h)}_{x,y} : x, y \in h\mathbb{Z}\}$ defined by $A^{(h)}_{hx,h(x+1)} = A^{(h)}_{hx,h(x-1)} = \frac{1}{2h^2}$, $A^{(h)}_{hx,hx} = -\frac{1}{h^2}$, $A^{(h)}_{x,y} = 0$ otherwise. For each $x \in h\mathbb{Z}$, let $(W^{(h,x)})_{x\in h\mathbb{Z}}$ be independent Wiener processes satisfying

$$\mathbb{E}[W^{(h,x)}(t)] \equiv 0 \quad\text{and}\quad \mathbb{E}[W^{(h,x)}(s)W^{(h,x)}(t)] = (s\wedge t)\frac{1}{h}.$$

Note that the diffusion of the independent Wiener processes changes as $h \to 0$. Also, for $f \in C^2(\mathbb{R})$, $\lim_{h\to 0} A^{(h)} f = \frac{1}{2}\frac{d^2}{dx^2} f$. The operator $A^{(h)}$ is the 'discrete Laplacian' on the lattice $h\mathbb{Z}$ and its limit is the operator $\frac{1}{2}\frac{d^2}{dx^2}$ (the Laplacian on $\mathbb{R}$). Formally, the limiting equation of (5) as $h \to 0$, when $A^{(h)}$ is used in place of A and $W^{(h,\cdot)}$ is used in place of $W^{(\cdot)}$, is equation (1), where ξ is space time 'white noise' and the final term of (1) is defined according to the theory of martingale measures due to Walsh [18].

Equation (1), with $\gamma > 1$, but with different conditions for the space variable, has been well studied; the main contributions are Mueller [8], Mueller and Sowers [9], Mueller [10] and Mueller [11], and also Krylov [4]. The works [8]–[11] consider the equation with non-negative and continuous initial condition $u(0,x)$ and Dirichlet boundary conditions $u(t,0) = u(t,J) = 0$ and consider the solution for $t > 0$ and $0 \le x \le J$.

Shiga [17] considers the equation on $\mathbb{R}$ for different ranges of γ. For $\gamma \ge 1$, he points to the *strong positivity property*, established by Mueller [7]; if the initial condition is non-negative and strictly positive on a set of positive measure, then the solution is strictly positive up to explosion time of the L^∞ norm.

Shiga [17] considers approximate equations with the truncation $(u \wedge n)^\gamma$. The approximating equation is

$$(6) \qquad \begin{cases} u_t^{(n)} = \frac{1}{2} u_{xx}^{(n)} + (u^{(n)} \wedge n)^\gamma \xi, \\ u^{(n)}(0,x) = u_0(x) \wedge n. \end{cases}$$

According to [17, Theorem 2.3], equation (6) has a unique solution, which is non-negative for n finite. Therefore, any solution to (1) obtained as a limit of $u^{(n)}$'s satisfying (6) will be non-negative. Shiga considers state space $\mathbb{R}$; the arguments for $\mathbb{S}^1$ are the same. Walsh [18, Theorem 3.2 and Corollary 3.4] proves existence, uniqueness, and regularity of solutions for equations similar to (6). His regularity results depend on the initial condition.

In [8], existence and uniqueness of solution is shown for equation (1) for $1 \le \gamma < 3/2$. Solutions to (1) agree with solutions to (6) up to time $\sigma_n = \inf\{t : \sup_x u(t,x) \ge n\}$. There is existence, uniqueness and continuity up to time $\sigma = \lim_{n\to\infty} \sigma_n$ and then it is shown that $\mathbb{P}(\sigma = \infty) = 1$ for $\gamma < 3/2$, where $\mathbb{P}$ is used to denote the probability measure.

In [9], Mueller and Sowers study equation (1), again with Dirichlet boundary conditions and the same conditions on the initial condition. In [9], $\gamma > 3/2$ is considered and, with σ defined in the same way, it is shown that there exists a $\gamma_0 \ge 3/2$ such that for $\gamma > \gamma_0$, $\mathbb{P}(\sigma < \infty) > 0$. The line of approach is to couple the solution to a branching process, where large peaks are regarded as particles in the branching process and offspring are peaks that are higher by some factor. It is shown that, for $\gamma > \gamma_0$, the expected number of offspring is greater than 1. It follows that the branching process survives with positive probability, which corresponds to $\sigma < \infty$. The event $\{\sigma < \infty\}$ corresponds to $\{\lim_{t\uparrow\sigma} \|u(t,\cdot)\|_\infty = \infty\}$. In Mueller [11], the techniques of [9] are sharpened to show that for all $\gamma > 3/2$, there is explosion of $\|u(t,\cdot)\|_\infty$ in finite time with positive probability.

The work of Mueller and Sowers [9] and Mueller [11] shows that the L^∞ spatial norm explodes for $\gamma > 3/2$ with positive probability, so that any technique for proving existence of solution that relies on long time existence of the L^∞ spatial norm will fail. Mueller [10] shows local existence and uniqueness for (1) (with Dirichlet boundary conditions) with unbounded initial conditions, indicating that L^p solutions could exist beyond the explosion time of the L^∞ norm. Furthermore, consideration of the one-dimensional SODE (2) might suggest that there is a well defined solution with long time existence of L^p norm for some $0 < p < \infty$, since the SODE has a well defined solution with probability 1.

In this article, the equation is considered on $\mathbb{S}^1$, the unit circle. That is, the space variable takes its values in $[0,1]$ where 0 and 1 are identified. Instead of taking Dirichlet boundary conditions, the identification $u(t,0) = u(t,1)$ is made and $\frac{d^2}{dx^2}$ is taken as the Laplacian on $\mathbb{S}^1$. While there is not a full proof of comparison results in this article, $\mathbb{P}(\sigma < \infty)$ (probability of explosion of L^∞ norm) should be *less* with Dirichlet boundary conditions than on the circle. This is stated as Theorem 32 (in the appendix) and a sketch of proof is given there.

Suppose that there exists a solution to (1), taken on the unit circle, with non-negative initial condition satisfying $\int_{\mathbb{S}^1} u(0,x)\,dx = C$ for some $C > 0$. Let $U(t) = \int_{\mathbb{S}^1} u(t,x)\,dx$. Then $\{U(t) : t \geq 0\}$ is a non-negative local martingale and, from a general result about non-negative local martingales (given below), it satisfies $\sup_{n\geq 1} n\mathbb{P}(\sup_t U(t) > n) \leq K < \infty$ for some K. It follows that $\int_{\mathbb{S}^1} u(t,x)\,dx$ is bounded almost surely in the time variable. Furthermore, the increasing process of U is simply

$$\langle U\rangle(t) = \int_0^t \int_{\mathbb{S}^1} u(s,x)^{2\gamma}\,dx\,ds.$$

Mueller and Sowers [9] followed by Mueller [11] show that there is explosion with positive probability of the L^∞ norm for $\gamma > 3/2$. This article shows existence of solutions in appropriate L^p spaces for all $\gamma > 1$; if the 'total mass process' is a well defined local martingale, then its increasing process is well defined and hence the natural value to use for p when searching for L^p solutions is $p = 2\gamma$.

The SPDE with $0 < \gamma < 1$**.** It is worth pointing out work and results for the SPDE with $0 < \gamma < 1$. Various aspects of this situation are dealt with by Shiga [17], Mueller, Mytnik and Perkins [12] and Mytnik and Neuman [13]. In [12], it is shown that for $0 < \gamma < 3/4 - \epsilon$ there is neither pathwise uniqueness nor uniqueness in law for any $\epsilon > 0$. The case of $3/4 < \gamma < 1$ is considered in [13], where it is shown that there is pathwise uniqueness for $\gamma > 3/4$. A drift term is also included. In [12] it is pointed out that the exponent $3/4$ is sharp for the border between uniqueness and non-uniqueness.

Shiga [17] points to two properties of this SPDE. One of these, the *strong positivity property* for $\gamma \geq 1$, has already been discussed. For $\gamma < 1$, he also discusses the *compact support property* for $\gamma = 1/2$: with state space $\mathbb{R}$, if the initial condition has compact support, then the solution will have compact support for all $t > 0$.

In this paper, only the case of $\gamma > 1$ is considered.

Organisation. Section 2 gives a brief resumé of the main well known results for martingale inequalities, which are the basis of the techniques used in the article. Section 3 gives necessary material, which extends the Walsh notion of SPDE to the situation here, where the stochastic integrand is not necessarily square integrable. Section 4 is the main section, where the existence result is proved. In Section 5, the problem of uniqueness is dealt with, while Section 6 deals with L^p norms of the solution.

The appendix gives a sketch of the proof that the explosion of the L^∞ norm for the problem on the circle is less than or equal to explosion time for the problem with Dirichlet boundary conditions.

2. Martingale inequalities.

This section gives some basic results about non-negative continuous local martingales. Throughout, $\mathbb{P}$ will be used to denote the generic probability measure over the probability space on which the processes and random variables under discussion are defined, and $\mathbb{E}$ will denote expectation with respect to $\mathbb{P}$.

LEMMA 1. *Let M be a non-negative continuous local martingale satisfying $M(0) = x > 0$. Let $\tau_n = \inf\{t : M(t) \geq n\}$. Then*

$$\mathbb{P}(\tau_n < \infty) \leq 1 \wedge \frac{x}{n}.$$

Proof. This is well known and follows from the gambler's ruin problem. The proof is included since it is short.

Let $\tau_n = \inf\{t : M(t) = n\}$. Then τ_n is a stopping time with respect to the natural filtration of M and the stopped process $M^{(\tau_n)}$ is a martingale. It follows that, for each $n \geq 1$,

$$x = \mathbb{E}[M^{(\tau_n)}(t)] = \mathbb{E}[M^{(\tau_n)}(t)\mathbf{1}_{[t,\infty)}(\tau_n)] + n\mathbb{P}(t > \tau_n)$$

and since $\mathbb{E}[M^{(\tau_n)}(t)\mathbf{1}_{[t,\infty)}(\tau_n)] \geq 0$ for all $t \geq 0$, the conclusion follows. ∎

LEMMA 2. *Let M be a non-negative continuous local martingale with initial condition $M(0) = x$. Then for all $\alpha \in (0,1)$,*

$$x^\alpha \le \mathbb{E}\Big[\sup_{0<s<\infty} M(s)^\alpha \;\Big|\; M(0) = x\Big] \le \frac{x^\alpha}{1-\alpha}.$$

Proof. Again, this is well known; it is a straightforward consequence of Lemma 1. It is included because heavy use is made of it in the proof of the main result.

Let $\tau_n = \inf\{t : M(t) \ge n\}$. Then

$$\mathbb{P}(\tau_n < t) = \mathbb{P}\Big(\sup_{0<s<t} M(s) \ge n\Big).$$

Let $\widetilde{M}$ denote the process such that $\widetilde{M}(t) = M(t)\mathbf{1}_{\{\sup_{0\le s<\infty} M(s)<\infty\}}$. Then the process $\widetilde{M}$ is equivalent to M, since from Lemma 1,

$$\mathbb{P}\Big(\sup_{0\le s<\infty} M(s) < \infty\Big) = 1.$$

Let M now denote this equivalent process and set

$$X = \sup_{0<s<\infty} M(s).$$

Then, from Lemma 1, for all $\alpha \in (0,1)$,

$$\begin{aligned} x^\alpha \le \mathbb{E}[X^\alpha] &= \int_0^\infty \mathbb{P}(X^\alpha \ge y)\,dy = \int_0^\infty \mathbb{P}(X \ge y^{1/\alpha})\,dy \\ &\le \int_0^\infty \Big(1 \wedge \frac{x}{y^{1/\alpha}}\Big)\,dy = x^\alpha + x\int_{x^\alpha}^\infty y^{-1/\alpha}\,dy = x^\alpha + \frac{\alpha}{1-\alpha}x^\alpha \\ &= \frac{x^\alpha}{1-\alpha}. \ \blacksquare \end{aligned}$$

COROLLARY 3. *Let M be a non-negative continuous local martingale. Then for all $\alpha \in (0,1)$,*

$$\mathbb{E}[M(0)^\alpha] \le \mathbb{E}\Big[\sup_{0<s<\infty} M(s)^\alpha\Big] \le \frac{1}{1-\alpha}\mathbb{E}[M(0)^\alpha] \tag{7}$$

Proof. Immediate from Lemma 2. ■

LEMMA 4. *Let M be a non-negative continuous local martingale M. For $\alpha \in (0,1)$, there exists a strictly positive constant $c(\alpha)$, which does not depend on M, such that*

$$\mathbb{E}[\langle M\rangle(\infty)^{\alpha/2}] \le \frac{2-\alpha}{c(\alpha)(1-\alpha)}\mathbb{E}[M(0)^\alpha]. \tag{8}$$

Here, $c(\alpha)$ is the strictly positive constant which emerges in the usual Burkholder–Davis–Gundy inequality which states that for all local martingales N such that $N(0)=0$,

$$c(\alpha)\mathbb{E}[\langle N\rangle(t)^{\alpha/2}] \le \mathbb{E}\Big[\sup_{0\le s\le t}|N(s)|^{\alpha}\Big] \le C(\alpha)\mathbb{E}[\langle N\rangle(t)^{\alpha/2}]. \tag{9}$$

We will denote by $K(\alpha)$ the multiplier in (8):

$$K(\alpha)=\frac{2-\alpha}{c(\alpha)(1-\alpha)}. \tag{10}$$

Proof of Lemma 4. Let $A_x=\{|\sup_{0<s<\infty}M_s-x|<x\}$ and let $\mathbf{1}_B$ denote the indicator function for a set B. Let A_x^c denote the complement of A_x. Note that if $y,x>0$ and $|y-x|>x$ then $y>2x$ so that $|y-x|=y-x<y$. Using the Burkholder–Davis–Gundy inequality, from Lemma 2 we get

$$\begin{aligned} c(\alpha)\mathbb{E}[\langle M\rangle(\infty)^{\alpha/2}\mid M(0)=x] &\le \mathbb{E}\Big[\sup_{0<s<\infty}|M(s)-x|^{\alpha}\ \Big|\ M(0)=x\Big] \\ &\le x^{\alpha}+\mathbb{E}\Big[\sup_{0<s<\infty}|M(s)-x|^{\alpha}\mathbf{1}_{A_x^c}\ \Big|\ M(0)=x\Big] \\ &\le x^{\alpha}+\mathbb{E}\Big[\sup_{0<s<\infty}|M(s)|^{\alpha}\ \Big|\ M(0)=x\Big] \\ &\le x^{\alpha}\Big(1+\frac{1}{1-\alpha}\Big)=\frac{2-\alpha}{1-\alpha}x^{\alpha}, \end{aligned}$$

which yields (8). ■

3. Wiener sheet, function spaces and stochastic integrals. A priori, if there is a well defined solution to (1), the stochastic integrand will not be square integrable: a solution (if it exists) will satisfy

$$\mathbb{E}\Big[\Big(\int_0^{\infty}\int_{\mathbb{S}^1}u(s,x)^{2\gamma}\,dx\,ds\Big)^{\alpha/2}\Big] \le \frac{2-\alpha}{c(\alpha)(1-\alpha)}\mathbb{E}\Big[\Big(\int_{\mathbb{S}^1}u_0(x)\,dx\Big)^{\alpha}\Big]$$

(from Lemma 4), but this upper bound explodes as $\alpha\uparrow 1$. Therefore, a notion of the stochastic integral which is slightly more general than that of Walsh [18] is required, since Walsh develops an L^2 theory and defines a stochastic integal $\int_0^t\int_0^x f(s,y)\,W(dy,ds)$ for f adapted and satisfying $\mathbb{E}[\int_0^t\int_0^x f^2(s,x)\,dx\,ds]<\infty$ for finite t and x. For a solution to the SPDE, $f=u^{\gamma}$, which does not a priori satisfy this property. This section provides a suitable extension to the Walsh theory. The differences from the Walsh construction are minor, but necessary for the problem in hand.

The formal definition of the Wiener sheet (Brownian sheet) is found in Walsh [18]. It was introduced into the literature earlier by T. Kitagawa [3]. The approach taken here to the construction of a stochastic integral with respect to a Wiener sheet largely follows the approach of Walsh, with gentle modification to accommodate the situation where second moments of the stochastic integral may not exist.

The probability space on which the driving Wiener sheet lives (and hence on which the equation under consideration is defined) is now given.

DEFINITION 5 (Wiener sheet). Let $\mathcal{B}(A)$ denote the Borel σ-field of a space A. Let $E = \mathbb{S}^1 \times \mathbb{R}_+$, $\mathcal{E} = \mathcal{B}(\mathbb{S}^1 \times \mathbb{R}_+)$, and λ Lebesgue measure defined on $(E, \mathcal{E})$. A *Wiener sheet* is a random set function W defined on the sets $A \in \mathcal{E}$ of finite λ-measure such that

(1) $W(A) \sim N(0, \lambda(A))$ for all $A \in \mathcal{E}$,
(2) for $A, B \in \mathcal{E}$ such that $A \cap B = \emptyset$, $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

LEMMA 6. *The Wiener sheet is well defined.*

Proof. This is proved in Walsh [18, Ch. 1, p. 269]. ■

DEFINITION 7 (Filtrations and probability space for the SPDE). Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$ denote the filtered probability space on which the Wiener sheet (Definition 5) is defined. That is, for $t > 0$, $\widetilde{\mathcal{F}}_t$ is the σ-field

$$\sigma(\{W([0,x] \times [0,s]) : 0 \le x < 1,\ 0 \le s \le t\}) \tag{11}$$

augmented by the $\widetilde{\mathbb{P}}$-null sets so that $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$ satisfies the usual conditions. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}_0, \widehat{\mathbb{P}})$ denote a probability space independent of $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}})$ and let $u_0 : \mathbb{S}^1 \times \widehat{\Omega} \to \mathbb{R}_+$, the initial condition for (1), be measurable with respect to $\widehat{\mathcal{F}}_0$. Let $\widehat{\mathcal{F}}_0$ be complete (i.e. contain all the $\widehat{\mathbb{P}}$-null sets). Let $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ be the filtered probability space, satisfying the usual conditions, defined by

$$\Omega = \widehat{\Omega} \times \widetilde{\Omega}, \quad \mathcal{F}_t = \widehat{\mathcal{F}}_0 \otimes \widetilde{\mathcal{F}}_t, \quad \mathbb{P} = \widehat{\mathbb{P}} \times \widetilde{\mathbb{P}} \quad \text{and} \quad \mathcal{F} = \bigvee_{t \ge 0} \mathcal{F}_t;$$

this is the probability space on which (u_0, W) is defined.

DEFINITION 8. A function $f(s, x, \omega)$ is *elementary* if it is of the form

$$f(s, x, \omega) = X(\omega)\mathbf{1}_{\{0\}}(s)\mathbf{1}_A(x)$$

where X is $\mathcal{F}_0$-measurable or, for $0 \le a \le b$,

$$f(s, x, \omega) = X(\omega)\mathbf{1}_{(a,b]}(s)\mathbf{1}_A(x)$$

where X is bounded and $\mathcal{F}_a$-measurable and $A \in \mathcal{B}(\mathbb{S}^1)$. The function f is *simple* if it is the sum of elementary functions. The class of simple functions will be denoted by $\mathcal{S}$.

DEFINITION 9. The *predictable* σ-field $\mathcal{P}$ is the σ-field generated by $\mathcal{S}$. A function is *predictable* if it is $\mathcal{P}$-measurable.

DEFINITION 10 (Function spaces). Let $g \in \mathcal{P}$. For $\alpha \in (0,2]$, the following function spaces will be employed, with $p > 1$ (mostly $p = 2\gamma$):

$$(12)\qquad \begin{aligned} \mathcal{S}_{p,\alpha;K} &= \Big\{ g \in \mathcal{P} : \|g\|_{p,\alpha} := \mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} |g(s,x)|^p \, dx \, ds\Big)^{\alpha/2}\Big]^{1/p} < K \Big\}, \\ \mathcal{S}_{p,\alpha} &= \bigcup_{K>1} \mathcal{S}_{p,\alpha;K}. \end{aligned}$$

The space $\mathcal{S}_{p,\alpha}$ is equipped with the metric $d_{p,\alpha}$ defined by

$$(13)\qquad d_{p,\alpha}(g,h) = \|g-h\|_{p,\alpha}$$

for $0 < \alpha < 2$. Two functions $g, h \in \mathcal{P}$ are said to be (p,α)-*equivalent* if $d_{p,\alpha}(g,h) = 0$.

NOTE. Observe that $\|\cdot\|_{p,\alpha}$ is not a norm, since it satisfies $\|cf\|_{p,\alpha} = |c|^{\alpha/2}\|f\|_{p,\alpha}$, which does not equal $|c|\,\|f\|_{p,\alpha}$ unless $\alpha = 2$. The distance $d_{p,\alpha}$ is a metric for all $\alpha \le 2$; this will be proved below. For $\alpha = 2$, $p > 1$, $d_{p,2}$ is clearly a metric. This will be used for solutions to approximating equations whose moments are all well defined. For $\alpha \in (0,2)$, the following lemma shows that $d_{p,\alpha}$ is a metric.

LEMMA 11. *For $p \ge 2$, the quantity $d_{p,\alpha}$ defined in* (13) *is a metric for $\alpha \in (0,2)$:*

(i) *The triangle inequality holds: for any $f, g, h \in \mathcal{S}_{p,\alpha}$,*

$$d_{p,\alpha}(f,g) \le d_{p,\alpha}(f,h) + d_{p,\alpha}(h,g).$$

(ii) $d_{p,\alpha}(f,g) = 0$ *implies that $f = g$ (up to equivalence).*
(iii) $d_{p,\alpha}(f,g) = d_{p,\alpha}(g,f)$.

Proof. Since (ii) is true by definition, and (iii) is clear, it only remains to prove the triangle inequality. Let

$$b_1 = \Big(\int_0^\infty \int_{\mathbb{S}^1} |f-g|^p(t,x) \, dx \, dt\Big)^{1/p},$$
$$b_2 = \Big(\int_0^\infty \int_{\mathbb{S}^1} |g-h|^p(t,x) \, dx \, dt\Big)^{1/p}.$$

Then, using $\|f+g\| \le \|f\| + \|g\|$ for L^p norms and that $(a_1 + \cdots + a_n)^{\alpha/2} \le a_1^{\alpha/2} + \cdots + a_n^{\alpha/2}$ for non-negative $a_1, \ldots, a_n$ and $0 < \alpha < 2$, together with

Hölder's inequality, we get

$$\begin{aligned} d_{p,\alpha}(f,h) &= \mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |(f-g)+(g-h)(t,x)|^p\,dx\,dt\Big)^{\alpha/2}\Big]^{1/p} \\ &\le \mathbb{E}\Big[\Big(\Big(\int_0^\infty\int_{\mathbb{S}^1} |f-g|^p\,dx\,dt\Big)^{1/p} + \Big(\int_0^\infty\int_{\mathbb{S}^1} |g-h|^p\,dx\,dt\Big)^{1/p}\Big)^{p\alpha/2}\Big]^{1/p} \\ &\le \mathbb{E}\Big[\Big(\Big(\int_0^\infty\int_{\mathbb{S}^1} |f-g|^p\,dx\,dt\Big)^{\alpha/(2p)} + \Big(\int_0^\infty\int_{\mathbb{S}^1} |g-h|^p\,dx\,dt\Big)^{\alpha/(2p)}\Big)^{p}\Big]^{1/p} \\ &\le \mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |f-g|^p\,dx\,dt\Big)^{\alpha/2}\Big]^{1/p} + \mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |g-h|^p\,dx\,dt\Big)^{\alpha/2}\Big]^{1/p} \\ &= d_{p,\alpha}(f,g) + d_{p,\alpha}(g,h). \end{aligned}$$

The second to third line follows since for non-negative A and B, and $p > 1$,

$$\mathbb{E}[(A+B)^p]^{1/p} \le \mathbb{E}[A^p]^{1/p} + \mathbb{E}[B^p]^{1/p}. \ \blacksquare$$

LEMMA 12. *For $p > 1$ and $\alpha \in (0,1)$, the space $\mathcal{S}_{p,\alpha}$ equipped with the metric $d_{p,\alpha}$ is complete.*

Proof. Consider a Cauchy sequence $(u^{(n)})_{n\ge 0}$ in $\mathcal{S}_{p,\alpha}$. There is a subsequence $(u^{(n_k)})_{k\ge 1}$ such that $d_{p,\alpha}(u^{(n_k)}, u^{(n_{k+1})}) \le e^{-k}$. Let

$$G = |u^{(n_0)}| + \lim_{N\to\infty} \sum_{k=1}^{N} |u^{(n_k)} - u^{(n_{k-1}))}|.$$

The limit is pointwise well defined $\lambda_{\mathbb{R}_+} \otimes \lambda_{\mathbb{S}^1} \otimes \mathbb{P}$-almost surely, where $\lambda_{\mathbb{R}_+}$ and $\lambda_{\mathbb{S}^1}$ denote Lebesgue measure over the time and spatial variables respectively. This is seen as follows (we omit the differentials for simplicity):

$$\begin{aligned} &\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} \Big(|u^{(n_0)}| + \sum_{k=1}^{\infty} |u^{(n_k)} - u^{(n_{k-1})}|\Big)^p\Big)^{\alpha/2}\Big]^{1/p} \\ &\quad\le \mathbb{E}\Big[\Big(\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_0)}|^p\Big)^{1/p} + \sum_{k=1}^{\infty}\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p\Big)^{1/p}\Big)^{p\alpha/2}\Big]^{1/p} \\ &\quad\le \mathbb{E}\Big[\Big(\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_0)}|^p\Big)^{\alpha/(2p)} + \sum_{k=1}^{\infty}\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p\Big)^{\alpha/(2p)}\Big)^{p}\Big]^{1/p} \\ &\quad\le \mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_0)}|^p\Big)^{\alpha/2}\Big]^{1/p} + \sum_{k=1}^{\infty}\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_k)} - u^{(n_{k-1})}|^p\Big)^{\alpha/2}\Big]^{1/p} \\ &\quad= \mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} |u^{(n_0)}|^p\Big)^{\alpha/2}\Big]^{1/p} + \sum_{k=1}^{\infty} d_{p,\alpha}(u^{(n_{k-1})}, u^{(n_k)}) < \infty. \end{aligned}$$

It follows that G is well defined almost surely and hence that $u^{(n_0)} + \sum_{k=1}^{N}(u^{(n_k)} - u^{(n_{k-1})})$ converges pointwise almost surely to a limit u such that $|u| \le G$. Now, from the dominated convergence theorem,

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} |u^{(n_k)} - u|^p\Big)^{\alpha/2}\Big] \xrightarrow{k\to\infty} 0.$$

Hence the space is complete. ■

Now the stochastic integral with respect to the Wiener sheet may be constructed.

REMARK. Although the construction is essentially the same as in Walsh [18], the stochastic integral here is constructed over the whole time range $[0,\infty)$. The functions of interest (solutions to (1)) decay as $t \to \infty$ and the definition presents no difficulty, as seen from the description below.

Let $\mathcal{C}$ denote the class of functions $g \in \mathcal{P}$ such that there are

- $m_0 < \infty$, disjoint sets $\{B_j : j = 1, \dots, m_0\}$ and $\mathcal{F}_0$-measurable random variables $f_1, \dots, f_{m_0}$,
- a collection $0 = t_0 < t_1 < \dots < t_n < \infty$,
- sets $\{A_{i,j}: i=1, \dots, n,\ j=1, \dots, m_i\}$ where for each i, the sets $A_{i,1}, \dots, A_{i,m_i}$ are disjoint and $m_j < \infty$ for each $j \in \{0, 1, \dots, n-1\}$ and $\bigcup_{j=1}^{m_i} A_{i,j} = \mathbb{S}^1$,
- a collection $(g_{i,j} : i \in \{0, \dots, n-1\},\ j \in \{1, \dots, m_i\})$ of random variables such that $g_{i,j}$ is $\mathcal{F}_{t_i}$-measurable for each $j \in \{1, \dots, m_i\}$,

and g is given by

$$g(s,x,\omega) = \sum_{j=1}^{m_0} f_j(\omega)\mathbf{1}_{\{0\}}(s)\mathbf{1}_{B_j}(x) + \sum_{i=0}^{n-1}\sum_{j=1}^{m_i} g_{i,j}(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(s)\mathbf{1}_{A_{i,j}}(x)$$

and satisfies

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} g(t,x)^2\,dx\,dt\Big)^{\alpha/2}\Big] = \mathbb{E}\Big[\Big(\sum_{i=1}^{n}(t_{i+1}-t_i)\sum_{j=1}^{m_i}|A_{ij}|g_{ij}^2\Big)^{\alpha/2}\Big] < \infty.$$

For $g \in \mathcal{C}$, the stochastic integral is defined as

$$I(g)(t) = \sum_{i=0}^{n-1}\sum_{j=1}^{m_i} g_{i,j} W((t\wedge t_i, t\wedge t_{i+1}] \times A_{ij}).$$

It is clear that, for $g \in \mathcal{C}$, the stochastic integral $I(g)$ is a continuous local martingale with quadratic variation given by

$$\langle I(g)\rangle(t) = \int_0^t \int_{\mathbb{S}^1} g(s,x)^2\,dx\,ds.$$

LEMMA 13. *Let $\alpha \in (0,1)$. The space of continuous local martingales M with $\mathbb{E}[\sup_t |M(t)|^\alpha] < \infty$ with metric $D_\alpha(M,N) = \mathbb{E}[\sup_t |M(t) - N(t)|^\alpha]$*

is complete in the following sense. Let $M^{(n)}$ denote a sequence of local martingales satisfying $\sup_n \mathbb{E}[\sup_t |M^{(n)}(t)|^\alpha] < K$ *for some $K < \infty$ such that*

$$\lim_{n\to\infty} \sup_{N\geq n} \mathbb{E}\Big[\sup_t |M^{(N)}(t) - M^{(n)}(t)|^\alpha\Big] = 0. \tag{14}$$

Then there is a continuous local martingale M with $\mathbb{E}[\sup_t |M(t)|^\alpha] < K$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_t |M^{(n)}(t) - M(t)|^\alpha\Big] = 0.$$

Proof. Firstly, for completeness, consider a subsequence $M^{(n_k)}$ such that

$$\lim_{m\to\infty} \mathbb{E}\Big[\Big(\sum_{k=1}^m \sup_{0<t<\infty} |M^{(n_k)}(t) - M^{(n_{k-1})}(t)|\Big)^\alpha\Big] < \infty$$

for $\alpha < 1$. Such a subsequence exists, by hypothesis, since for $\alpha < 1$,

$$\Big(\sum_{k=1}^m \sup_{0<t<\infty} |M^{(n_k)}(t) - M^{(n_{k-1})}(t)|\Big)^\alpha \leq \sum_{k=1}^m \sup_{0<t<\infty} |M^{(n_k)}(t) - M^{(n_{k-1})}(t)|^\alpha,$$

and, by (14), a subsequence $(M^{(n_k)})_{k\geq 1}$ can be extracted such that

$$\sum_{k=1}^\infty \mathbb{E}\Big[\sup_{0<t<\infty} |M^{(n_k)}(t) - M^{(n_{k-1})}(t)|^\alpha\Big] < \infty.$$

Let $G = \sup_t M^{(n_0)}(t) + \sum_{k=1}^\infty \sup_t |M^{(n_k)}(t) - M^{(n_{k-1})}(t)|$. Then $\mathbb{E}[G^\alpha] < \infty$ so that $G < \infty$ almost surely and hence $M = M^{(n_0)} + \sum_{k=1}^\infty (M^{(n_k)} - M^{(n_{k-1})})$ exists almost surely and converges almost surely uniformly in t. In particular, since each $M^{(n_k)}$ is continuous almost surely, the following argument shows that M is continuous almost surely. It is necessary and sufficient to show that for each $T < \infty$ and all $\epsilon > 0$ there exists a $\delta(T, \omega, \epsilon) > 0$ almost surely (we write just $\delta(\epsilon)$) such that

$$\sup_{0\leq t\leq T} \sup_{|t-s|<\delta(\epsilon)} |M(t) - M(s)| < \epsilon. \tag{15}$$

Now, for any ϵ, there is a $k > 0$ such that $\sup_t |M(t) - M^{(n_k)}(t)| < \epsilon/3$, and for such n_k there exists a $\delta(\epsilon)$ such that $\sup_{|t-s|<\delta(\epsilon))} |M^{(n_k)}(t) - M^{(n_k)}(s)| < \epsilon/3$. Using this $\delta(\epsilon)$, it follows that (15) holds.

Finally, the local martingale property is established. For a fixed $\alpha \in (0, 1)$, choose a sequence $M^{(n_j)}$ such that, for each j,

$$\mathbb{E}\Big[\sup_t |M^{(n_j)}(t) - M(t)|^\alpha\Big] < \frac{1}{j^2}.$$

Let

$$\tau_N = \inf\{t : |M(t)| > N\}, \qquad \tau_N^{(j)} = \inf\{t : |M^{(n_j)}(t)| > N\}.$$

Then (clearly) $\lim_{N\to\infty}\tau_N = \infty$ almost surely. For $t > s > 0$,

$$\begin{aligned}
&\mathbb{E}[M(t\wedge\tau_N)\,|\,\mathcal{F}_s] \\
&\quad = \mathbb{E}[M^{(n_j)}(t\wedge\tau_N^{(j)})\,|\,\mathcal{F}_s] + \mathbb{E}[M(t\wedge\tau_N) - M^{(n_j)}(t\wedge\tau_N^{(j)})\,|\,\mathcal{F}_s] \\
&\quad = M^{(n_j)}(s\wedge\tau_N^{(j)}) + \mathbb{E}[M(t\wedge\tau_N) - M^{(n_j)}(t\wedge\tau_N^{(j)})\,|\,\mathcal{F}_s].
\end{aligned}$$

Now, $M^{(n_j)}(t\wedge\tau_N^{(j)}) \xrightarrow{j\to\infty} M(t\wedge\tau_N)$ almost surely for all $t > 0$. Indeed, let $r_N = \liminf_{j\to\infty}\tau_N^{(j)}$; then $M^{(n_j)}(t\wedge r_N) \xrightarrow{j\to\infty} M(t\wedge r_N)$ almost surely and there is a subsequence j_k such that $M^{(n_{j_k})}(r_N\wedge t) = N$ for $r_N < t$, so that $\tau_N \le r_N$ and hence we have equality, since (by construction) $M(t) < N$ for all $t < r_N$. From this, the convergence is clear. Since $|M(t\wedge\tau_N) - M^{(n_j)}(t\wedge\tau_N^{(j)})| < 2N$, it follows from the dominated convergence theorem that the second term converges to 0 as $j\to\infty$ and hence

$$\mathbb{E}[M(t\wedge\tau_N)\,|\,\mathcal{F}_s] = M(s\wedge\tau_N),$$

so M is a continuous local martingale. ∎

For $g \in \mathcal{S}_{2,\alpha}$, the stochastic integral may now be constructed.

Definition 14 (Stochastic integral). If $\|g\|_{2,\alpha} = K$, consider an approximating sequence $g^{(n)} \in \mathcal{C}$ such that $\|g^{(n)}\|_{2,\alpha} \le 2K$ for each n and $\lim_{n\to\infty}\|g^{(n)} - g\|_{2,\alpha} = 0$. The stochastic integral $I(g)$ is defined as the limit of $I(g^{(n)})$, in the sense of convergence of local martingales of Lemma 13.

Lemma 15. *The stochastic integral of Definition* 14 *is well defined.*

Proof. It follows from the Burkholder–Davis–Gundy inequality that for $\alpha < 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_t |I(g^{(n)})(t) - I(g)(t)|^\alpha\Big] \\
&\qquad\qquad \le C(\alpha)\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1}(g^{(n)} - g)^2(t,x)\,dx\,dt\Big)^{\alpha/2}\Big] \xrightarrow{n\to\infty} 0
\end{aligned}$$

where $C(\alpha)$ is the universal constant from the Burkholder–Davis–Gundy inequality. The result now follows by Lemma 13. ∎

The following small lemma helps in the definition of a stochastic integral.

Lemma 16. *Let $g^{(n)}$ be a non-decreasing sequence of non-negative functions, each satisfying $\|g^{(n)}\|_{p,\alpha} < K$ for a given $K < \infty$, $p > 1$ and $\alpha \in (0,1)$. Then $g^{(n)}$ is a Cauchy sequence in $\mathcal{S}_{p,\alpha}$.*

Proof. Firstly, it is an easy consequence of Fatou's lemma that the limit $g = \lim_{n\to\infty} g^{(n)}$ satisfies $\|g\|_{p,\alpha} < K$, so that $g \in \mathcal{S}_{p,\alpha;K}$. To conclude, it remains to show that $d_{p,\alpha}(g^{(n)}, g) \xrightarrow{n\to\infty} 0$; but this follows directly by dominated convergence: The functions $g - g^{(n)}$ are non-negative, converging pointwise to 0, with dominating function g. ∎

4. Definition and existence of solution. Equation (1) is understood as the equivalent Stochastic Integral Equation (SIE) given by

$$u(t,x) = P_t u_0(x) + \int_0^t \int_{\mathbb{S}^1} p(t-r, x-y) u(r,y)^\gamma \, W(dr, dy) \tag{16}$$

$\mathbb{P}$-a.s. for $0 \leq t < \infty$ and $x \in \mathbb{S}^1$, where $x - y$ is taken modulo 1 and $p : [0,\infty) \times \mathbb{S}^1 \to \mathbb{R}_+$ satisfies

$$\begin{cases} p_t = \frac{1}{2} p_{xx}, \\ p(0,\cdot) = \delta_0(\cdot), \end{cases}$$

where δ_0 denotes the Dirac delta function with unit mass at 0 and $P_t f(x) = \int_{\mathbb{S}^1} p(t, x-y) f(y) \, dy$. The initial condition u_0 is taken to be a measurable mapping, independent of W, which satisfies

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(\int_{\mathbb{S}^1} (u_0(x) \wedge n) \, dx\Big)^2\Big] = \mathbb{E}\Big[\Big(\int_{\mathbb{S}^1} u_0(x) \, dx\Big)^2\Big] < \infty.$$

(As stated before, the condition on u_0 excludes Dirac masses, which the techniques do not deal with.) Equation (16) is the *mild form* of (1); a function u that satisfies (16) is known as a *mild solution* to (1).

These conditions are imposed for two reasons: firstly, this ensures that the proofs of Walsh [18] may be used to provide existence and uniqueness for solutions to the approximating equations. Secondly, existence of a second moment of $\int_{\mathbb{S}^1} u_0(x) \, dx$ is used at several places in the proof of the main theorem.

An optimal description of the *state space* (a space $\mathcal{S}$ such that if $u_0 \in \mathcal{S}$ then $u(t,\cdot) \in \mathcal{S}$ for all $t \geq 0$ $\mathbb{P}$-almost surely) is not obtained in this article. That is (of course) a serious and very interesting question. Mueller and Sowers [9] proved explosion of the L^∞ spatial norm for sufficiently large γ, while Mueller [11] showed that there was explosion of the L^∞ norm for all $\gamma > 3/2$. The nature of the explosion is unknown. The aim of this article is simply to show existence of a solution to (16) which is a non-negative (generalised) function u, which satisfies (16) and such that

- $\mathbb{E}[(\sup_{0\leq t<\infty} \int_{\mathbb{S}^1} u(t,x) \, dx)^\alpha] < \infty$ for all $\alpha \in (0,1)$ and
- $\mathbb{E}[(\int_0^\infty \int_{\mathbb{S}^1} u^{2\gamma}(t,x) \, dx \, dt)^{\alpha/2}] < \infty$ for all $\alpha \in (0,1)$.

A priori, if there is a well defined non-negative solution such that all the terms are well defined (and, in particular, the stochastic term is well defined), then $U(t) := \int_{\mathbb{S}^1} u(t,x) \, dx$ will be a non-negative local martingale and hence

$\mathbb{E}[\sup_{0<t<\infty} U(t)^\alpha] < \infty$ for all $\alpha \in (0,1)$. Furthermore, if there is a well defined solution, then the increasing process of this non-negative local martingale is $\langle U\rangle(t) = \int_0^t \int_{\mathbb{S}^1} u(s,x)^{2\gamma}\,dx\,ds$ and will therefore (by the Burkholder–Davis–Gundy inequality) satisfy $\mathbb{E}[\langle U\rangle(\infty)^{\alpha/2}] < \infty$ for all $\alpha \in (0,1)$.

Existence of solution is established by considering suitable approximating functions $(u^{(n)})_{n\ge 1}$ where, for each n, $u^{(n)}$ is well defined and solves

$$u^{(n)}(t,x) = P_t(u_0 \wedge n)(x) + \int_0^t \int_{\mathbb{S}^1} p(t-s,x-y)(u^{(n)}(s,y)\wedge n)^\gamma\, W(ds,dy). \tag{17}$$

The functions $u^{(n)}$ are considered as mappings $u^{(n)} : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to \mathbb{R}_+$.

Lemma 17 below forms the basis for obtaining $L^{2\gamma}$ norms. It is proved by a straightforward application of the Burkholder–Davis–Gundy inequality and is key for establishing uniform bounds necessary to find a convergent subsequence of $(u^{(n)})_{n\ge 1}$.

LEMMA 17. *Recall that $\gamma > 1$. For each $n \ge 1$, there exists a unique solution to* (17) *in $\mathcal{S}_{2\gamma,2}$ equipped with metric $d_{2\gamma,2}$. For $\alpha \in (0,1)$, let*

$$\widetilde{K}(\alpha) := K(\alpha)\mathbb{E}[U^\alpha(0)] \tag{18}$$

where $K(\alpha) > 0$ is from (10). *Then, for $\alpha \in (0,1)$, $\widetilde{K}(\alpha) < \infty$ and*

$$\sup_n \mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} (u^{(n)}(s,y)\wedge n)^{2\gamma}\,dy\,ds\Big)^{\alpha/2}\Big] < \widetilde{K}(\alpha) < \infty. \tag{19}$$

Proof. The first statement (unique solution to (17) for n finite) follows similarly to Walsh [18, Theorem 3.2, p. 313] and is standard; the initial condition $u_0(x)\wedge n$ is bounded and therefore the Walsh proof holds, with minor modifications to deal with the space $\mathbb{S}^1$ instead of $[0,L]$.

With the truncation at n in the stochastic term, existence of moments (and hence the solution in $(\mathcal{S}_{2\gamma,2}, d_{2\gamma,2})$) does not present a problem. Walsh restricted his construction to finite time intervals $[0,T]$ and did not consider the whole real line $[0,\infty)$. Let $u^{(n,T)}$ denote the function that provides the unique solution up to time T and $u^{(n,T)}(t,\cdot) \equiv 0$ for all $t > T$ and let $u^{(n)} = \lim_{T\to\infty} u^{(n,T)}$. Then $u^{(n)}$ is well defined and provides the unique solution up to time T for all $T \in \mathbb{R}_+$.

To prove the second statement, let

$$U^{(n)}(t) = \int_{\mathbb{S}^1} u^{(n)}(t,x)\,dx \tag{20}$$

and note that $U^{(n)}$ is a non-negative martingale that satisfies

$$U^{(n)}(t) = \int_{\mathbb{S}^1} (u_0(x)\wedge n)\,dx + \int_0^t \int_{\mathbb{S}^1} (u^{(n)}(s,y)\wedge n)^\gamma\, W(dy,ds).$$

It is straightforward that for finite n, $U^{(n)}$ is a martingale, since it is a stochastic integral in the sense of Walsh with bounded integrand. Its increasing process is

$$\langle U^{(n)}\rangle(t) = \int_0^t \int_{\mathbb{S}^1} (u^{(n)}(s,y)\wedge n)^{2\gamma}\, dy\, ds.$$

From Lemma 4, for all $\alpha < 1$,

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} (u^{(n)}(s,y)\wedge n)^{2\gamma}\, dy\, ds\Big)^{\alpha/2}\Big] \le \widetilde{K}(\alpha),$$

thus proving the second statement of Lemma 17. ■

Attention is now turned to establishing the convergence necessary to show that the limit of the stochastic terms in the equation is the stochastic term in the limiting equation. The key to this is equation (21) of Theorem 18 below; convergence of $(u^{(n_j)}\wedge n_j)_{j\ge 1}$ in the space $\mathcal{S}_{2\gamma,\alpha}$ for all $\alpha\in(0,1)$ is a direct corollary of this.

THEOREM 18. *Let $v^{(n)} = u^{(n)}\wedge n$. There exists a subsequence $(u^{(n_j)})_{j\ge 1}$ which satisfies, for all $\alpha\in(0,1)$,*

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} (v^{(n_j)}(s,x)^\gamma - v^{(n_k)}(s,x)^\gamma)^2\, dx\, ds\Big)^{\alpha/2}\Big] \xrightarrow{j,k\to\infty} 0. \tag{21}$$

Therefore,

$$\lim_{j\to\infty}\lim_{k\to\infty} d_{2,\alpha}(v^{(n_j)\gamma}, v^{(n_k)\gamma}) = 0$$

and hence $\lim_{j\to\infty}\lim_{k\to\infty} d_{2\gamma,\alpha}(v^{(n_j)}, v^{(n_k)}) = 0$ for all $\alpha\in(0,1)$. Since $\mathcal{S}_{2\gamma,\alpha}$ is complete *for $\alpha\in(0,1)$ by Lemma* 13, *there is a $u\in\mathcal{S}_{2\gamma,\alpha}$ which satisfies*

$$\lim_{j\to\infty} d_{2\gamma,\alpha}(v^{(n_j)}, u) = 0.$$

Proof. The proof is in stages and is based on the total mass processes $U^{(n)}$, which are non-negative local martingales.

STEP 1: *A Banach space, weak convergence and convex combinations.* Recall the definition of the 'total mass process' $U^{(n)}$, given by (20). The first step is to show that there is a subsequence $(U^{(n_j)})_{j\ge 1}$, a Mazur sequence of convex combinations of $U^{(n_j)}$,

$$\widetilde{V}_k = \sum_{j=k}^{f(k)} \alpha_{kj} U^{(n_j)}, \tag{22}$$

and a U such that $\|\widetilde{V}_k - U\| \xrightarrow{k\to\infty} 0$ in a suitable sense in a suitable space. In subsequent steps, it will be shown that this limit is a local martingale (Step 2), and that $\lim_{t\to\infty} U(t) = 0$ $\mathbb{P}$-a.s. (Step 3).

In Step 4, *strong* convergence is established; if $U(t) \xrightarrow{t\to\infty} 0$ $\mathbb{P}$-a.s. for a subsequence $(n_j)_{j\geq 1}$, then $\mathbb{E}[\sup_t |U^{(n_j)}(t) - U(t)|^\alpha] \xrightarrow{j\to\infty} 0$. Finally (Step 5), this is used to establish the result.

For Step 1, it should be pointed out that the term 'weak convergence' does *not* refer to the sense usually used by probabilists, i.e. weak convergence of probability measures. Here it refers to weak convergence in a reflexive Banach space and the Banach–Alaoglu theorem. This has been used in SPDEs in the context of Galerkin approximations (see, for example, Rockner and Liu [15] and Lototsky and Rozovsky [6]), but is not the standard use of the term in the probabilistic literature. The application here of 'weak convergence' in the functional-analytic sense is different from the standard use in SPDEs.

The construction of the Banach space which enables a weak limit to be obtained is based on the observation that, for each $n \in \mathbb{N}$ and all $\alpha \in (0,1)$,

$$\sup_n \mathbb{E}[\langle U^{(n)}, U^{(n)}\rangle^{\alpha/2}(\infty)] = \sup_n \mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} v^{(n)}(s,x)^{2\gamma}\,dx\,ds\Big)^{\alpha/2}\Big] \leq K(\alpha)\mathbb{E}[U(0)^\alpha].$$

This is simply equation (8); the constant $K(\alpha)$ is given by (10). The upper bound is uniform; it does not depend on n and is the basis of the construction of a reflexive Banach space. The Banach–Alaoglu theorem, which is found in Rudin [16, p. 69, Theorem 3.15] then gives weak compactness, hence the existence of a weakly convergent subsequence and weak limit.

Recall that $v^{(n)} = u^{(n)} \wedge n$ and let

$$\widehat{W}^{(n)}(t) = \int_{\mathbb{S}^1} v^{(n)}(t,x)\,dx, \tag{23}$$

$$E^{(n)}(t) = \int_{\mathbf{S}^1} (u^{(n)}(t,x) - n)\mathbf{1}_{\{u^{(n)}(t,x)\geq n\}}\,dx, \tag{24}$$

so that $U^{(n)} = \widehat{W}^{(n)} + E^{(n)}$. It is relatively easy to establish a suitable space and a weakly convergent subsequence of $\widehat{W}^{(n)}$ with limit U in that space. The harder part is to show that $E^{(n)} \to 0$ in an appropriate sense.

By Jensen's inequality, using $\int_0^\infty \frac{dt}{(1+t)^2} = 1$ and the obvious fact that $\frac{1}{(1+t)^2} \leq 1$ for all $t \in \mathbb{R}_+$, we get

$$\sup_n \int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[\widehat{W}^{(n)}(t)^{\gamma\alpha}]\,dt \leq K(\alpha)\mathbb{E}[U(0)^\alpha].$$

Now take $\alpha = \frac{1+\gamma}{2\gamma} < 1$; then $K(\frac{1+\gamma}{2\gamma}) < \infty$ and, explicitly,

$$\sup_n \int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[\widehat{W}^{(n)1+(\gamma-1)/2}(t)]\,dt \leq K\Big(\frac{1+\gamma}{2\gamma}\Big)\mathbb{E}[U(0)^{(1+\gamma)/(2\gamma)}] < \infty,$$

where $(\gamma-1)/2 > 0$ (strict inequality for $\gamma > 1$). This leads naturally to the Banach space $\mathcal{R}$ with norm

$$\|F\|_{\mathcal{R}} = \left(\int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[|F|^{(1+\gamma)/2}(t)]\,dt\right)^{2/(1+\gamma)}. \tag{25}$$

Since $(1+\gamma)/2 > 1$ (strict inequality for $\gamma > 1$), this is a *reflexive* Banach space and hence any sequence bounded in norm has a weakly convergent subsequence. The linear functionals Λ_G are defined as

$$\Lambda_G(F) = \int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[F(t)G(t)]\,dt$$

for G satisfying

$$\int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[|G|^{(\gamma+1)/(\gamma-1)}(t)]dt < \infty.$$

The Banach space is therefore an L^p space where $p = (1+\gamma)/2 > 1$; its dual is the space L^q where $q = \frac{\gamma+1}{\gamma-1} < \infty$.

Now, the Banach–Alaoglu theorem may be invoked; the Banach space is reflexive and hence is weakly compact. Therefore, there is a subsequence $(n_j)_{j\in\mathbb{N}}$ such that $(\widehat{W}^{(n_j)})_{j\geq 1}$ has a weak limit U. That is, for each G belonging to the dual space,

$$\lim_{j\to\infty} \Lambda_G(\widehat{W}^{(n_j)}) = \Lambda_G(U).$$

Constraints on the subsequence. Without loss of generality, a $\beta \in (0,1)$ is chosen and the subsequence is taken to satisfy

$$\sum_{j=1}^\infty \frac{1}{n_j^\beta} < \infty. \tag{26}$$

This is a technical hypothesis, which will be used to establish that convex combinations of the extra pieces $E^{(n)}$ converge to 0 appropriately.

Mazur's lemma may now be invoked (this is stated as Theorem 3.13 in Rudin [16, p. 67], although he does not refer to it as Mazur's lemma). $\widehat{W}^{(n_j)}$ is a weakly convergent subsequence in a Banach space, hence there exist convex combinations $\{\alpha_{kj} : j \in \{k,\dots,f(k)\}\}$, where for finite k, $f(k)$ is finite, such that $\sum_{j=k}^{f(k)} \alpha_{kj} = 1$, $\alpha_{kj} \geq 0$ for each k and j, and

$$\widehat{V}_k := \sum_{j=k}^{f(k)} \alpha_{kj}\widehat{W}^{(n_j)} \tag{27}$$

converges *strongly* to U in $\mathcal{R}$, that is,

$$\lim_{k\to\infty}\int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[|\widehat{V}_k(t)-U(t)|^{(1+\gamma)/2}]\,dt = 0. \tag{28}$$

Next, it is necessary to show that the extra pieces $E^{(n_j)}$ converge to 0 in a suitable sense. Although these extra pieces should, in principle, be small (and indeed are small), the proof presented here is somewhat cumbersome (and a much shorter proof is probably achievable). Set

$$D_k(t) = \sum_{j=k}^{f(k)} \alpha_{kj} E^{(n_j)}(t)$$

so that the D_k's are the Mazur convex combinations of the extra pieces $E^{(n_j)}$, and set

$$\widetilde{V}_k(t) = \widehat{V}_k(t) + D_k(t) = \sum_{j=k}^{f(k)} \alpha_{kj} U^{(n_j)}(t).$$

Then, for each $k\in\mathbb{N}$, $\widetilde{V}_k$ is a non-negative local martingale. The next step is to establish that $D_k(t) \xrightarrow{k\to\infty} 0$ in a certain sense. Let

$$\widetilde{u}_k(t,x) = \sum_{j=k}^{f(k)} \alpha_{kj} u^{(n_j)}(t,x),$$

$$d_k(t,x) = \sum_{j=k}^{f(k)} \alpha_{kj}(u^{(n_j)}(t,x)-n_j)\mathbf{1}_{\{u^{(n_j)}(t,x)\ge n_j\}}.$$

Now let $C_j = \sup_t U^{(n_j)}(t)$ so that for $\alpha\in(0,1)$, $\mathbb{E}[C_j^\alpha] < \frac{1}{1-\alpha}\mathbb{E}[U(0)^\alpha] < \infty$ from (7). Also,

$$\sup_t \int_{\mathbb{S}^1} \mathbf{1}_{\{u^{(n_j)}(t,x)\ge n_j\}}\,dx \le 1\wedge \frac{1}{n_j}\sup_t U^{(n_j)}(t) = 1\wedge\frac{C_j}{n_j}.$$

This simply uses the result that, for a non-negative function f, if $\int f\le C$, then $\int \mathbf{1}_{\{f\ge n\}} \le C/n$. Hence

$$\sup_t \int_{\mathbb{S}^1} \mathbf{1}_{\{d_k(t,x)>0\}}\,dx \le \sup_t \sum_{j=k}^{f(k)} \int_{\mathbb{S}^1} \mathbf{1}_{\{u^{(n_j)}(t,x)\ge n_j\}}\,dx \le \sum_{j=k}^{f(k)} \frac{C_j}{n_j},$$

from which, for any $\alpha\in(0,1)$,

$$\mathbb{E}\Big[\sup_t\Big(\int_{\mathbb{S}^1} \mathbf{1}_{\{d_k(t,x)>0\}}\,dx\Big)^\alpha\Big] \le \frac{1}{1-\alpha}\mathbb{E}[U(0)^\alpha]\sum_{j=k}^{f(k)}\frac{1}{n_j^\alpha}.$$

Here the following inequality is used: for $\alpha \in (0,1)$ and any collection of non-negative numbers $(b_j)_{j\geq 1}$, $(\sum_j b_j)^\alpha \leq \sum_j b_j^\alpha$.

Now, recall that the subsequence has been chosen to satisfy (26) for a particular $\beta \in (0,1)$. Hence $\sum_{j=1}^\infty 1/n_j^\alpha < \infty$ for all $\alpha \geq \beta$ and so $\lim_{k\to\infty} \sum_{j=k}^{f(k)} 1/n_j^\alpha = 0$ for all $\alpha \geq \beta$. Furthermore, if for a sequence of random variables X_k satisfying $|X_k| \leq 1$, $\lim_{k\to\infty} \mathbb{E}[|X_k|^\beta] = 0$ for some $\beta > 0$, then for all $p > 0$, $\lim_{k\to\infty} \mathbb{E}[|X_k|^p] = 0$. Hence

$$\lim_{k\to\infty} \mathbb{E}\Big[\sup_t \Big(\int_{\mathbb{S}^1} \mathbf{1}_{\{d_k(t,x)>0\}}\, dx\Big)^p\Big] = 0 \qquad \forall p > 0. \tag{29}$$

Now, set

$$g_k(s,x) := \sum_{j=k}^{f(k)} \alpha_{kj} v^{(n_j)}(s,x)^\gamma. \tag{30}$$

Then $\widetilde{u}_k(t,x)$ satisfies

$$\widetilde{u}_k(t,x) = P_t \widetilde{u}_k(0,x) + \int_0^t\int p(t-s, x-y) g_k(s,y)\, W(dy,ds)$$

where $\widetilde{u}_k(0,x) = \sum_j \alpha_{kj} u^{(n_j)}(0,x)$. Let

$$\sigma_{k,N} := \inf\{t : \widetilde{V}_k(t) \geq N\} \tag{31}$$

and define the process M as

$$M_{s,t}(x) := \begin{cases} P_{t-s}\widetilde{u}_k(s,x), & 0 \leq s \leq t, \\ \widetilde{u}_k(t,x), & s \geq t. \end{cases}$$

Then, for fixed (t,x), $\{M_{s,t}(x) : s \in \mathbb{R}_+\}$ is (clearly) a local martingale (in s).

It is straightforward that

$$\int_{\mathbb{S}^1} p(s,x)^2\, dx \leq 3 + \frac{1}{2\sqrt{\pi s}}. \tag{32}$$

This follows from taking $p(t,x) \leq 1 + \widetilde{p}(t,x)$, where $\widetilde{p}(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$. Then, using the Burkholder–Davis–Gundy inequality with $p = 2$, we get

$$\mathbb{E}\Big[\sup_{0\leq t\leq\infty} |M(t) - M(0)|^p\Big] \leq C(p)\mathbb{E}[\langle M, M\rangle^{p/2}(\infty)]$$

for a local martingale M, so that

$$\mathbb{E}\Big[\sup_{0\leq t<\infty} |M(t)|^p\Big] \leq C(p)\mathbb{E}[|M(0)|^p] + C(p)\mathbb{E}[\langle M, M\rangle^{p/2}(\infty)]$$

for a constant $C(p) < \infty$:

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{S}^1} \sup_{0\le s\le t} M_{s\wedge\sigma_{k,N},t}(x)^2\,dx\Big]\\
&\quad\le C(2)\mathbb{E}\Big[\int_{\mathbb{S}^1}(P_t\widetilde{u}_k(0,x))^2\,dx\Big] + C(2)\mathbb{E}\Big[\int_{\mathbb{S}^1}\int_0^{\sigma_{k,N}\wedge t}\int_{\mathbb{S}^1} p_{t-s}(x-y)^2 g_k(s,y)^2\,dy\,ds\,dx\Big]\\
&\quad\le C(2)\mathbb{E}\Big[\int_{\mathbb{S}^1}(P_t\widetilde{u}_k(0,x))^2\,dx\Big] + C(2)\mathbb{E}\Big[\int_0^{\sigma_{k,N}\wedge t}\Big(3+\frac{1}{2\sqrt{\pi(t-s)}}\Big)\int_{\mathbb{S}^1} g_k(s,y)^2\,dy\,ds\Big].
\end{aligned}$$

Let $A_N = \{\sigma_{k,N-1} < \infty,\, \sigma_{k,N} = \infty\}$ and let $Y(s) = \mathbb{P}(A_N|\mathcal{F}_s)$. Then $Y(s)$ can be computed quite easily:

$$Y(s) = \begin{cases} 0, & s \ge \sigma_{k,N},\\ 1 - \frac{\widetilde{V}_k(s)}{N}, & \sigma_{k,N-1} \le s < \sigma_{k,N},\\ \frac{\widetilde{V}_k(s)}{N(N-1)}, & s < \sigma_{k,N-1}. \end{cases}$$

This is seen as follows: there are two cases, $\sigma_{k,N-1} = 0$ and $\sigma_{k,N-1} > 0$. For any stopping time $\rho_1 \le \sigma_{k,N-1}$ consider the first case: $\rho_1 = \sigma_{k,N-1} = 0$; then $\widetilde{V}_k(0)$ is $\mathcal{F}_{\rho_1}$-measurable and

$$\mathbb{P}(\sigma_{k,N} = \infty,\, \sigma_{k,N-1} < \infty \,|\, \mathcal{F}_{\rho_1}) = \begin{cases} 0, & \widetilde{V}_k(0) \ge N,\\ \frac{\widetilde{V}_k(0)}{N}, & N-1 \le \widetilde{V}_k(0) < N. \end{cases}$$

For $\sigma_{k,N-1} > 0$,

$$\begin{aligned}
&\mathbb{P}(\sigma_{k,N} = \infty,\, \sigma_{k,N-1} < \infty \,|\, \mathcal{F}_{\rho_1})\\
&\qquad = \mathbb{E}\big[\mathbb{E}[\mathbf{1}_{\{\sigma_{k,N}=\infty\}}\mathbf{1}_{\{\sigma_{k,N-1}<\infty\}} \,|\, \mathcal{F}_{\sigma_{k,N-1}}]\mathcal{F}_{\rho_1}\big]\\
&\qquad = \mathbb{E}\Big[\Big(1 - \frac{N-1}{N}\Big)\mathbf{1}_{\{\sigma_{k,N-1}<\infty\}} \,\Big|\, \mathcal{F}_{\rho_1}\Big]\\
&\qquad = \frac{1}{N}\mathbb{P}(\sigma_{k,N-1} < \infty \,|\, \mathcal{F}_{\rho_1}) = \frac{\widetilde{V}_k(\rho_1)}{N(N-1)}
\end{aligned}$$

and for any stopping time ρ_2 with $\sigma_{k,N-1} < \rho_2 \le \sigma_{k,N}$,

$$\mathbb{P}(\sigma_{k,N} = \infty,\, \sigma_{k,N-1} < \infty \,|\, \mathcal{F}_{\rho_2}) = \mathbb{P}(\sigma_{k,N} = \infty \,|\, \mathcal{F}_{\rho_2}) = 1 - \frac{\widetilde{V}_k(\rho_2)}{N},$$

from which the result follows.

Now expressions for $\mathbb{P}(\sigma_{k,N} < \infty)$ and $\mathbb{P}(0 < \sigma_{k,N} < \infty)$ are derived which will be used in the argument. They follow from the martingale property of $\widetilde{V}_k$:

$$\mathbb{E}[\widetilde{V}_k(0)] = \mathbb{E}[\widetilde{V}_k(\sigma_{k,N})] = N\mathbb{P}(0 < \sigma_{k,N} < \infty) + \mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)\ge N\}}],$$

so that

$$\mathbb{P}(0 < \sigma_{k,N} < \infty) = \frac{1}{N}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)<N\}}]. \tag{33}$$

From this,

$$\mathbb{P}(\sigma_{k,N} < \infty) = \frac{1}{N}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)<N\}}] + \mathbb{P}(\widetilde{V}_k(0) \geq N). \tag{34}$$

Itô's formula may be applied to $M_{s,t}(x)^2 Y(s)$. For $0 \leq s \leq t$,

$$\partial_s M_{s,t}(x) = \int p(t-s; x-y) g_k(s,y)\, W(dy, ds),$$

$$\partial_s \widetilde{V}_k(s) = \int g_k(s,y)\, W(dy, ds),$$

$$\partial_s Y(s) = \begin{cases} \frac{\partial_s \widetilde{V}_k(s)}{N(N-1)}, & 0 \leq s < \sigma_{k,N-1}, \\ -\frac{\partial_s \widetilde{V}_k(s)}{N}, & \sigma_{k,N-1} \leq s < \sigma_{k,N}. \end{cases}$$

Also, recall (32) and that $p(t,x) \leq 1 + \frac{1}{\sqrt{2\pi}\sqrt{t}}$ so that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{S}^1} \widetilde{u}_k(t,x)^2\, dx\Big)\mathbf{1}_{A_N}\Big] \\
&\leq \mathbb{E}\Big[\int_{\mathbb{S}^1} (P_t \widetilde{u}_k(0,x))^2\, dx\, \mathbf{1}_{A_N}\Big] + \int_0^t \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} p(t-s, x-y)^2 \mathbb{E}[Y(s) g_k(s,y)^2]\, dy\, dx\, ds \\
&\quad + 2\int_0^t \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} p(t-s, x-y) \mathbb{E}\Big[M_{s,t}(x) g_k(s,y)^2 \Big(\frac{\mathbf{1}_{[0,\sigma_{k,N-1})}(s)}{N(N-1)} - \frac{\mathbf{1}_{[\sigma_{k,N-1},\sigma_{k,N})}(s)}{N}\Big)\Big]\, ds\, dy\, dx \\
&\leq \mathbb{E}\Big[\int_{\mathbb{S}^1} (P_t \widetilde{u}_k(0,x))^2\, dx\, \mathbf{1}_{A_N}\Big] + \frac{1}{N}\mathbb{E}\Big[\int_0^t \Big(3 + \frac{1}{2\sqrt{\pi(t-s)}}\Big)\mathbf{1}_{[0,\sigma_{k,N-1}]}(s)\, d\langle \widetilde{V}_k, \widetilde{V}_k\rangle(s)\Big] \\
&\quad + \mathbb{E}\Big[\int_0^t \Big(3 + \frac{1}{2\sqrt{\pi(t-s)}}\Big)\mathbf{1}_{[\sigma_{k,N-1},\sigma_{k,N}]}(s)\, d\langle \widetilde{V}_k, \widetilde{V}_k\rangle(s)\Big] \\
&\quad + \frac{2}{N}\mathbb{E}\Big[\int_0^t \Big(1 + \frac{1}{\sqrt{2\pi}\sqrt{t-s}}\Big)\mathbf{1}_{[0,\sigma_{k,N-1}]}(s)\, d\langle \widetilde{V}_k, \widetilde{V}_k\rangle(s)\Big]
\end{aligned}$$

where $Y(s) \leq 1/N$ on $0 \leq s \leq \sigma_{k,N-1}$ (since $\widetilde{V}_k(s) \leq N-1$ on this interval) and $Y(s) \leq 1$ for $\sigma_{k,N-1} \leq s < \sigma_{k,N}$. The last inequality was obtained by $\int_{\mathcal{S}^1} M_{s,t}(x)\, dx = \widetilde{V}_k(s) \leq N-1$ for $s < \sigma_{k,N-1}$ and using $p(t-s,z) \leq 1 + \frac{1}{\sqrt{2\pi}\sqrt{t-s}}$. Also, note $\frac{d}{ds}\langle \widetilde{V}_k, \widetilde{V}_k\rangle(s) = \int_{\mathbb{S}^1} g_k(s,y)^2\, dy$.

Easy calculus gives $\int_s^\infty \frac{1}{(1+t)^2}\frac{1}{\sqrt{t-s}}\, dt \leq 2$, so that for all $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1} \widetilde{u}_k(t,x)^2\, dx\, dt\Big)\mathbf{1}_{A_N}\Big] \\
&\qquad \leq \mathbb{E}\Big[\int_{\mathbb{S}^1} (P_\epsilon \widetilde{u}_k(0,x))^2\, dx\Big] \\
&\qquad\quad + \frac{15}{N}\mathbb{E}[\langle \widetilde{V}_k, \widetilde{V}_k\rangle(\sigma_{k,N-1})] + 7\mathbb{E}[\langle \widetilde{V}_k, \widetilde{V}_k\rangle(\sigma_{k,N}) - \langle \widetilde{V}_k, \widetilde{V}_k\rangle(\sigma_{k,N-1})] \\
&\qquad \leq C_\epsilon
\end{aligned}$$

for some constant $C_\epsilon < \infty$ which does not depend on N. The justification is as follows: for $s > \epsilon$, $\int_{\mathbb{S}^1}(P_\epsilon \widetilde{u}_k(0,x))^2\,dx \geq \int_{\mathbb{S}^1}(P_s\widetilde{u}_k(0,x))^2\,dx$. Secondly, $P_\epsilon\widetilde{u}_k(0,x) \leq \big(1+\frac{1}{\sqrt{2\pi\epsilon}}\big)U(0)$ and (by hypothesis) $\mathbb{E}[U(0)^2] < \infty$.

Using (33) and (34), we get

$$\begin{aligned}
&\mathbb{E}[\langle \widetilde{V}_k,\widetilde{V}_k\rangle(\sigma_{k,N}) - \langle \widetilde{V}_k,\widetilde{V}_k\rangle(\sigma_{k,N-1})] = \mathbb{E}[\widetilde{V}_k^2(\sigma_{k,N}) - \widetilde{V}_k^2(\sigma_{k,N-1})]\\
&= N^2\mathbb{P}(0<\sigma_{k,N}<\infty) - (N-1)^2\mathbb{P}(0<\sigma_{k,N-1}<\infty) - \mathbb{E}[\widetilde{V}_k(0)^2\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&= N\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)<N\}}] - (N-1)\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)<N-1\}}] - \mathbb{E}[\widetilde{V}_k(0)^2\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&= N\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N}>0\}}] - (N-1)\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}>0\}}] - \mathbb{E}[\widetilde{V}_k(0)^2\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&= \mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}>0\}}] + N\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N}>0\}\setminus\{\sigma_{k,N-1}>0\}}] - \mathbb{E}[\widetilde{V}_k(0)^2\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&= \mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}>0\}}] + N\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&\quad - \mathbb{E}[\widetilde{V}_k(0)^2\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}]\\
&= \mathbb{E}[(N-\widetilde{V}_k(0))\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}}] + \mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\sigma_{k,N-1}>0\}}]\\
&\leq \mathbb{E}[\widetilde{V}_k(0)(\mathbf{1}_{\{\sigma_{k,N-1}=0\}\setminus\{\sigma_{k,N}=0\}} + \mathbf{1}_{\{\sigma_{k,N-1}>0\}})]\\
&\leq \mathbb{E}[\widetilde{V}_k(0)] \leq \mathbb{E}[U(0)].
\end{aligned}$$

To simplify notation, let $X = \int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1}\widetilde{u}_k(t,x)^2\,dx\,dt$ and let $c_N = \mathbb{E}[X\mathbf{1}_{A_N}]$ (so that, for each $N \in \mathbb{N}$, $c_N \leq C_\epsilon$). Note

$$c_N = \mathbb{E}[X \mid \sigma_{k,N} = \infty,\ \sigma_{k,N-1} < \infty]\mathbb{P}(\sigma_{k,N} = \infty,\ \sigma_{k,N-1} < \infty),$$

so that, by a straightforward application of Jensen's inequality, for $\alpha \in (0,2)$,

$$c_N^{\alpha/2} \geq \mathbb{E}[X^{\alpha/2} \mid \sigma_{k,N} = \infty, \sigma_{k,N-1} < \infty]\mathbb{P}^{\alpha/2}(\sigma_{k,N} = \infty, \sigma_{k,N-1} < \infty),$$

giving

$$\mathbb{E}[X^{\alpha/2}\mathbf{1}_{\{\sigma_{k,N}=\infty,\,\sigma_{k,N-1}<\infty\}}] \leq c_N^{\alpha/2}\mathbb{P}^{1-\alpha/2}(\sigma_{k,N} = \infty,\ \sigma_{k,N-1} < \infty).$$

Now, $\mathbb{P}(A_N) = \mathbb{P}(\sigma_{k,N-1} < \infty) - \mathbb{P}(\sigma_{k,N} < \infty)$ can be estimated quite easily using (34):

$$\begin{aligned}
&\mathbb{P}(A_N) = \mathbb{P}(\sigma_{k,N-1} < \infty) - \mathbb{P}(\sigma_{k,N} < \infty)\\
&= \mathbb{E}\left[\widetilde{V}_k(0)\left(\frac{1}{N-1}\mathbf{1}_{\{\widetilde{V}_k(0)<N-1\}} - \frac{1}{N}\mathbf{1}_{\{\widetilde{V}_k(0)<N\}}\right)\right] + \mathbb{P}(N-1 \leq \widetilde{V}_k(0) < N)\\
&= \frac{1}{N(N-1)}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)<N-1\}}] - \frac{1}{N}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{N-1\leq\widetilde{V}_k(0)<N\}}] + \mathbb{P}(N-1\leq\widetilde{V}_k(0)<N)\\
&= \frac{\mathbb{E}[\widetilde{V}_k(0)]}{N(N-1)} - \frac{1}{N-1}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)\geq N-1\}}] + \frac{1}{N}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)\geq N\}}] + \mathbb{P}(N-1\leq\widetilde{V}_k(0)<N)\\
&= \frac{\mathbb{E}[\widetilde{V}_k(0)]}{N(N-1)} - \frac{\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{\widetilde{V}_k(0)\geq N\}}]}{N(N-1)} - \frac{1}{N-1}\mathbb{E}[\widetilde{V}_k(0)\mathbf{1}_{\{N-1\leq\widetilde{V}_k(0)<N\}}] + \mathbb{P}(N-1\leq\widetilde{V}_k(0)<N)\\
&\leq \frac{\mathbb{E}[\widetilde{V}_k(0)]}{N(N-1)},
\end{aligned}$$

so that

$$\mathbb{P}(A_N) \le 1 \wedge \frac{\mathbb{E}[U(0)]}{N(N-1)}.$$

Therefore, taking $N_1 = \lceil 1 + \sqrt{\mathbb{E}[U(0)]}\rceil$ (rounding up to the nearest larger integer) yields

$$\mathbb{E}[X^{\alpha/2}] \le \sum_{N=1}^{N_1} c_N^{\alpha/2} + \mathbb{E}[U(0)]^{1-\alpha/2} \sum_{N=N_1+1}^{\infty} \frac{c_N^{\alpha/2}}{(N-1)^{2-\alpha}},$$

and since $c_N < C_\epsilon < \infty$ for each $N \in \mathbb{N}$ and $\epsilon > 0$, there is a constant $C(\alpha,\epsilon) < \infty$ for $\alpha \in (0,1)$ and $\epsilon > 0$ such that

$$\mathbb{E}[X^{\alpha/2}] \le C(\alpha,\epsilon) < \infty.$$

Hence, for all $\alpha \in (0,1)$ and $\epsilon > 0$,

$$\mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1} \widetilde{u}_k(t,x)^2\,dx\,dt\Big)^{\alpha/2}\Big] \le C(\alpha,\epsilon) < \infty.$$

This, together with (29), the fact that $d_k \le \widetilde{u}_k$ and Hölder's inequality gives, for $0 < \beta < \alpha < 1$ and all $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2} D_k(t)\,dt\Big)^{\beta/2}\Big] \\
&\qquad\le \mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1} \widetilde{u}_k(t,x)\mathbf{1}_{\{d_k(t,x)>0\}}\,dx\,dt\Big)^{\beta/2}\Big] \\
&\qquad\le \mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1} \widetilde{u}_k(t,x)^2\,dx\,dt\Big)^{\beta/2}\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}\int_{\mathbb{S}^1} \mathbf{1}_{\{d_k(t,x)>0\}}\,dx\,dt\Big)^{\beta/2}\Big] \qquad (35)\\
&\qquad\le C(\alpha,\epsilon)^{\beta/\alpha}\mathbb{E}\Big[\Big(\sup_t \int_{\mathbb{S}^1} \mathbf{1}_{\{d_k(t,x)>0\}}\,dx\Big)^{\alpha\beta/(2(\alpha-\beta))}\Big]^{(\alpha-\beta)/\alpha} \xrightarrow{k\to\infty} 0. \qquad (36)
\end{aligned}$$

This shows that $D_k \xrightarrow{k\to\infty} 0$ and gives a sense in which this convergence holds; from this and using (28), for $0 < \beta < 1$ and $\epsilon > 0$ we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2}|\widetilde{V}_k(t) - U(t)|\,dt\Big)^{\beta/2}\Big] \\
&\quad\le \mathbb{E}\Big[\Big(\int_0^\infty \frac{1}{(1+t)^2}|\widehat{V}_k(t) - U(t)|\,dt\Big)^{\beta/2}\Big] + \mathbb{E}\Big[\Big(\int_\epsilon^\infty \frac{1}{(1+t)^2} D_k(t)\,dt\Big)^{\beta/2}\Big] \\
&\quad\xrightarrow{k\to\infty} 0.
\end{aligned}$$

Using dominated convergence and the fact that $\sup_k \mathbb{E}[\sup_t \widetilde{V}_k(t)^\alpha] < \infty$ for

$\alpha \in (0,1)$ and $0 \le D_k(t) \le \widetilde{V}_k(t)$, it is straightforward that

$$\mathbb{E}\Big[\Big(\int_0^\infty \frac{1}{(1+t)^2}|\widetilde{V}_k(t) - U(t)|\,dt\Big)^{\alpha/2}\Big] \xrightarrow{k\to\infty} 0 \quad \forall \alpha \in (0,1). \tag{37}$$

STEP 2: *Showing that the weak limit is a local martingale.*

LEMMA 19. *The limit U may be chosen such that $U(t)$ is a local martingale.*

Proof. From (37), for any $N \ge 1$,

$$\int_0^\infty \frac{1}{(1+t)^2}\mathbb{E}[|\widetilde{V}_k(t) - U(t)| \wedge N]\,dt \xrightarrow{k\to\infty} 0. \tag{38}$$

A standard fact about L^1 convergence is that it implies that there is a subsequence which gives convergence pointwise a.s. and that this convergence is almost uniform. Indeed, from the construction, there is a subsequence $(k_i)_{i\in\mathbb{N}}$ such that $|\widetilde{V}_{k_i}(t) - U(t)| \xrightarrow{i\to\infty} 0$ pointwise $\mathbb{P} \times \frac{dt}{(1+t)^2}$ a.s. and convergence is almost uniform and also $D_{k_i}(t) \xrightarrow{i\to\infty} 0$ $\mathbb{P}$-a.s., with convergence almost uniform, hence $|\widehat{V}_{k_i} - U| \xrightarrow{i\to\infty} 0$ pointwise with convergence almost uniform. Let

$$U(t) = \liminf_{i\to\infty} \widetilde{V}_{k_i}(t). \tag{39}$$

Then U satisfies (37) (with k_i relabelled as k): for each $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^\infty \frac{1}{(1+t)^2}|\widetilde{V}_{k_i}(t) - U(t)|\,dt\Big)^{\alpha/2}\Big] \\
&\quad\le \epsilon^{\alpha/2} + \mathbb{E}\Big[\Big(\sup_t \widetilde{V}_{k_i}(t)^{\alpha/2} + \sup_t U(t)^{\alpha/2}\Big)\Big(\int_0^\infty \frac{1}{(1+t)^2}\mathbf{1}_{\{|\widetilde{V}_{k_i}(t)-U(t)|>\epsilon\}}\,dt\Big)^{\alpha/2}\Big] \\
&\quad\le \epsilon^{\alpha/2} + \Big(\mathbb{E}\Big[\sup_t \widetilde{V}_{k_i}(t)^{\alpha}\Big]^{1/2} + \mathbb{E}\Big[\sup_t U(t)^{\alpha}\Big]^{1/2}\Big)\mathbb{E}\Big[\int_0^\infty \frac{1}{(1+t)^2}\mathbf{1}_{\{|\widetilde{V}_{k_i}(t)-U(t)|>\epsilon\}}\,dt\Big]^{\alpha/2} \\
&\quad\xrightarrow{i\to\infty} \epsilon^{\alpha/2}
\end{aligned}$$

using Hölder's inequality, together with the universal constant $\widetilde{K}(\alpha)$ such that $\sup_i \mathbb{E}[\sup_t \widetilde{V}_k(t)^\alpha] < \widetilde{K}(\alpha) < +\infty$ for all $\alpha \in (0,1)$. Also,

$$\widetilde{V}_{k_i}(t) \le \sup_s \widetilde{V}_{k_i}(s),$$

so $\liminf_i \widetilde{V}_{k_i}(t) \le \liminf_i \sup_s \widetilde{V}_{k_i}(s)$ so that

$$\sup_t \liminf_i \widetilde{V}_{k_k}(t) \le \liminf_i \sup_t \widetilde{V}_{k_i}(t),$$

and hence (clearly by Fatou)

$$\mathbb{E}\Big[\sup_t \liminf_i \widetilde{V}_{k_i}(t)^\alpha\Big] \le \liminf_i \mathbb{E}\Big[\sup_t \widetilde{V}_{k_i}(t)^\alpha\Big].$$

Since this is true for all $\epsilon > 0$, the statement holds: U satisfying (39) also satisfies (37) and (28).

From this point onwards, the collection of Mazur combinations $\widetilde{V}_{k_i} := \sum_j \alpha_{k_i,j} U^{(n_j)}$ for $i \ge 1$ is considered and is relabelled as $\widetilde{V}_k$.

Recall that $\sigma_{k,N} = \inf\{t : \widetilde{V}_k(t) \ge N\}$ and let $\widetilde{\sigma}_N = \limsup_{k\to\infty} \sigma_{k,N}$. From the definition,

$$\{\widetilde{\sigma}_N \le t\} = \bigcup_{j=1}^{\infty} \bigcap_{k\ge j} \{\sigma_{k,N} \le t\} \tag{40}$$

and

$$\liminf_{k\to\infty} \widetilde{V}_k(t \wedge \sigma_{k,N}) = \begin{cases} U(t), & \{\widetilde{\sigma}_N > t\}, \\ N, & \{\widetilde{\sigma}_N \le t\}. \end{cases} \tag{41}$$

To establish that U is a local martingale, it suffices to show that, for any $A \in \mathcal{F}_s$ with $s < t$,

$$\mathbb{E}[(U(t)\mathbf{1}_{\{t<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{t\ge\widetilde{\sigma}_N\}})\mathbf{1}_A] = \mathbb{E}[(U(s)\mathbf{1}_{\{s<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{s\ge\widetilde{\sigma}_N\}})\mathbf{1}_A].$$

But, for $s < t$ and $A \in \mathcal{F}_s$,

$$\begin{aligned} &\mathbb{E}\Big[\liminf_{k\to\infty} \widetilde{V}_k(\sigma_{k,N} \wedge t)\mathbf{1}_A\Big] \\ &\quad = \liminf_{k\to\infty} \mathbb{E}[\widetilde{V}_k(\sigma_{k,N} \wedge t)\mathbf{1}_A] = \liminf_{k\to\infty} \mathbb{E}[\widetilde{V}_k(\sigma_{k,N} \wedge s)\mathbf{1}_A] \\ &\quad = \mathbb{E}\Big[\liminf_{k\to\infty} \widetilde{V}_k(\sigma_{k,N} \wedge s)\mathbf{1}_A\Big] = \mathbb{E}[(U(s)\mathbf{1}_{\{s<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{s\ge\widetilde{\sigma}_N\}})\mathbf{1}_A]. \end{aligned}$$

The equalities where expectation and limits are exchanged hold by bounded convergence (for each k, $\widetilde{V}_k(\sigma_{k,N} \wedge t) \le N$) and by plugging in (41).

At the same time,

$$\mathbb{E}\Big[\liminf_{k\to\infty} \widetilde{V}_k(\sigma_{k,N} \wedge t)\mathbf{1}_A\Big] = \mathbb{E}[(U(t)\mathbf{1}_{\{t<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{t\ge\widetilde{\sigma}_N\}})\mathbf{1}_A].$$

From this, for $N < \infty$, the process $\widetilde{U}_N$ defined by $\widetilde{U}_N(t) := U(t)\mathbf{1}_{\{t<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{t\ge\widetilde{\sigma}_N\}}$ is a martingale. For $t < \widetilde{\sigma}_N$, clearly $U(t) < N$.

By construction, $\widetilde{\sigma}_N$ is a *predictable* stopping time, as seen in Lemma 20 below. From this,

$$\mathbb{E}[\mathbf{1}_{\{\widetilde{\sigma}_N<\infty\}}(\widetilde{U}_N(\widetilde{\sigma}_N) - \widetilde{U}_N(\widetilde{\sigma}_N-))] = 0$$

because $\mathcal{N}$ defined by

$$\mathcal{N}(t) := \int_0^t \mathbf{1}_{\{\widetilde{\sigma}_N\}}(s)\, dU_N(s) = (\widetilde{U}_N(\widetilde{\sigma}_N) - \widetilde{U}_N(\widetilde{\sigma}_N-))\mathbf{1}_{[\widetilde{\sigma}_N,\infty)}(t)$$

is a bounded martingale (by virtue of it being a stochastic integral where $\mathbf{1}_{\{\widetilde{\sigma}_N\}}$ is adapted) and hence

$$\mathbb{E}[\mathbf{1}_{\{\widetilde{\sigma}_N<\infty\}}(\widetilde{U}_N(\widetilde{\sigma}_N) - \widetilde{U}_N(\widetilde{\sigma}_N-))] = \mathbb{E}[\mathcal{N}(\infty)] = \mathbb{E}[\mathcal{N}(0)] = 0.$$

Hence, on $\{\widetilde{\sigma}_N < \infty\}$, $\lim_{t\uparrow\widetilde{\sigma}_N} U(t) = N$.

On the set $\{\widetilde{\sigma}_N < \infty\}$, U satisfies $\lim_{t\to\widetilde{\sigma}_N} U(t) = N$, hence $U(t)\mathbf{1}_{\{t<\widetilde{\sigma}_N\}} + N\mathbf{1}_{\{t\geq\widetilde{\sigma}_N\}} = U(t\wedge\widetilde{\sigma}_N)$. Thus U is a local martingale and

$$\widetilde{\sigma}_N = \sigma_N = \inf\{t : U(t) \geq N\}. \ \blacksquare$$

Lemma 20. *The stopping times $\widetilde{\sigma}_N$ defined above are* predictable *with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ from Definition* 7.

Discussion. This lemma and surrounding issues are well known to those with understanding of the Vancouver school, as exemplified by John Walsh; indeed, this is necessary background for undertaking research in the field. One important feature of the St Flour course [18] of 1984 by John Walsh is the great level of care which he gives to results about the Wiener sheet, which establish that key continuity properties of the filtration for the Wiener process in $\mathbb{R}^d$ for d finite carry over to the Wiener sheet filtration $(\mathcal{F}_t)_{t\geq 0}$ of Definition 7.

One of these properties is that any stopping time with respect to this filtration (the completed filtration generated by the Wiener sheet) is predictable, which follows in exactly the same way as the well known result for stopping times with respect to the filtration generated by the Wiener process in $\mathbb{R}^d$. The result for the Wiener process with state space $\mathbb{R}^d$ is a direct consequence of the martingale representation theorem, which carries over to the Wiener sheet. This will be discussed later.

This is well known to all researchers in the field. At the same time, it is *necessary* background and of such importance that it is useful to give an outline here.

Before discussing the martingale representation theorem, another proof that stopping times with respect to the Wiener sheet filtration are predictable will be given.

Aside. That all stopping times with respect to the completed filtration generated by the Wiener process in $\mathbb{R}^d$ are *predictable* is standard and is part of any serious introduction to Brownian motion and continuous martingales. It is a direct consequence of the Martingale Representation Theorem: from that theorem, martingales with respect to this filtration (denoted by $(\mathcal{G}_t)_{t\geq 0}$) are continuous. Hence, consider a stopping time τ with respect to $(\mathcal{G}_t)_{t\geq 0}$ and the martingale $M_t = \mathbb{E}[\mathbf{1}_{\{\tau<\infty\}}|\mathcal{G}_t]$. Then $\tau_N := \inf\{t : M_t \geq 1 - 1/N\} \wedge N$ is a sequence of stopping times which (by continuity of M) satisfies $\tau_N < \tau$

for $\tau > 0$; it is a non-decreasing sequence of stopping times which satisfies $\lim_{N\to\infty} \tau_N = \tau$, hence τ is predictable.

The same holds for the filtration generated by the Wiener sheet and it is clear that the proof for the Wiener process carries over to the Wiener sheet with very little modification. This will be stated as Theorem 21 and an outline of the proof given.

Firstly, Lemma 20 will be established by showing that all stopping times with respect to the Wiener sheet filtration are predictable. This simply requires an appeal to the standard properties of the Wiener sheet which are presented in detail by Walsh [18], together with a theorem by Blumenthal and Meyer, which is stated and proved in Kallenberg [2].

For a sequence $(\tau_k)_{k\geq 0}$ of stopping times with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$, it does not necessarily hold that $\tau := \limsup_k \tau_k$ is a stopping time with respect to $(\mathcal{G}_t)_{t\geq 0}$; such a statement is only true in general if the filtration $(\mathcal{G}_t)_{t\geq 0}$ is right continuous. Let $\mathcal{G}_t^+ = \bigwedge_{s>t} \mathcal{G}_s$. Then, for a sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{G}_t)_{t\geq 0}$ stopping times, it is standard (and easy to show) that $\limsup_n \tau_n =: \tau$ is a stopping time with respect to $(\mathcal{G}_t^+)_{t\geq 0}$. If, in addition, the filtration is generated by a *continuous* Markov process, the stopping time τ constructed in this way is predictable. A careful reading of Lemma 13 shows that martingale constructions leading to counterexamples are prohibited.

Proof of Lemma 20. By construction, the time $\widetilde{\sigma}_N$ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ (Definition 7) since this filtration is right continuous (this is discussed by Walsh [18]) and $\widetilde{\sigma}_N = \limsup_k \sigma_{k,N}$ where each $\sigma_{k,N}$ is a stopping time. By construction, therefore, $\widetilde{\sigma}_N$ is an $(\mathcal{F}_t)_{t\geq 0}$ stopping time.

A note on the construction. Now the question of whether, for each k and N, $\sigma_{k,N}$ is indeed an $(\mathcal{F}_t)_{t\geq 0}$ stopping time, is considered, and this again goes back to the care and attention to such important details found in the treatment by John Walsh [18]. For each $n < \infty$, $(u \wedge n)^\gamma$ is Lipschitz and bounded in u. The unique solution for (17) (with n finite) may be constructed along the lines of Walsh's equation (3.12) on p. 315. His $f(u,t)$ in equation (3.5) is the $(u \wedge n)^\gamma$ here, which clearly satisfies the Lipschitz condition which he requires. The additional $-V$ in the equation of Walsh alters the Green's function by an e^{-t}, but does not affect the technique or convergence result, nor do the other differences. If one takes $L = 1$ in Walsh, his reflecting boundary conditions mean that (modulo the e^{-t} resulting from his $-V$ term) the Green's functions are similar.

The solution $u^{(n)}$ to (17) is therefore unique and, crucially, is the limit of a sequence of functions, each of which is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, a property which holds in the limit (sequence of approximating functions for fixed n). Furthermore, similarly to Corollary 3.4 of Walsh (p. 318), since $u^{(n)}(0,x) := u(0,x) \wedge n$ is bounded, the corollary may be applied to establish that $u^{(n)}(t,x)$ has a modification that is Hölder continuous in both t and x with exponent $1/4 - \epsilon$ for all $\epsilon > 0$. Hence for each $k < \infty$, since $f(k)$ in the definition of $\{\alpha_{kj} : j = k, \ldots, f(k)\}$ is finite, $\widetilde{V}_k(t) := \sum_{j=k}^{f(k)} \alpha_{kj} \int_{\mathbb{S}^1} u^{(n_j)}(t,x)\,dx$ is a continuous martingale adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Hence $\widetilde{\sigma}_{k,N}$ is indeed an $(\mathcal{F}_t)_{t\geq 0}$ stopping time.

The result that all stopping times with respect to this filtration are predictable follows directly from a theorem of Blumenthal and Meyer, which is Theorem 25.20 in Kallenberg [2, p. 501].

Theorem of Blumenthal and Meyer (from Kallenberg [2]). *Let X be a canonical Feller process with arbitrary distribution and fix any optional time τ* (Kallenberg's definition of 'optional time' corresponds to 'stopping time' here). *Then these conditions are equivalent:*

(i) *τ is predictable,*
(ii) *τ is accessible,*
(iii) *$X_\tau = X_{\tau-}$ on $\{\tau < \infty\}$.*

It is therefore necessary and sufficient to show that the Wiener sheet is a canonical Feller process and that it is *continuous*.

Firstly, the necessary *continuity* properties of the Wiener sheet are standard and fully dealt with by Walsh [18]; his Proposition 1.4 on p. 275 is applicable here. Walsh gives the Kolmogorov criterion (Corollary 1.2 on p. 273 and Corollary 1.3 on p. 274). The R_1 of Walsh is $[0,1]^n$. In the setting here, $n = 2$ and the Walsh t is $(s,x) \in [0,1]^2$ here. His W_t is $W([0,x] \times [0,s])$ here. By rescaling, his result can easily be extended to deal with $(s,x) \in [0,T] \times [0,1]$ for any finite $T \in \mathbb{R}_+$. Then, in t and x for $(t,x) \in [0,T] \times [0,1]$, $W([0,t] \times [0,x])$ has a continuous modification with modulus of continuity

$$\Delta(\delta) \leq C\rho(\delta) + Y\sqrt{\delta}$$

for a constant $C < 32\sqrt{2}$, Y a well defined non-negative random variable satisfying $\mathbb{E}[e^{Y^2/(16T)}] < \infty$ and $\rho(\delta)$ a continuous function which satisfies:

$$\frac{\rho(|h|)}{\sqrt{|h| \log(1/|h|)}} \to 1.$$

Here $|h| = \sqrt{h_1^2 + h_2^2}$.

Here $(\mathcal{F}_t)_{t\geq 0}$ is the completed filtration generated by the initial condition u_0 and the Wiener sheet $W([0,x] \times [0,s])$, $x \in [0,1]$, $0 \leq s \leq t$, which are independent of each other. Let $X_t = W([0,\cdot] \times [0,t])$. Let $\mathcal{S}$ denote the space of *continuous* functions $f : [0,1] \to \mathbb{R}$, endowed with the topology of uniform convergence and metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$; then $\mathcal{S}$ is a locally compact separable metric space.

Firstly, *continuity* of X_t with this norm follows directly from the modulus of continuity provided by Walsh (above); the process $\{X(t) : t \in \mathbb{R}_+\}$ defined has a continuous modification and, for each $T \in \mathbb{R}_+$,

$$\sup_{0\le t\le T} \limsup_{s\searrow 0} \frac{d(X_{t-s\vee 0}, X_t)}{\sqrt{s\log(1/s)}} \le C < \infty.$$

Hence statement (iii) of the Blumenthal–Meyer theorem as stated in Kallenberg is satisfied for any $(\mathcal{F}_t)_{t\ge 0}$ stopping time τ. Therefore any $(\mathcal{F}_t)_{t\ge 0}$ stopping time is predictable, since the theorem states that any time satisfying statement (iii) also satisfies (i), provided that X is a canonical Feller process in the sense of Kallenberg.

While X clearly satisfies the Markov property at an informal level, some details need to be added to show that $X(t)$ defined in this way is a well defined Feller process in the sense used by Kallenberg, to which the theorem can be applied.

The conditions stated on p. 369 of Kallenberg [2] are:

(F1) $T_t C_0(S) \subset C_0(S)$ (the continuous functions vanishing at ∞) for $t \ge 0$.
(F2) $T_t f(s) \to f(s)$ as $t \to 0$ for $f \in C_0(S)$ and $s \in S$.
(F3) For $f \in C_0(S)$, $T_t f \to f$ as $t \to 0$.

These conditions should be satisfied by

$$T_t f(s) := \mathbb{E}[f(s + W([0,t] \times [0,\cdot])], \qquad f \in C_0(S).$$

For $t > 0$, T_t is clearly a contraction operator on $C_0(S)$ endowed with the norm $\|f\| = \sup_s |f(s)|$. Furthermore, clearly $T_t T_s = T_{t+s}$. It has been established that $\sup_{x\in[0,1]} |W([0,t] \times [0,x))| \xrightarrow{t\to 0} 0$ almost surely. Therefore, using continuity of f, the fact that $|f(s)| \xrightarrow{|s|\to\infty} 0$, boundedness of f, together with some straightforward analysis, it is clear that for $f \in C_0(S)$, conditions (F1)–(F3) hold.

Finally, it is also clear that $\{X(t) : t \ge 0\}$ (trivially) satisfies the shift operator condition stated on p. 380 of Kallenberg [2] (by elementary properties of the Wiener sheet) and hence is a *canonical Feller process* according to the definition given there.

All conditions of the Blumenthal–Meyer theorem are therefore satisfied.

Now Lemma 20 follows; there are no strange and startling surprises when one moves from the filtration generated by a Wiener process in $\mathbb{R}^d$ to a filtration generated by the Wiener sheet: stopping times with respect to the completed filtration generated by the Wiener sheet are indeed predictable. ■

Continuity of martingales with respect to $\mathcal{F}_t$. Of course, having established that $\widetilde{U}_N$ is a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, it follows automatically that it is a *continuous* martingale, in just the same way as martingales with respect to the filtration generated by finite-dimensional Brownian motion are continuous martingales. The mantra 'stopping times with respect to filtrations generated by continuous Markov processes are predictable' is well known to a wider audience and continuity of martingales, based on this, is an easy one-liner, but continuity of martingales with respect to the Wiener sheet filtration can be established more directly, using the arguments for the standard Wiener process filtration found in Revuz and Yor [14, Ch. 5, Sect. 3].

THEOREM 21. *All local martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ from Definition* 7 *are* continuous *martingales. Any local martingale M with respect to this filtration has a representation*

$$M(t) = M(0) + \int_0^t \int_{\mathbb{S}^1} \psi(s,x)\, W(dx,ds)$$

where ψ is locally in $L^2(\mathbb{R}_+ \times \mathbb{S}^1)$.

Proof. This follows in exactly the same way as in the standard approach for Brownian motion, found (for example) in Revuz and Yor [14, Ch. 5, Sect. 3]. The differences are very minor and are as follows. Let $\mathcal{I}$ denote the set of step functions with compact support in $\mathbb{R}_+ \times \mathbb{S}^1$, that is,

$$f(t,x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_{jk} \mathbf{1}_{(t_{i-1},t_i]}(t)\mathbf{1}_{(x_{k-1},x_k]}(x).$$

Let $\mathcal{E}^f$ be defined by

$$\mathcal{E}^f(t,x) = \exp\bigg\{\int_0^t\int_0^x f(s,y)\, W(dy,ds) - \frac{1}{2}\int_0^t\int_0^x f(s,y)^2\, dx\, ds\bigg\}.$$

Then Lemma 3.1 of Revuz and Yor [14, Ch. 5, bottom of p. 198] carries over, with only minor modifications to the proof to accommodate the two-parameter process. For any $Y \in L^2(\mathcal{F}_\infty, \mathbb{P})$ which is orthogonal to every $\mathcal{E}^f(\infty, 1)$ consider any finite sequence $0 < t_1 < \cdots < t_n < \infty$ and $0 < x_1 < \cdots < x_m < 1$ with $t_0 = x_0 = 0$ and consider the function

$$\psi((z_{ij})_{1\leq i\leq n,\, 1\leq j\leq m}) = \mathbb{E}\Big[\exp\Big\{\sum_{i,j} z_{ij} W((t_{i-1},t_i],(x_{j-1},x_j])\Big\}\cdot Y\Big]$$

and the argument proceeds in the same way as in [14] to establish that $Y \cdot \mathbb{P}$ is the zero measure and hence the set $\{\mathcal{E}^f_\infty : f \in \mathcal{I}\}$ is total in $L^2(\mathcal{F}_\infty, \mathbb{P})$. From this, the equivalent statement of [14, Proposition 3.2, p. 199] carries over: for any $F \in L^2(\mathcal{F}_\infty, \mathbb{P})$, there exists a unique predictable process ϕ satisfying $\int_0^\infty \int_{\mathbb{S}^1} \phi(s,x)^2\, dx\, ds < \infty$ such that

$$F = \mathbb{E}[F] + \int_0^\infty \int_{\mathbb{S}^1} \phi(s,x)\, W(dx, ds).$$

The proof is virtually the same. Then also [14, Corollary 3.3, p. 200] follows: for the filtration $(\mathcal{F}_t)_{t\geq 0}$, every optional process is predictable.

The punch-line, which is [14, Theorem 3.4, p. 200], carries through with the proof virtually unaltered: Every $(\mathcal{F}_t)_{t\geq 0}$ local martingale M has a version which may be written

$$M(t) = C + \int_0^t \int_{\mathbb{S}^1} \psi(s,x)\, W(dx, ds)$$

where ψ is a predictable process which is locally in $L^2(\mathbb{R}_+ \times \mathbb{S}^1)$. In particular, any $(\mathcal{F}_t)_{t\geq 0}$ martingale has a continuous version. ■

STEP 3: *Showing that the weak limit U satisfies $U(t) \xrightarrow{t\to\infty} 0$ $\mathbb{P}$-a.s.*

LEMMA 22. *$U(t) \to 0$ $\mathbb{P}$-a.s.*

Proof. Recall the definition of g_k given by (30). Then $\widetilde{V}_k(t) = \widetilde{V}_k(0) + \int_0^t \int_{\mathbb{S}^1} g_k(s,x)\, W(dx, ds)$ is a non-negative local martingale with $\widetilde{V}_k(0) \leq U(0)$, and therefore

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} g_k(s,x)^2\, dx\, ds\Big)^{\alpha/2}\Big] \leq K(\alpha)\mathbb{E}[U(0)^\alpha] < \infty, \qquad \forall \alpha \in (0,1),$$

where $K(\alpha)$ is from (10). From (30), Jensen's inequality gives

$$g_k(s,x) = \sum_j \alpha_{kj} v^{(n_j)\gamma}(s,x) \geq \Big(\sum_j \alpha_{kj} v^{(n_j)}(s,x)\Big)^\gamma,$$

and hence (another application of Jensen)

$$\int_{\mathbb{S}^1} g_k(s,x)\, dx \geq \Big(\sum_j \alpha_{kj} \int_{\mathbb{S}^1} v^{(n_j)}(s,x)\, dx\Big)^\gamma = \widehat{V}_k(s)^\gamma$$

(recall equations (23) and (27) which define $\widehat{V}_k$), so that

$$\begin{aligned} K(\alpha)\mathbb{E}[U(0)^\alpha] &\geq \mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} g_k^2(s,x)\, dx\, dt\Big)^{\alpha/2}\Big] \\ &\geq \mathbb{E}\Big[\Big(\int_0^\infty \Big(\int_{\mathbb{S}^1} g_k(s,x)\, dx\Big)^2 ds\Big)^{\alpha/2}\Big] \geq \mathbb{E}\Big[\Big(\int_0^\infty \widehat{V}_k(s)^{2\gamma}\, ds\Big)^{\alpha/2}\Big]. \end{aligned}$$

Recall (28). Using the standard fact that L^1 convergence implies that there is a subsequence which converges pointwise and almost uniformly, there is a subsequence $(k_i)_{i\geq 1}$ such that $\mathbb{P}(d\omega)\times\frac{dt}{(1+t)^2}$-a.s., $\lim_{i\to\infty}\widehat{V}_{k_i}(t)=U(t)$. This subsequence has already been relabelled as $(\widehat{V}_k)_{k\geq 1}$. Then, for each $T<\infty$ and $N<\infty$, using continuity of sample paths of U (which is a continuous local martingale) and the fact that $U(t)=\liminf_k \widetilde{V}_k(t)\geq \liminf_k \widehat{V}_k(t)$ with equality $\mathbb{P}(d\omega)\times\frac{dt}{(1+t)^2}$ almost everywhere, we obtain for all $N<\infty$ and $T<\infty$,

$$\mathbb{E}\Big[\int_0^T\Big|\liminf_{k\to\infty}(\widehat{V}_k(s)^{2\gamma}\wedge N)-(U(s)^{2\gamma}\wedge N)\Big|\,ds\Big]=0,$$

and therefore

$$\int_0^T\liminf_{k\to\infty}(\widehat{V}_k(s)^{2\gamma}\wedge N)\,ds=\int_0^T(U(s)^{2\gamma}\wedge N)\,ds\qquad \mathbb{P}\text{-a.s.},$$

so that for $\alpha\in(0,1)$ and all $T<\infty$ and $N<\infty$,

$$\mathbb{E}\Big[\Big(\int_0^T\liminf_{k\to\infty}(\widehat{V}_k(s)^{2\gamma}\wedge N)\,ds\Big)^{\alpha/2}\Big]=\mathbb{E}\Big[\Big(\int_0^T(U(s)^{2\gamma}\wedge N)\,ds\Big)^{\alpha/2}\Big].$$

Hence

$$\begin{aligned}K(\alpha)\mathbb{E}[U(0)^\alpha]&\geq\liminf_{k\to\infty}\mathbb{E}\Big[\Big(\int_0^\infty\widehat{V}_k(s)^{2\gamma}ds\Big)^{\alpha/2}\Big]\\&\geq\mathbb{E}\Big[\Big(\int_0^\infty\liminf_{k\to\infty}\widehat{V}_k(s)^{2\gamma}\,ds\Big)^{\alpha/2}\Big]\geq\mathbb{E}\Big[\Big(\int_0^T U(s)^{2\gamma}\wedge N\,ds\Big)^{\alpha/2}\Big]\end{aligned}$$

for each T and N. By Fatou's lemma,

$$\lim_{T\to\infty}\lim_{N\to\infty}\mathbb{E}\Big[\Big(\int_0^T U(s)^{2\gamma}\wedge N\,ds\Big)^{\alpha/2}\Big]\geq\mathbb{E}\Big[\Big(\int_0^\infty U(s)^{2\gamma}\,ds\Big)^{\alpha/2}\Big]$$

so that

$$\infty>K(\alpha)\mathbb{E}[U(0)^\alpha]\geq\mathbb{E}\Big[\Big(\int_0^\infty U(s)^{2\gamma}\,ds\Big)^{\alpha/2}\Big]\qquad\forall\alpha\in(0,1).$$

Now U is a local martingale and hence $U(t)\xrightarrow{t\to\infty}U(\infty)$ (a random variable) $\mathbb{P}$-a.s. From this, $U(\infty)=0$ $\mathbb{P}$-a.s., otherwise there is a contradiction. The result follows. ■

Step 4: *Weak to strong.* The next task is to show that when Mazur combinations of $U^{(n_j)}$ converge to a local martingale U, then the original sequence $U^{(n_j)}$ converges to U and that if $U(t)\xrightarrow{t\to\infty}0$ $\mathbb{P}$-a.s. then $\lim_{j\to\infty}\mathbb{E}[\sup_t|U^{(n_j)}(t)-U(t)|^\alpha]=0$ for $\alpha\in(0,1)$.

LEMMA 23. *The sequence* $(U^{(n_j)})_{j\geq 1}$ *of martingales satisfies*

$$\lim_{j\to\infty} \mathbb{E}[\langle U^{(n_j)} - U, U^{(n_j)} - U\rangle^{\alpha/2}(\infty)] = 0 \quad \forall \alpha \in (0,1),$$

and equivalently (*by the Burkholder–Davis–Gundy inequality*),

$$\lim_{j\to\infty} \mathbb{E}\Big[\sup_{t\in\mathbb{R}_+} |U^{(n_j)}(t) - U(t)|^{\alpha}\Big] = 0 \quad \forall \alpha \in (0,1).$$

Proof. The definition of 'weak convergence' introduced implies clearly that for any $A \in \mathcal{F}$ and any stopping time ρ such that $\mathbb{E}[U(\rho)] = \mathbb{E}[U(0)]$,

$$\lim_{j\to\infty} \mathbb{E}[U^{(n_j)}(\rho)\mathbf{1}_A] = \mathbb{E}[U(\rho)\mathbf{1}_A].$$

It has been established that the weak limit U is a local martingale, hence the 'weak convergence' of martingales to a local martingale is equivalent to convergence of a sequence of probability measures over $(\Omega, \mathcal{F})$ to a probability measure over $(\Omega, \mathcal{F})$.

The process $\big(\frac{U^{(n_j)}(t)}{U^{(n_j)}(0)}\big)_{t\geq 0}$ may be considered as the Radon–Nikodym derivative of a probability measure over $(\Omega, \mathcal{F})$, say $\mathbb{Q}^j$. The definition of weak convergence implies that these measures converge to a measure, say $\mathbb{Q}$, over $(\Omega, \mathcal{F})$, with Radon–Nikodym derivative defined by the process $\big(\frac{U(t)}{U(0)}\big)_{t\geq 0}$. In general, convergence of measures does not imply convergence of their Radon–Nikodym derivatives, but in the case of non-negative martingales, where the limit is a non-negative local martingale, the reasoning that leads to Girsanov's theorem can be adapted to show convergence of these Radon–Nikodym derivatives in a relatively straightforward manner.

The notions that are used and the basic results may be found in Revuz and Yor [14, Ch. 8, Sect. 1, pp. 325–333].

The main result is the following. If Λ is a non-negative continuous local martingale satisfying $\Lambda(0) = 1$, let $\widetilde{L}$ denote the continuous local martingale such that

$$\Lambda(t) = \exp\big\{\widetilde{L}(t) - \tfrac{1}{2}\langle \widetilde{L}, \widetilde{L}\rangle(t)\big\} =: \mathcal{E}(\widetilde{L})(t),$$

where $\mathcal{E}$ (as in [14]) denotes an exponential martingale. Let P denote the measure under which Λ is a martingale and let $Q = \mathcal{E}(\widetilde{L}) \cdot P$ (notation taken from Revuz and Yor, which is clear). Then, if M is a continuous P-martingale, $\widetilde{M} := M - \langle M, \widetilde{L}\rangle$ is the corresponding continuous Q-martingale.

The next part is unnecessary, but is inserted for convenience in the computations later; only local martingales driven by U need be considered. Let $\mathcal{G}_t = \sigma(\{U(s) : 0 \leq s \leq t\})$, so that $\mathcal{G}_t \subseteq \mathcal{F}_t$. Let $\widetilde{U}^{(n_j)}(t) = \mathbb{E}[U^{(n_j)}(t) \,|\, \mathcal{G}_t]$. Then it is clear that $\widetilde{U}^{(n_j)}(t)$ is a martingale with respect to $(\mathcal{G}_t)_{t\geq 0}$; to show the martingale property, consider stopping times $s \le t$ and, using $\mathcal{G}_s \subseteq \mathcal{F}_s$,

$$\begin{aligned}\mathbb{E}[\widetilde{U}^{(n_j)}(t) \mid \mathcal{G}_s] &= \mathbb{E}\big[\mathbb{E}[U^{(n_j)}(t) \mid \mathcal{G}_t] \bigm| \mathcal{G}_s\big] = \mathbb{E}[U^{(n_j)}(t) \mid \mathcal{G}_s] \\ &= \mathbb{E}\big[\mathbb{E}[U^{(n_j)}(t) \mid \mathcal{F}_s] \bigm| \mathcal{G}_s\big] = \mathbb{E}[U^{(n_j)}(s) \mid \mathcal{G}_s] = \widetilde{U}^{(n_j)}(s).\end{aligned}$$

By the martingale representation theorem, $\widetilde{U}^{(n_j)}$ has a representation

$$\widetilde{U}^{(n_j)}(t) = U^{(n_j)}(0) + \int_0^t b_j(s)\, dU(s)$$

where $b_j(s)$ is a predictable process with respect to $(\mathcal{G}_s)_{s\ge 0}$. Furthermore, $U^{(n_j)} - \widetilde{U}^{(n_j)} = Z_j$, where Z_j satisfies $\langle Z_j, U\rangle(t) \equiv 0$. Weak convergence $U^{(n_j)} \rightharpoonup U$ implies $\widetilde{U}^{(n_j)} \rightharpoonup U$ and $Z_j \rightharpoonup 0$.

This implies in particular that, for any $N > 1$, $\widetilde{U}^{(n_j)}(\sigma_N) \rightharpoonup U(\sigma_N)$, so that, on $\{\sigma_N = \infty\}$, $\widetilde{U}^{(n_j)}(\infty) \rightharpoonup 0$. Weak convergence to 0 of a non-negative sequence implies strong convergence to 0. Since this is true for all N, we have $\widetilde{U}^{(n_j)}(\infty) \to 0$ (strong convergence to 0).

Now, $\langle U^{(n_j)}, U^{(n_j)}\rangle(\infty)$ has the law of the first hitting time of 0 of a Wiener process (the Dambis–Dubins–Schwartz Brownian motion; see Revuz and Yor [14, Ch. 5, Sect. 1, Theorem 1.6, p. 181]) with initial condition $U^{(n_j)}(0)$. Also, the limiting law of $\langle \widetilde{U}^{(n_j)}, \widetilde{U}^{(n_j)}\rangle(\infty)$ is the law of the first hitting time of 0 of a Wiener process with initial value $U(0)$. Since

$$\langle U^{(n_j)}, U^{(n_j)}\rangle(\infty) = \langle \widetilde{U}^{(n_j)}, \widetilde{U}^{(n_j)}\rangle(\infty) + \langle Z_j, Z_j\rangle(\infty),$$

the limits of the laws of $\langle U^{(n_j)}, U^{(n_j)}\rangle(\infty)$ and $\langle \widetilde{U}^{(n_j)}, \widetilde{U}^{(n_j)}\rangle(\infty)$ are the same and

$$\langle U^{(n_j)}, U^{(n_j)}\rangle(\infty) \ge \langle \widetilde{U}^{(n_j)}, \widetilde{U}^{(n_j)}\rangle(\infty) \quad \forall j \in \mathbb{N},$$

therefore $\langle Z_j, Z_j\rangle(\infty) = \langle U^{(n_j)} - \widetilde{U}^{(n_j)}, U^{(n_j)} - \widetilde{U}^{(n_j)}\rangle(\infty) \xrightarrow{j\to\infty} 0$ so that $U^{(n_j)} - \widetilde{U}^{(n_j)} \to 0$ *strongly.*

Now let L^j and L denote the local martingales such that

$$\begin{aligned}\widetilde{U}^{(n_j)}(t) &= \widetilde{U}^{(n_j)}(0) \exp\big\{L^j(t) - \tfrac{1}{2}\langle L^j, L^j\rangle(t)\big\}, \\ U(t) &= U(0) \exp\big\{L(t) - \tfrac{1}{2}\langle L, L\rangle(t)\big\}.\end{aligned}$$

Let $\mathbb{Q}^j$ denote the measure over $(\Omega, \mathcal{F})$ with Radon–Nikodym derivative with respect to $\mathbb{P}$ defined by the process $\big(\frac{\widetilde{U}^{(n_j)}(t)}{\widetilde{U}^{(n_j)}(0)}\big)_{t\ge 0}$. For a non-negative integrable, $\mathcal{F}_T$-measurable X, consider the sequence of $\mathbb{Q}^j$ martingales $\mathcal{M}^j$ defined by $\mathcal{M}^j_X(t) = \mathbb{E}_{\mathbb{Q}^j}[X \mid \mathcal{F}_t]$. Let $M_X(t) = \mathbb{E}_{\mathbb{P}}[X \mid \mathcal{F}_t]$ denote the corresponding $\mathbb{P}$-martingale. Then, from the Girsanov transformation (by direct application of the computations in Revuz and Yor [14, Ch. 8, Sect. 1, pp. 325–333]),

$$d\mathcal{M}^j_X(t) = dM_X(t) - d\langle M_X, L^j\rangle(t)$$

where $dL^j = \frac{d\widetilde{U}^{(n_j)}}{\widetilde{U}^{(n_j)}}$. Now let L^* denote any $\mathbb{P}$-martingale which is a limit point of the sequence $\sum_j \alpha_{kj} L^j$. Then $\sum_j \alpha_{kj} \mathcal{M}_X^j \to \mathcal{M}_X^*$ along this sequence, where

$$d\mathcal{M}_X^*(t) = dM_X(t) - d\langle M_X, L^*\rangle(t).$$

This holds for all bounded, $\mathcal{F}_T$-measurable X and almost all $t \in [0, T]$, $\mathbb{P}$-a.s. In other words, the family of regular conditional probability distributions $(\mathbb{Q}^j(\cdot \mid \mathcal{F}_t))_{t\in\mathbb{R}}$ has a weak limit which is a family of regular conditional probability distributions $(\mathbb{Q}^*(\cdot \mid \mathcal{F}_t))_{t\in\mathbb{R}}$ such that, for any $t < T < \infty$, the conditional measure, conditioned on $\mathcal{F}_t$, over $\mathcal{F}_T$ has Radon–Nikodym derivative

$$\exp\bigl\{(L^*(T) - L^*(t)) - \tfrac{1}{2}(\langle L^*, L^*\rangle(T) - \langle L^*, L^*\rangle(t))\bigr\}$$

with respect to $\mathbb{P}(\cdot \mid \mathcal{F}_t)$; for each bounded X, $\mathcal{M}_X^*$ is a $\mathbb{Q}^*$-martingale.

Considering $t = 0$, we have $\mathbb{Q}^* = \mathbb{Q}$, hence any limit point L^* of $\sum_k \alpha_{kj} L^j$ is $L^* = L$, where $U(t) = U(0)\exp\bigl\{L(t) - \frac{1}{2}\langle L, L\rangle(t)\bigr\}$.

To add some detail: Let $L^{*k} = \sum_j \alpha_{kj} L^j$; then for each k, L^{*k} is a continuous local martingale. Consider the stopping times $\rho_{N,k} = \inf\{t : \langle L^{*k}, L^{*k}\rangle(t) = N\}$. Then existence of weak limits of $L^{*k}(\cdot \wedge \rho_{N,k})$ is straightforward. The argument above then shows that there is a *unique* limit point, which is $L(\cdot \wedge \rho_N)$, where $\rho_N = \inf\{t : \langle L, L\rangle(\rho_N) \geq N\}$. This holds for all N and hence in the limit. Since the L^{*k} are already convex combinations, L^{*k} converges to L strongly.

Therefore, letting L^j denote the (local) martingale such that

$$\widetilde{U}^{(n_j)}(t) = \widetilde{U}^{(n_j)}(0)\exp\bigl\{L^j(t) - \tfrac{1}{2}\langle L^j, L^j\rangle(t)\bigr\},$$

we have both $\widetilde{U}^{(n_j)} \rightharpoonup U$ and $L^j \rightharpoonup L$.

Now let D^j denote the martingale which defines the Radon–Nikodym derivative $\frac{d\mathbb{Q}^j}{d\mathbb{Q}}$. Following the treatment of Revuz and Yor, we get

$$D^j(t) = \exp\bigl\{(L^j - L)(t) - \tfrac{1}{2}\langle L^j - L, L^j - L\rangle(t)\bigr\} = \mathcal{E}(L^j - L)(t).$$

Let $\widetilde{\mathbb{Q}}^j$ denote the measure over $(\Omega, \mathcal{F})$ whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is given by $\frac{d\widetilde{\mathbb{Q}}^j}{d\mathbb{P}}$. Also, let $\widetilde{\mathbb{Q}}^{j,\alpha}$ denote the measure whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is defined by the martingale $D^{j,\alpha}$:

$$D^{j,\alpha}(t) = \exp\biggl\{\alpha(L^j - L)(t) - \frac{\alpha^2}{2}\langle L^j - L, L^j - L\rangle(t)\biggr\}.$$

If M is a martingale under $\mathbb{P}$ then, by Girsanov, the corresponding $\widetilde{\mathbb{Q}}^{j,\alpha}$-martingale is

$$M - \alpha\langle M, L^j - L\rangle \xrightarrow{j\to\infty} M,$$

where, throughout the argument, $\rightharpoonup$ denotes convergence of Mazur convex combinations, so that $D^{j,\alpha} \rightharpoonup 1$ for each $\alpha \in (0,1]$.

Now, making the expansion

$$D^{j,\alpha}(t) = 1+\alpha(L^j-L)(t)+\frac{\alpha^2}{2}\big((L^j-L)^2(t)-\langle L^j-L, L^j-L\rangle(t)\big)+\sum_{k=3}^{\infty}\alpha^k\mathcal{N}_k(t)$$

where each $\mathcal{N}_k$ is a martingale, since $D^{j,\alpha} \rightharpoonup 1$ for all $\alpha \in (0,1]$, each term in the expansion converges (weakly) to 0. In particular,

$$(L^j - L)^2(t) - \langle L^j - L, L^j - L\rangle(t) \rightharpoonup 0.$$

Now, from the definition of $\widetilde{U}^{(n_j)}$ and L^j, there is an adapted function a_j such that $(L^j - L)(t) = \int_0^t a_j(s)\,dL_s$ for $t \geq 0$ so that $a_j \rightharpoonup 0$ (the Mazur combinations converge to 0: $\lim_{k\to\infty}\sum_j \alpha_{kj}a_j = 0$). Let

$$F(s,r) = \lim_{k\to\infty}\sum_j \alpha_{kj}a_j(s)a_j(r).$$

Then F is non-negative definite (it may be viewed as a covariance operator) and

$$\begin{aligned} 0 &= \lim_{k\to\infty}\sum_j \alpha_{kj}\{(L^j - L)(t) - \langle L^j - L, L^j - L\rangle(t)\} \\ &= 2\int_0^t\int_0^s F(s,r)\,dL(r)\,dL(s). \end{aligned}$$

Hence $\int_0^t F(t,r)\,dL(r) \equiv 0$ for all $t \in \mathbb{R}_+$. Let $M^j = L^j - L$. Then

$$M^j(t)^2 = 2\int_0^t M^j(s) \circ dM^j(s),$$

where $\circ$ denotes Stratonovich integration. Therefore, going back to the definition of Stratonovich integration, $\overline{M(t)^2} := \lim_{k\to\infty}\sum_j \alpha_{kj}M^j(t)^2$ is the limit as mesh tends to zero of

$$\sum_i \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_0^{s_i} F(r,u)\,dL_u\,dL_r + \sum_i \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_0^{s_{i+1}} F(r,u)\,dL_u\,dL_r,$$

which is

$$\begin{aligned} \sum_i \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_0^{r} F(r,u)\,dL_u\,dL_r &- \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_{s_i}^{r} F(r,u)\,dL_u\,dL_r \\ &+ \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_0^{r} F(r,u)\,dL_u\,dL_r + \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_r^{s_{i+1}} F(r,u)\,dL_u\,dL_r, \end{aligned}$$

and using $\int_0^r F(r,u)\,dL_u = 0$, this gives

$$\sum_i -\frac{1}{2}\int_{s_i}^{s_{i+1}}\int_{s_i}^{r} F(r,u)\,dL_u\,dL_r + \frac{1}{2}\int_{s_i}^{s_{i+1}}\int_{s_i}^{r} F(r,u)\,dL_u\,dL_r = 0.$$

Hence $\overline{M(t)^2} = 0$ and since $\overline{M}(t)^2 - \overline{\langle M,M\rangle}(t)$ is a martingale, we have $\overline{\langle M,M\rangle} \equiv 0$ so that $\overline{a^2(t)} \equiv 0$ for all $t \in \mathbb{R}_+$. From this, $a_j \to 0$ strongly, so that $L^j \to L$ strongly. In particular, this implies $\langle L^j - L, L^j - L\rangle(t) \to 0$ for all $t < \infty$.

The results so far give strong convergence of L^j to L and of $\langle L^j, L^j\rangle$ to $\langle L, L\rangle$, hence (in particular) strong convergence of $\widetilde{U}^{(n_j)}(\sigma_N)$ to $U(\sigma_N)$. Let $\tau_N^{(n_j)} = \inf\{t : \widetilde{U}^{(n_j)}(t) \geq N\}$. Then $\liminf_{j\to\infty} \tau_N^{(n_j)} \geq \sigma_N$, and since for each N and each j, $\mathbb{P}(\tau_N^{(n_j)} < \infty) \leq \mathbb{P}(\sigma_N < \infty)$, we have

$$\bigcap_{k\geq 1}\bigcup_{j\geq k}\{\tau_N^{(n_j)} < \infty\} = \bigcup_{k\geq 1}\bigcap_{j\geq k}\{\tau_N^{(n_j)} < \infty\} = \{\sigma_N < \infty\}.$$

Therefore

$$\begin{aligned}&\mathbb{E}[(\widetilde{U}^{(n_j)}(\tau_N^{(n_j)}) - U(\sigma_N))^2]\\ &\quad = N^2\mathbb{P}\big((\{\tau_N^{(n_j)} < \infty\}\backslash\{\sigma_N < \infty\}) \cup (\{\sigma_N < \infty\}\backslash\{\tau_N^{(n_j)} < \infty\})\big) \xrightarrow{j\to\infty} 0.\end{aligned}$$

By Doob's L^p inequality with $p = 2$,

$$\mathbb{E}\Big[\sup_t\big(\widetilde{U}^{(n_j)}(t\wedge\tau_N^{(n_j)}) - U(t\wedge\sigma_N)\big)^2\Big] \leq 4\mathbb{E}[(\widetilde{U}^{(n_j)}(\tau_N^{(n_j)}) - U(\sigma_N))^2] \xrightarrow{j\to\infty} 0.$$

Now, for $0 < \alpha < 1$,

$$\begin{aligned}\mathbb{E}\Big[\sup_t|\widetilde{U}^{(n_j)}(t) - U(t)|^\alpha\Big] &\leq \mathbb{E}\Big[\sup_t \widetilde{U}^{(n_j)\alpha}(t)\mathbf{1}_{\{\tau_N^{(n_j)}<\infty\}}\Big]\\ &\quad + \mathbb{E}\Big[\sup_t U^\alpha(t)\mathbf{1}_{\{\sigma_N<\infty\}}\Big] + \mathbb{E}\Big[\sup_t|\widetilde{U}^{(n_j)}(t\wedge\tau_N^{(n_j)}) - U(t\wedge\sigma_N)|^\alpha\Big].\end{aligned}$$

For the first of these terms, using $\mathbb{P}(\tau_N^{(n_j)} < \infty) \leq 1 \wedge \frac{\mathbb{E}[U(0)]}{N}$ we get

$$\begin{aligned}&\mathbb{E}\Big[\sup_t \widetilde{U}^{(n_j)\alpha}(t)\mathbf{1}_{\{\tau_N^{(n_j)}<\infty\}}\Big]\\ &\qquad \leq \mathbb{E}\Big[\sup_t \widetilde{U}^{(n_j)(1+\alpha)/2}(t)\Big]^{2\alpha/(1+\alpha)}\mathbb{P}(\tau_N^{(n_j)} < \infty)^{(1-\alpha)/2} < \frac{K(\alpha)}{N^{(1-\alpha)/2}}\end{aligned}$$

for a constant $K(\alpha)$. The same bound holds for the second term. The third converges to 0 from above as $j \to \infty$. This is true for all N, hence

$$\lim_{j\to\infty}\mathbb{E}\Big[\sup_t|\widetilde{U}^{(n_j)}(t) - U(t)|^\alpha\Big] = 0 \quad \forall\alpha \in (0,1)$$

and, by the Burkholder–Davis–Gundy inequality,

$$\lim_{j\to\infty}\mathbb{E}[\langle\widetilde{U}^{(n_j)} - U, \widetilde{U}^{(n_j)} - U\rangle^{\alpha/2}(\infty)] = 0 \quad \forall\alpha \in (0,1).$$

Hence, from the above,

$$\lim_{j\to\infty} \mathbb{E}[\langle U^{(n_j)} - U, U^{(n_j)} - U\rangle^{\alpha/2}(\infty)] = 0 \quad \forall \alpha \in (0,1)$$

and, by Burkholder–Davis–Gundy,

$$\lim_{j\to\infty} \mathbb{E}\Big[\sup_{0\le t<\infty} |U^{(n_j)}(t) - U(t)|^{\alpha}\Big] = 0 \quad \forall \alpha \in (0,1). \ \blacksquare$$

STEP 5: *Convergence of solutions to the approximating SPDEs.* Recall

$$U^{(n_j)}(t) = U^{(n_j)}(0) + \int_0^t \int_{\mathbb{S}^1} v^{(n_j)}(s,x)^{\gamma}\, W(ds,\, dx)$$

and let f denote the function in the Martingale Representation Theorem 21 such that

$$U(t) = U(0) + \int_0^t \int_{\mathbb{S}^1} f(s,x)\, W(dx, ds).$$

From the previous lemma,

$$\mathbb{E}[\langle U^{(n_j)} - U, U^{(n_j)} - U\rangle^{\alpha/2}(\infty)] \xrightarrow{j\to\infty} 0 \quad \forall \alpha \in (0,1).$$

Equivalently,

$$\mathbb{E}\Big[\Big(\int_0^{\infty} \int_{\mathbb{S}^1} (v^{(n_j)}(t,x)^{\gamma} - f(t,x))^2\, dx\, dt\Big)^{\alpha/2}\Big] \xrightarrow{j,k\to\infty} 0.$$

Therefore $(v^{(n_j)\gamma})_{j\ge 0}$ is a Cauchy sequence in $d_{2,\alpha}$ for $\alpha \in (0,1)$. Let $u = f^{1/\gamma}$. Since

$$\mathbb{E}\Big[\Big(\int_0^{\infty} \int_{\mathbb{S}^1} (v^{(n_j)}(t,x)^{\gamma} - f(t,x))^2\, dx\, dt\Big)^{\alpha/2}\Big] \ge \mathbb{E}\Big[\Big(\int_0^{\infty} \int_{\mathbb{S}^1} (v^{(n_j)}(t,x) - u(t,x))^{2\gamma}\, dx\, dt\Big)^{\alpha/2}\Big],$$

$(v^{(n_j)})_{j\ge 0}$ is a Cauchy sequence in $\mathcal{S}_{2\gamma,\alpha}$. The space is complete (by Lemma 12) and the limit $u \in \mathcal{S}_{2\gamma,\alpha}$ of $(v^{(n_j)})_{j\ge 1}$ satisfies

$$\lim_{j\to\infty} d_{2,\alpha}(v^{(n_j)\gamma}, u^{\gamma}) = 0, \qquad \lim_{j\to\infty} d_{2\gamma,\alpha}(v^{(n_j)}, u) = 0.$$

This concludes the proof of Theorem 18. $\blacksquare$

The main result may now be stated and proved; the proof is a straightforward consequence of the preceding results.

THEOREM 24. *The limiting object u provides a solution to* (1).

Proof. Consider the space of test functions

$$\mathcal{T} = \Big\{\phi : C^\infty(\mathbb{R}_+ \times \mathbb{S}^1) : \sup_{t,x}(|\phi(t,x)| + |\phi_t(t,x)| + |\phi_{xx}(t,x)|) \le 1\Big\}$$

The function $u^{(n_j)}$ satisfies (17) (with $n = n_j$) if and only if for all $\phi \in \mathcal{T}$,

$$\int_{\mathbb{S}^1} u^{(n_j)}(t,x)\phi(t,x)\,dx - \int_0^t\int_{\mathbb{S}^1} u^{(n_j)}(s,x)\phi_s(s,x)\,dx\,ds - \frac{1}{2}\int_0^t\int_{\mathbb{S}^1} \phi_{xx}(s,x)u^{(n_j)}(s,x)\,dx\,ds = \int_{\mathbb{S}^1} u_0(x)\phi(0,x)\,dx + \int_0^t\int_{\mathbb{S}^1} \phi(s,x)v^{(n_j)\gamma}(s,x)W(dx,ds)$$

where (as earlier) $v^{(n)} = u^{(n)} \wedge n$. A function u satisfies equation (16) (driven by W) if and only if for all $\phi \in \mathcal{T}$,

$$\int_{\mathbb{S}^1} u(t,x)\phi(t,x)\,dx - \int_0^t\int_{\mathbb{S}^1} u(s,x)\phi_s(s,x)\,dx\,ds - \frac{1}{2}\int_0^t\int_{\mathbb{S}^1} \phi_{xx}(s,x)u(s,x)\,dx\,ds = \int_{\mathbb{S}^1} u_0(x)\phi(0,x)\,dx + \int_0^t\int_{\mathbb{S}^1} \phi(s,x)u^\gamma(s,x)\,W(dx,ds). \tag{42}$$

From the foregoing, it is clear that

$$\int_0^\infty\int_{\mathbb{S}^1} \phi_s(s,x)u^{(n_j)}(s,x)\,dx\,ds \xrightarrow{j\to\infty} \int_0^\infty\int_{\mathbb{S}^1} \phi_s(s,x)u(s,x)\,dx\,ds$$

and

$$\int_0^\infty\int_{\mathbb{S}^1} \phi_{xx}(s,x)u^{(n_j)}(s,x)\,ds\,dx \xrightarrow{j\to\infty} \int_0^\infty\int_{\mathbb{S}^1} \phi_{xx}(s,x)u(s,x)\,ds\,dx$$

$\mathbb{P}$-almost surely. For the last term,

$$\mathbb{E}\Big[\sup_{0\le t<\infty}\Big|\int_0^t\int_{\mathbb{S}^1} \phi(s,x)v^{(n_j)}(s,x)^\gamma\,W^{(n_j)}(ds,dx) - \int_0^t\int_{\mathbb{S}^1} \phi(s,x)u(s,x)^\gamma\,W(ds,dx)\Big|^\alpha\Big] \le C(\alpha)\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}^1} \phi(s,x)^2(v^{(n_j)}(s,x)^\gamma - u(s,x)^\gamma)^2\,dx\,ds\Big)^{\alpha/2}\Big] \xrightarrow{j\to\infty} 0$$

by Theorem 18 and the definition of the stochastic integral. The result follows. ∎

5. Uniqueness. This section is devoted to establishing uniqueness of the solution, in the sense given in the following theorem.

THEOREM 25 (Uniqueness). *Let u and v denote two solutions to equation* (1) *in $\mathcal{S}_{2\gamma,\alpha}$* (Definition 10) *for all $\alpha \in (0,1)$. Suppose that $u(0,\cdot) = v(0,\cdot)$. Then $d_{2\gamma,\alpha}(u,v) \equiv 0$ for all $\alpha \in (0,1)$.*

Before proving the theorem, some remarks are in order. Firstly, a standard Gronwall approach was beyond the capabilities of the author; it did not seem possible to get round the difficulties posed by the non-linearity u^γ in the equation, where $\gamma > 1$. Therefore, a different approach is used.

In the previous section, existence of *strong* solutions (using the term *strong* in the sense of Revuz and Yor [14, Ch. 9, Sect. 1, Definition 1.5, p. 367]); that is, solutions which are progressively measurable with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, was established. Here, suppose there is non-uniqueness of solution. For any two solutions u and v, $(u(t,\cdot), v(t,\cdot))$ should define a Feller process (this will be established), even if one is considering an enlarged filtration $(\mathcal{H}_t)_{t\geq 0}$ where, for each t, $\mathcal{H}_t \supseteq \mathcal{F}_t$ and $\mathcal{H}_t \perp \sigma(W([0,x]\times[t,T]) : 0 \leq x \leq 1, T \geq t)$. The aim is therefore to establish that solutions determine a unique infinitesimal generator which corresponds to the solutions. Having established that they form a Feller process (in a suitable sense) with a well defined, unique, infinitesimal generator, uniqueness of solution is a direct consequence.

The formalism is taken from Kallenberg [2] and some work is required to ensure that the formalism is satisfied, since the state space is *infinite*-dimensional, rather than *finite*-dimensional. The whole proof would be straightforward if the Markov chain had a finite-dimensional state space. The basic ideas are straighforward, but require some detail to extend to infinite dimensions. This Markov process has generator (say) $\mathcal{L}$. For suitable functionals F and initial conditions u_0, v_0 for u and v respectively, set $F(t; u_0, v_0) = \mathbb{E}[F(u(t,\cdot), v(t,\cdot))]$. Clearly, for $u_0 = v_0$, $F(t; u_0, u_0) = \mathbb{E}[F(u(t,\cdot).u(t,\cdot))]$ is *a* solution. Consider F of the form $F(u,v) = \mathcal{V}_1(u+v)\mathcal{V}_2(u-v)$ where $\mathcal{V}_2(0) = 0$; by uniqueness of the corresponding semigroup, $F(t; u_0, u_0) = \mathbb{E}[\mathcal{V}_1(2u(t))\mathcal{V}_2(0)] \equiv 0$ is *the* solution when $u_0 = v_0$ and hence $u(t,\cdot) = v(t,\cdot)$.

Fourier expansions of the solutions u and v are taken. Equation (42) gives explicit equations for the Fourier coefficients and from this the infinitesimal generator can be computed in a straightforward manner.

The proof is now given.

Proof of Theorem 25. Any function $u \in \mathcal{S}_{2\gamma,\alpha}$ for some $\alpha \in (0,1)$ satisfies $\int_0^\infty \|u(t,\cdot)\|_{2\gamma}^{2\gamma}\, dt < \infty$ $\mathbb{P}$-almost surely. On the set of $\mathbb{P}$-measure 1 where this holds, clearly $\|u(t,\cdot)\|_{2\gamma} < \infty$ for Lebesgue almost all $t \in \mathbb{R}_+$ and hence $\|u(t,\cdot)\|_2 < \infty$ on this set (since $\gamma > 1$). Denote by $\mathcal{D} \subseteq \Omega \times \mathbb{R}_+$ the set

$$\mathcal{D} = \bigcup_{K>0} \Big\{ \int_0^\infty \|u(t,\cdot)\|_{2\gamma}^{2\gamma}\, dt + \int_0^\infty \|v(t,\cdot)\|_{2\gamma}^{2\gamma}\, dt < K \Big\}.$$

Then $\mathbb{P}(\mathcal{D}) = 1$. For $\omega \in \mathcal{D}$, let

$$\mathcal{T}(\omega) = \bigcup_{K>0} \{t : \|u(t,\cdot)\|_{2\gamma} + \|v(t,\cdot)\|_{2\gamma} < K\}.$$

Then $\mathbb{P}$-a.s., $\mathcal{T}(\omega)$ is a set of full Lebesgue measure. Let

$$\Xi = \{(\omega, t) : \omega \in \mathcal{D},\ t \in \mathcal{T}(\omega)\}.$$

For $\gamma > 1$ (the situation under consideration here) and $f : \mathbb{S}^1 \to \mathbb{R}_+$, $\|f\|_2 \le \|f\|_{2\gamma}$ by Hölder's inequality.

Let

$$\lambda_j(t) = \int_{\mathbb{S}^1} e^{-i2\pi jx} u(t,x)\, dx, \qquad \mu_j(t) = \int_{\mathbb{S}^1} e^{-i2\pi jx} v(t,x)\, dx \qquad \forall j \in \mathbb{Z}.$$

Since $\sup_t \int_{\mathbb{S}^1} u(t,x)\, dx < \infty$ $\mathbb{P}$-a.s. and $\sup_t \int_{\mathbb{S}^1} v(t,x)\, dx < \infty$ $\mathbb{P}$-a.s., it follows directly that $\max_j \sup_t(|\lambda_j(t)| + |\mu_j(t)|) < \infty$ $\mathbb{P}$-a.s.

Let $\widehat{u}_N(t,x) = \sum_{j=-N}^N \lambda_j(t) e^{ij2\pi x}$ and $\widehat{v}_N(t,x) = \sum_{j=-N}^N \mu_j(t) e^{ij2\pi x}$. On $\mathcal{D}$, let

$$\widehat{u} = \begin{cases} \lim_{N\to\infty} \widehat{u}_N, & \text{limit well defined,} \\ 0, & \text{otherwise,} \end{cases}$$

define $\widehat{v}$ analagously, and let $\widehat{u} \equiv \widehat{v} \equiv 0$ on $\Omega \setminus \mathcal{D}$.

Justification of the Fourier transform. Firstly, by Carleson's theorem [1], the Fourier expansion of any L^2 function converges almost everywhere, hence for $(\omega,t) \in \Xi$, $\widehat{u}_N$ and $\widehat{v}_N$ converge to u and v respectively for almost all $x \in \mathbb{S}^1$. Secondly, norm convergence of $\widehat{u}_N$ and $\widehat{v}_N$ is standard on Ξ in the sense that

$$\lim_{N\to\infty} \int_{\mathbb{S}^1} |u - \widehat{u}_N|^2(t,x)\, dx = \lim_{N\to\infty} \int_{\mathbb{S}^1} |v - \widehat{v}_N|^2(t,x)\, dx = 0, \qquad (\omega,t) \in \Xi,$$

and hence

$$\|u(t,\cdot) - \widehat{u}(t,\cdot)\|_2 = 0 \quad \text{and} \quad \|v(t,\cdot) - \widehat{v}(t,\cdot)\|_2 = 0 \qquad \forall (\omega,t) \in \Xi,$$

since for $(\omega,t) \in \mathcal{D}$, $u, v \in L^2(\mathbb{S}^1)$. This is the Riesz–Fisher theorem.

Since $u(t,x) - \widehat{u}(t,x) = 0$ $\mathbb{P} \times dt \times dx$-almost everywhere, $\widehat{u} = \widehat{v} = 0$ on the set where the sequence $(\widehat{u}_N, \widehat{v}_N)$ does not converge and the bounds $\mathbb{E}[(\int_0^\infty \|u\|_{2\gamma}^{2\gamma}(t)\, dt)^\alpha] < \infty$ and $\mathbb{E}[(\int_0^\infty \|v\|_{2\gamma}^{2\gamma}(t)\, dt)^\alpha] < \infty$ hold for $\alpha \in (0,1)$, it follows that for all $\alpha \in (0,1)$,

$$\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}} |u(t,x)-\widehat{u}(t,x)|^{2\gamma}\,dx\,dt\Big)^{\alpha/2}\Big]=0,$$

$$\mathbb{E}\Big[\Big(\int_0^\infty\int_{\mathbb{S}} |v(t,x)-\widehat{v}(t,x)|^{2\gamma}\,dx\,dt\Big)^{\alpha/2}\Big]=0$$

and hence

$$d_{2\gamma,\alpha}(v,u)=d_{2\gamma,\alpha}(\widehat{v},\widehat{u}) \quad \forall\alpha\in(0,1).$$

Note that

$$\|\widehat{u}(t,\cdot)\|_2^2=\sum_{-\infty}^{\infty}\lambda_j(t)\lambda_{-j}(t), \qquad \|\widehat{v}(t,\cdot)\|_2^2=\sum_{-\infty}^{\infty}\mu_j(t)\mu_{-j}(t) \quad \forall(\omega,t)\in\Xi.$$

The Fourier coefficients. Let $\lambda_{j0}:=\lambda_j(0)$ and $\mu_{j0}:=\mu_j(0)$ so that $u(0,x)=\sum_{j=-\infty}^{\infty}\lambda_{j0}e^{ij2\pi x}$ and $v(0,x)=\sum_{j=-\infty}^{\infty}\mu_{j0}e^{ij2\pi x}$. By integration over the space variable and using the fact that both u and v satisfy the equation $w_t=\frac{1}{2}w_{xx}+w^\gamma\xi$, we get

$$\begin{aligned}\lambda_n(t)&=\lambda_{n0}-\tfrac{1}{2}n^2\int_0^t\lambda_n(s)\,ds+M_n(t),\\ \mu_n(t)&=\lambda_{n0}-\tfrac{1}{2}n^2\int_0^t\mu_n(s)\,ds+N_n(t),\end{aligned}\tag{43}$$

where

$$M_n(t)=\int_0^t\int_{\mathbb{S}^1} e^{-inx}u(s,x)^\gamma\,W(dx,ds),$$

$$N_n(t)=\int_0^t\int_{\mathbb{S}^1} e^{-inx}v(s,x)^\gamma\,W(dx,ds).$$

This is a straightforward consequence of (42) with appropriate choice of test functions ϕ. Note that the quadratic variations are:

$$\begin{aligned}\langle M_m,M_n\rangle(t)&=\int_0^t\int_{\mathbb{S}^1} e^{-i(n+m)x}v(s,x)^{2\gamma}\,dx\,ds,\\ \langle N_m,N_n\rangle(t)&=\int_0^t\int_{\mathbb{S}^1} e^{-i(n+m)x}u(s,x)^{2\gamma}\,dx\,ds,\\ \langle M_m,N_n\rangle(t)&=\int_0^t\int_{\mathbb{S}^1} e^{-i(n+m)x}u(s,x)v^\gamma(s,x)^\gamma\,dx\,ds.\end{aligned}\tag{44}$$

Spaces. Denote by $\mathcal{S}$ the space

$$\mathcal{S} = \{\gamma : \gamma_n = \gamma^*_{-n} \ \forall n \in \mathbb{Z}\}$$

where α^* denotes the complex conjugate of α, with metric

$$d_{\mathcal{S}}(\gamma,\delta) = \sqrt{\sum_{n=-\infty}^{\infty} \frac{1}{(1+|n|)^2}(\gamma_n-\delta_n)(\gamma_{-n}-\delta_{-n})}.$$

The important point about this weight is that if $d_{\mathcal{S}}(\gamma,0)\to\infty$ then $\max_n|\gamma_n|\to\infty$. For $(\gamma_1,\gamma_2),(\delta_1,\delta_2)\in\mathcal{S}^2$, the notation

$$d_{\mathcal{S}^2}((\gamma_1,\gamma_2),(\delta_1,\delta_2)) = \sqrt{d_{\mathcal{S}}(\gamma_1,\delta_1)^2 + d_{\mathcal{S}}(\gamma_2,\delta_2)^2}$$

will be used. The aim is to show that the stochastic evolution defined by (43) defines a Feller transition semigroup over the space $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$, which is now defined.

Definition 26. The space $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ is defined as the space of real-valued functions $\mathcal{U}$ over $\mathcal{S}^2$ which satisfy the following two conditions:

(1) they are continuous under the metric $d_{\mathcal{S}}$, that is, for any sequence $(\gamma_n,\delta_n)_{n\geq1}$,

$$d_{\mathcal{S}^2}((\gamma_n,\delta_n),(\gamma,\delta)) \xrightarrow{n\to\infty} 0 \implies \lim_{n\to\infty}|\mathcal{U}(\gamma_n,\delta_n)-\mathcal{U}(\gamma,\delta)| = 0.$$

(2) $\mathcal{U}(\gamma,\delta) \xrightarrow{|\gamma|+|\delta|\to\infty} 0$ where $|\gamma| := d_{\mathcal{S}}(\gamma,0)$.

A suitable metric on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ is now defined. Let

$$\mathcal{U}^{(N)}(\lambda,\mu) = \mathcal{U}(\widetilde{\lambda}^{(N)},\widetilde{\mu}^{(N)}) \tag{45}$$

where

$$\widetilde{\lambda}^{(N)}_j = \begin{cases}\lambda_j, & j\in\{-N,\dots,N\},\\ 0, & \text{otherwise.}\end{cases} \qquad \widetilde{\mu}^{(N)}_j = \begin{cases}\mu_j, & j\in\{-N,\dots,N\},\\ 0, & \text{otherwise.}\end{cases} \tag{46}$$

Set

$$\langle\!\langle \mathcal{U},\mathcal{V}\rangle\!\rangle = \frac{1}{e-1}\sum_{N=1}^{\infty} e^{-N}\langle\!\langle \mathcal{U}^{(N)},\mathcal{V}^{(N)}\rangle\!\rangle_N \tag{47}$$

where $\mathcal{U}^{(N)}$ and $\mathcal{V}^{(N)}$ are the N-approximations defined by (45) for $\mathcal{U}$ and $\mathcal{V}$ and, for $f,g:\mathbb{R}^{4N+2}\to\mathbb{R}$,

$$\langle\!\langle f,g\rangle\!\rangle_N = \int \frac{1}{(2\pi)^{2N+1}} e^{-|x|^2/2} f^{(N)}(x) g^{(N)}(x)\,dx. \tag{48}$$

Here $x = (x_1,\dots,x_{4N+2})\in\mathbb{R}^{4N+2}$ and the components of the vector x are the $4N+2$ real valued variables required to define $\lambda_{-N},\dots,\lambda_N$ and $\mu_{-N},\dots,\mu_N$, using $\lambda_j = \lambda^*_{-j}$, $\lambda_0 = x_1$, $\lambda_j = x_{2j}+ix_{2j+1}$ for $j=1,\dots,N$, $\mu_0 = x_{2N+2}$, $\mu_j = x_{2j+2N+1}+ix_{2j+2N+2}$ for $j=1,\dots,N$.

Clearly, (47) defines an inner product:

- $\langle\!\langle \mathcal{U}, \mathcal{V} \rangle\!\rangle = \langle\!\langle \mathcal{V}, \mathcal{U} \rangle\!\rangle$,
- $\langle\!\langle a\mathcal{U}, \mathcal{V} \rangle\!\rangle = a\langle\!\langle \mathcal{U}, \mathcal{V} \rangle\!\rangle$ for a scalar a, and for $\mathcal{U}, \mathcal{V}, \mathcal{W}$,

$$\langle\!\langle \mathcal{U}, \mathcal{V} + \mathcal{W} \rangle\!\rangle = \langle\!\langle \mathcal{U}, \mathcal{V} \rangle\!\rangle + \langle\!\langle \mathcal{U}, \mathcal{W} \rangle\!\rangle,$$

- $\langle\!\langle \mathcal{U}, \mathcal{U} \rangle\!\rangle \geq 0$, with equality if and only if $\mathcal{U} \equiv 0$.

Therefore, $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defines an inner product over a Hilbert space $\mathcal{H}$ such that $C_0(\mathcal{S}^2, d_{\mathcal{S}^2}) \subseteq \mathcal{H}$.

The following metric on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ will be used:

$$D(\mathcal{U}, \mathcal{V}) = \sqrt{\langle\!\langle \mathcal{U} - \mathcal{V}, \mathcal{U} - \mathcal{V} \rangle\!\rangle}, \tag{49}$$

and the norm

$$\|\mathcal{U}\| = \sqrt{\langle\!\langle \mathcal{U}, \mathcal{U} \rangle\!\rangle}. \tag{50}$$

Establishing an Itô formula. The next step is to establish that an Itô formula holds for functions $\mathcal{U}(\lambda, \mu)$ belonging to a suitable class. The class on which the Itô formula is established is a subset of $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ (Definition 26); it is the functions in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ which also satisfy conditions (51) and (53) below. The second of these implies that the function can be approximated by restriction to finite dimensions.

The conditions are

$$\begin{cases} \sup\limits_{\lambda,\mu \in \mathcal{S}} |\mathcal{U}(\lambda, \mu)| < \infty, \\ \sup\limits_{\lambda,\mu \in \mathcal{S}} \sum\limits_{n} n^2 (|\lambda_n \partial_{\lambda_n} \mathcal{U}(\lambda, \mu)| + |\mu_n \partial_{\mu_n} \mathcal{U}(\lambda, \mu)|) < \infty, \\ \sup\limits_{\lambda,\mu \in \mathcal{S}} \sum\limits_{m,n} (|\partial^2_{\mu_m \mu_n} \mathcal{U}(\lambda, \mu)| + |\partial^2_{\lambda_m \mu_n} \mathcal{U}(\lambda, \mu)| + |\partial^2_{\lambda_m \lambda_n} \mathcal{U}(\lambda, \mu)|) < \infty, \\ \sup\limits_{\lambda,\mu \in \mathcal{S}} \sum\limits_{n} (|\partial_{\mu_n} \mathcal{U}(\lambda, \mu)|^2 + |\partial_{\lambda_n} \mathcal{U}(\lambda, \mu)|^2) < \infty. \end{cases} \tag{51}$$

Here $\partial^p_{a_1 \dots a_p}$ denotes the pth partial derivative with respect to the arguments $a_1, \dots, a_p$. Let

$$\mathcal{W}_N = \mathcal{U} - \mathcal{U}^{(N)} \tag{52}$$

where $\mathcal{U}^{(N)}$ is defined by (45). The condition that ensures that $\mathcal{U}$ can be approximated by $\mathcal{U}^{(N)}$ is

(53)
$$\begin{cases} \lim_{N\to\infty} \sup_{\lambda,\mu\in\mathcal{S}} |\mathcal{W}_N(\lambda,\mu)| = 0, \\ \sup_{\lambda,\mu\in\mathcal{S}} \sum_n n^2\big(|\lambda_n\partial_{\lambda_n}\mathcal{W}_N(\lambda,\mu)| + |\mu_n\partial_{\mu_n}\mathcal{W}_N(\lambda,\mu)|\big) \xrightarrow{N\to\infty} 0, \\ \sup_{\lambda,\mu\in\mathcal{S}} \sum_{m\,n}\big(|\partial^2_{\mu_m\mu_n}\mathcal{W}_N(\lambda,\mu)| + |\partial^2_{\mu_m\lambda_n}\mathcal{W}_N(\lambda,\mu)| + |\partial^2_{\lambda_m\lambda_n}\mathcal{W}_N(\lambda,\mu)|\big) \xrightarrow{N\to\infty} 0. \end{cases}$$

For collections $(\lambda_j)_{j=-\infty}^{\infty}$ and $(\mu_j)_{j=-\infty}^{\infty}$ such that $\lambda_j = \lambda^*_{-j}$ (complex conjugate) and $\mu_j = \mu^*_{-j}$, set

$$f(\lambda;x) := \sum_j \lambda_j e^{i2\pi jx}$$

and consider μ and λ such that

$$f(\lambda,x) \geq 0 \quad \forall x\in\mathbb{S}^1, \qquad f(\mu,x)\geq 0 \quad \forall x\in\mathbb{S}^1,$$
$$\int_{\mathbb{S}^1} f(\lambda,x)^{2\gamma}\,dx < \infty, \qquad \int_{\mathbb{S}^1} f(\mu,x)^{2\gamma}\,dx < \infty.$$

For such λ and μ, set

$$F_m(\lambda,\mu) = \int_{\mathbb{S}^1} e^{-i2\pi mx} f(\lambda,x)^\gamma f(\mu,x)^\gamma\,dx. \tag{54}$$

Note that

$$\frac{d}{dt}\langle M_m, M_n\rangle(t) = F_{m+n}(\lambda(t),\lambda(t)), \quad \frac{d}{dt}\langle N_m,N_n\rangle(t) = F_{m+n}(\mu(t),\mu(t)),$$
$$\frac{d}{dt}\langle M_m, N_n\rangle(t) = F_{m+n}(\lambda(t),\mu(t)).$$

Let $\mathcal{L}$ be defined as

$$\begin{aligned} \mathcal{L}(\lambda,\mu) &= -\frac{1}{2}\sum_{n=-\infty}^{\infty} n^2(\lambda_n\partial_{\lambda_n} + \mu_n\partial_{\mu_n}) \\ &+ \frac{1}{2}\sum_{m,n=-\infty}^{\infty}\big(F_{m+n}(\lambda,\lambda)\partial^2_{\lambda_m\lambda_n} + F_{m+n}(\mu,\mu)\partial^2_{\mu_m\mu_n} + 2F_{m+n}(\lambda,\mu)\partial^2_{\lambda_m\mu_n}\big), \end{aligned} \tag{55}$$

where $\partial^p_{a_1,\dots,a_p}\mathcal{U}(\lambda,\mu)$ means the pth partial derivative of $\mathcal{U}$ with respect to the arguments labelled $a_1,\dots,a_p$.

Definition 27 (Domain of infinitesimal generator). Let $\mathcal{D}_*(\mathcal{L})$ be the functions $\mathcal{U}\in C_0(\mathcal{S}^2)$ which satisfy both (51) and (53). Let $\mathcal{D}(\mathcal{L})$ be the functions $\mathcal{U}$ such that for any sequence $(\mathcal{U}_n)$ such that $D(\mathcal{U},\mathcal{U}_n) \xrightarrow{n\to\infty} 0$, where $\mathcal{U}_n\in\mathcal{D}_*(\mathcal{L})$, $\mathcal{L}\mathcal{U}_n \to_{\mathcal{H}} \mathcal{Y}$ for some $\mathcal{Y}\in\mathcal{H}$, and let $\mathcal{L}\mathcal{U}$ be defined by $\mathcal{L}\mathcal{U} = \mathcal{Y}$. The space $\mathcal{D}(\mathcal{L})$ is the *domain of the infinitesimal generator* $\mathcal{L}$. The space $\mathcal{D}(\mathcal{L})$ is the functions $\mathcal{U} \in C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ on which $\mathcal{L}\mathcal{U}$ is well defined and bounded. From the definition, $\mathcal{L}\mathcal{U}$ is well defined for all $\mathcal{U} \in \mathcal{D}_*(\mathcal{L})$.

Lemma 28. *$\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ under the metric D defined by* (49).

Proof. This is clear. Functions in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ are bounded and the construction of the metric ensures the convergence. A function $\mathcal{U} \in C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$ may be approximated by the approximations $\mathcal{U}^{(N)}$ defined by (45), which may be further approximated by a smoothed version, with the smoothing decreasing as $N \to \infty$ to get a sequence of functions in $\mathcal{D}_*(\mathcal{L})$ with limit $\mathcal{U}$. ∎

Lemma 29. *Let $\mathcal{U} \in \mathcal{D}_*(\mathcal{L})$ and let $(\lambda(t), \mu(t))$ satisfy* (43) *with initial conditions $\lambda(0) = \lambda$, $\mu(0) = \mu$. Then Itô's formula may be applied to give*

$$(56)\qquad \mathcal{U}(\lambda(t), \mu(t)) - \mathcal{U}(\lambda, \mu) - \int_0^t (\mathcal{L}\mathcal{U})(\lambda(s), \mu(s))\, ds$$
$$= \sum_n \int_0^t (\partial_{\lambda_n}\mathcal{U})(\lambda(s), \mu(s))\, dM_n(s) + \sum_n \int_0^t (\partial_{\mu_n}\mathcal{U})(\lambda(s), \mu(s))\, dN_n(s)$$

where the right hand side is a martingale with quadratic variation process Q where

$$(57)\qquad Q(t) = \sum_{n_1,n_2} \int_0^t (\partial_{\lambda_{n_1}}\mathcal{U})(\lambda(s), \mu(s))(\partial_{\lambda_{n_2}}\mathcal{U})(\lambda(s), \mu(s)) F_{n_1+n_2}(\lambda(s), \lambda(s))\, ds$$
$$+ \sum_{n_1,n_2} \int_0^t (\partial_{\mu_{n_1}}\mathcal{U})(\lambda(s), \mu(s))(\partial_{\mu_{n_2}}\mathcal{U})(\lambda(s), \mu(s)) F_{n_1+n_2}(\mu(s), \mu(s))\, ds$$
$$+ 2\sum_{n_1,n_2} \int_0^t (\partial_{\lambda_{n_1}}\mathcal{U})(\lambda(s), \mu(s))(\partial_{\mu_{n_2}}\mathcal{U})(\lambda(s), \mu(s)) F_{n_1+n_2}(\lambda(s), \mu(s))\, ds.$$

Proof. Following the line of proof taken by Revuz and Yor [14, Theorem 3.3, p. 147], if $\mathcal{U}$ satisfies (51) and (53), Itô's formula may be applied to $\mathcal{U}^{(N)}(\lambda, \mu)$ defined by (45) for each $N < \infty$ to give

$$(58)\qquad \mathcal{U}^{(N)}(\lambda(t), \mu(t)) - \mathcal{U}^{(N)}(\lambda, \mu) - \int_0^t (\mathcal{L}\mathcal{U}^{(N)})(\lambda(s), \mu(s))\, ds$$
$$= \sum_{n=-N}^{N} \int_0^t (\partial_{\lambda_n}\mathcal{U}^{(N)}(\lambda(s), \mu(s))\, dM_n(s) + \sum_{n=-N}^{N} \int_0^t (\partial_{\mu_n}\mathcal{U}^{(N)})(\lambda(s), \mu(s))\, dN_n(s).$$

Clearly, from the hypotheses on $\mathcal{U}$, the left hand side of (58) converges to the left hand side of (56) as $N \to \infty$. Denote the left hand side of (56) (which is equal to the right hand side) by $\mathcal{M}^{(N)}$, which is a continuous martingale, bounded on any finite interval $[0, T]$ ($\mathcal{U}$ is bounded because $\mathcal{U} \in C_0(\mathcal{S}^2)$; $\mathcal{L}\mathcal{U}$ is bounded from the definition of $\mathcal{D}_*(\mathcal{L})$). Let $\mathcal{M}$ denote the limit of $\mathcal{M}^{(N)}$; clearly $\lim_{N\to\infty}\sup_{0\le t\le T}|\mathcal{M}(t)-\mathcal{M}^{(N)}(t)|=0$ $\mathbb{P}$-a.s. Since the space of *continuous* local martingales is *complete* (Lemma 13), $\mathcal{M}$ is a continuous martingale (it is bounded, hence a true martingale, and it is continuous).

Let $Q^{(N)}$ denote the quadratic variation of $\mathcal{M}^{(N)}$ and let Q^* denote the quadratic variation of the limiting martingale $\mathcal{M}$. Now Q^* has to be identified with Q. But this is straightforward: for all m,

$$|F_m(\lambda(t),\lambda(t))|\le\|u\|_{2\gamma}^{2\gamma}(t),\qquad |F_m(\mu(t),\mu(t))|\le\|v\|_{2\gamma}^{2\gamma}(t),$$
$$|F_m(\lambda(t),\mu(t))|\le\|u\|_{2\gamma}^{\gamma}(t)\|v\|_{2\gamma}^{\gamma}(t).$$

Recall that, for all $\alpha\in(0,1)$,

$$\mathbb{E}\Big[\Big(\int_0^\infty\|u\|_{2\gamma}^{2\gamma}(t)\,dt\Big)^{\alpha/2}\Big]<\infty,\qquad \mathbb{E}\Big[\Big(\int_0^\infty\|v\|_{2\gamma}^{2\gamma}(t)\,dt\Big)^{\alpha/2}\Big]<\infty.$$

Using the bounds of (51) and (53) which show that

$$\sup_{\lambda,\mu}|\partial_{\lambda_n}\mathcal{U}-\partial_{\lambda_n}\mathcal{U}^{(N)}|\xrightarrow{N\to\infty}0,$$

together with the boundedness hypotheses, it is straightforward to apply the dominated convergence theorem to show that $\lim_{N\to\infty}Q^{(N)}(t)=Q(t)$. ∎

Establishing the Markov property. The next step is to establish that $(\lambda(t),\mu(t))_{t\ge0}$ is a time homogeneous Markov process with infinitesimal generator $\mathcal{L}$.

Lemma 30. *$\mathcal{L}$ is the infinitesimal generator of a unique Feller transition semigroup $(\mathcal{Q}_t)_{t\ge0}$ on $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$.*

Proof. $\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$, by Lemma 28, in the sense described in that lemma. Furthermore, $\mathcal{D}_*(\mathcal{L})\subseteq\mathcal{D}(\mathcal{L})$ (by definition of $\mathcal{D}(\mathcal{L})$), hence $\mathcal{D}(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2,d_{\mathcal{S}^2})$. The operator $\mathcal{L}$ is *closed*, which follows from the definition of $\mathcal{D}(\mathcal{L})$, using the characterisation that a linear operator $\mathcal{L}:\mathcal{D}(\mathcal{L})\to\mathcal{H}$ is closed if and only if the domain $\mathcal{D}(\mathcal{L})$ endowed with the norm $\|\mathcal{U}\|+\|\mathcal{L}\mathcal{U}\|$ is a Banach space, i.e. a linear, normed, complete space, and this is clear.

Now suppose there exists a family $\mathcal{Q}$ of transition semigroups with $\mathcal{L}$ as infinitesimal generator. Note that for any $Q\in\mathcal{Q}$,

$$\frac{Q_h-I}{h}\xrightarrow{h\to0}\mathcal{L}.$$

Furthermore, if $f\in\mathcal{D}(\mathcal{L})$, then for any $Q\in\mathcal{Q}$ and all $t>0$,

$$\mathcal{L}Q_tf=Q_t\mathcal{L}f.$$

Suppose that $\mathcal{Q}$ has more than one element; consider two of them, $Q^{(1)}$ and $Q^{(2)}$. Let $f \in \mathcal{D}(\mathcal{L})$ and let $w(s) = Q_s^{(1)} Q_{t-s}^{(2)} f$. Then

$$\frac{d}{ds} w(s) = Q_s^{(1)} \mathcal{L} Q_{t-s}^{(2)} f - Q_s^{(1)} \mathcal{L} Q_{t-s}^{(2)} f = 0,$$

showing that w is constant on $[0, t]$, hence (taking $s = 0$ and t), $Q_t^{(1)} f = Q_t^{(2)} f$ for all $t > 0$. It follows that there is at most one $Q \in \mathcal{Q}$.

Since $\mathcal{L}$ is a closed operator, existence now follows from the Hille–Yosida theorem. The statement, taken from Kallenberg [2, Theorem 19.11, p. 375] is as follows:

Characterisation of generators (Hille–Yosida). *Let A be a linear operator on C_0 with domain $\mathcal{D}$. Then A is closable and its closure $\overline{A}$ is the generator of a Feller semigroup on C_0 if and only if the following conditions hold:*

- *$\mathcal{D}$ is dense in C_0,*
- *the range of $a_0 - A$ is dense in C_0 for some $a_0 > 0$,*
- *if $\sup_y f(y) \vee 0 \le f(x)$ for some $x \in \mathcal{S}$ then $Af(x) \le 0$.*

It has to be shown that $\mathcal{L}$ satisfies these properties. The first has already been established: the space $\mathcal{D}_*(\mathcal{L})$ is dense in $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$. The second and third can be shown to be satisfied by approximating $\mathcal{L}$ by operators $\mathcal{L}^{(N)}$ over *finite*-dimensional subspaces $\mathcal{S}_N = \mathbb{R}^{4N+1}$ which are generators of Feller processes. Therefore, they satisfy $\|(a_0 I - \mathcal{L}^{(N)})^{-1}\|_{\mathcal{O},N} \le 1/a_0$ for finite N (where $\|\cdot\|_{\mathcal{O},N}$ denotes the operator norm) and inherit the same property in the limit. Similarly, if $f \in C_0(\mathcal{S})$, with maximum at $x \in \mathcal{S}$, then $\mathcal{L}^{(N)} f(x) \le 0$ for all N and the same holds in the limit.

Putting in the details, let

$$\begin{aligned}(59)\qquad \mathcal{L}^{(N)}(\lambda, \mu) = {} & -\frac{1}{2} \sum_{n=-N}^{N} n^2 (\lambda_n \partial_{\lambda_n} + \mu_n \partial_{\mu_n}) \\ & + \frac{1}{2} \sum_{m,n=-N}^{N} \big(F_{m+n}(\widetilde{\lambda}^{(N)}, \widetilde{\lambda}^{(N)}) \partial^2_{\lambda_m, \lambda_n} + F_{m+n}(\widetilde{\mu}^{(N)}, \widetilde{\mu}^{(N)}) \partial^2_{\mu_m \mu_n} \\ & \qquad + 2F_{m+n}(\widetilde{\lambda}^{(N)}, \widetilde{\mu}^{(N)}) \partial^2_{\lambda_m \mu_n} \big)\end{aligned}$$

where $(\widetilde{\lambda}^{(N)}, \widetilde{\mu}^{(N)})$ are defined in (46). Then $\mathcal{L}^{(N)}$ is clearly the generator of a Feller process for any finite N and therefore, for every real $a_0 > 0$ and all $M, N < \infty$, $\|(a_0 I - \mathcal{L}^{(M)})^{-1}\|_{\mathcal{O};N} \le 1/a_0$, where $\|.\|_{\mathcal{O};N}$ denotes the operator norm when an operator is applied to functions $\mathcal{U}^{(N)}$ for $\mathcal{U} \in C_0(\mathcal{S})$. This holds, since (Hille–Yosida) it holds for any generator of a contraction semigroup. The result therefore holds in the limit.

The third condition similarly holds: if $\mathcal{U}(\lambda,\mu) = \sup_{l,m}\mathcal{U}(l,m) > 0$, then $\mathcal{L}^{(N)}\mathcal{U}(\lambda,\mu) \le 0$ for each N and hence $\mathcal{L}\mathcal{U}(\lambda,\mu) < \infty$.

It has therefore been established that the infinitesimal generator $\mathcal{L}$ generates a unique Feller transition semigroup on $C_0(\mathcal{S}^2, d_{\mathcal{S}^2})$, and furthermore, by Lemma 29, it is the transition semigroup of (u,v). ∎

Establishing that the solution to the Kolmogorov equation is identically zero for $u_0 = v_0$. The work so far in this section has simply been caused by establishing that properties which would be clear and plain and which would require no further justification for a Feller process with finite state space also hold for $(u(t,\cdot), v(t,\cdot))$, where the state space is infinite-dimensional.

Establishing the result. The remainder is now straightforward; the only issue to be resolved is to show that there exists a $\mathcal{U}(\lambda,\mu) \in \mathcal{D}(\mathcal{L})$ which is of a suitable form. Consider the function

$$\mathcal{U}(\lambda,\mu) = (\mathcal{V}_1(\lambda+\mu)\mathcal{V}_1(\lambda-\mu))\mathcal{V}_2(\lambda-\mu)$$

where

$$\mathcal{V}_1(\alpha) = \exp\Big\{-\sum_{n=0}^{\infty} e^{-n} g(\alpha_n\alpha_{-n})\Big\},$$

$$\mathcal{V}_2(\beta) = 1 - \exp\Big\{-\sum_{n=0}^{\infty} e^{-n} f(\beta_n\beta_{-n})\Big\},$$

where $g : \mathbb{C} \to \mathbb{C}$ is a function such that, restricted to $\mathbb{R}_+$, $g(x) = x\mathbf{1}_{[0,1]}(x) + (1+\log x)\mathbf{1}_{[1,\infty)}(x)$ and $f : \mathbb{R}_+ \to [0,2]$ is a non-decreasing function satisfying $f(x) = x$ for $x \in [0,1]$, $\lim_{x\to\infty} f(x) = 2$ and $f'(x) \le 1$, $|f''(x)| < 2$,

$$\sup_x\big((1+|x|)|f'(x)| + |x|\,|f''(x)|\big) < C$$

for a constant $C < \infty$. Such a choice of $\mathcal{U}$ is clearly in $C_0(\mathcal{S})$ (using the definition $|\gamma| = d_{\mathcal{S}}(0,\gamma)$), and furthermore it is straightforward to check that it also satisfies (51) and (53).

Let

$$\begin{aligned}\mathcal{F}(t;\lambda,\mu) &= \mathbb{E}_{(\lambda,\mu)}[\mathcal{U}(\lambda(t),\mu(t))]\\ &= \mathbb{E}_{\lambda,\mu}[\mathcal{V}_1(\lambda(t)+\mu(t))\mathcal{V}_1(\lambda(t)-\mu(t))\mathcal{V}_2(\lambda(t)-\mu(t))].\end{aligned}$$

Now, following Kallenberg [2, Ch. 19, Lemma 19.3, p. 369], it is a consequence of the fact that (u,v) is a Feller process that $\mathcal{F}(t,\lambda,\mu) \xrightarrow{\mu\to\lambda} \mathcal{F}(t,\lambda,\lambda)$ (for a Feller process X with transition semigroup T satisfying $T_tC_0 \subset C_0$ for $t \ge 0$, $X_t^x \xrightarrow{d} X_t^y$ as $x \to y$ for fixed $t \ge 0$; superscript denotes initial condition). Clearly, by construction, there is *a* solution $\mathcal{F}(t;\lambda,\lambda) \equiv 0$ for all $t \ge 0$ and, by uniqueness of the semigroup, this is the unique solution for the initial condition $u_0 = v_0 = \sum_{j=-\infty}^{\infty} \lambda_j e^{2\pi i j}$. Hence, from the definition of $\mathcal{V}_2$, for each n and each fixed $t \in \mathbb{R}_+$, $|\lambda_n(t) - \mu_n(t)| = 0$ $\mathbb{P}$-almost surely when $u_0 = v_0$. Since $\|u - v\|_2^2(t) = \sum_n |\lambda_n(t) - \mu_n(t)|^2$, it follows that for almost all $t > 0$ and all $0 \le N < \infty$, $\mathbb{E}[N \wedge \|u - v\|_2^{2\gamma}(t)] \equiv 0$ for Lebesgue almost all $t \ge 0$.

Let $U(t) = \int_{\mathbb{S}^1} u(t,x)\,dx$ and $V(t) = \int_{\mathbb{S}^1} v(t,x)\,dx$. Then $U - V$ is a *continuous* local martingale. Furthermore, $|U(t) - V(t)| = |\int_{\mathbb{S}^1}(u(t,x) - v(t,x))\,dx| \le (\int_{\mathbb{S}^1}(u(t,x) - v(t,x))^2\,dx)^{1/2}$ so that, $\mathbb{P}$-almost surely, $U(t) - V(t) = 0$ for Lebesgue almost all $t \ge 0$. Using continuity of $V(t) - U(t)$, it follows that $\mathbb{P}$-almost surely, $\sup_{0 \le t \le T} |U(t) - V(t)| = 0$ for any fixed $T < \infty$. Using the fact that $(U(t) - V(t))^2 - \int_0^t \int_{\mathbb{S}^1} (u(s,x)^\gamma - v(s,x)^\gamma)^2\,dx\,ds$ is a *continuous* local martingale, it follows that

$$\lim_{t\to\infty} \int_0^t \int_{\mathbb{S}^1} (u(s,x)^\gamma - v(s,x)^\gamma)^2\,dx\,ds = 0 \qquad \mathbb{P}\text{-almost surely.}$$

Together with the apriori bound

$$\mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} (u(s,x)^\gamma - v(s,x)^\gamma)^2\,dx\,ds\Big)^{\alpha/2}\Big] \le 2^{1+\alpha/2}\widetilde{K}(\alpha), \qquad \alpha \in (0,1),$$

for a universal constant $\widetilde{K}(\alpha) < \infty$ for $\alpha \in (0,1)$, depending only on α, this shows that for $0 < \alpha < 1$,

$$d_{2\gamma,\alpha}(u,v) \le \mathbb{E}\Big[\Big(\int_0^\infty \int_{\mathbb{S}^1} (u(s,x)^\gamma - v(s,x)^\gamma)^2\,dx\,ds\Big)^{\alpha/2}\Big] = 0,$$

thus completing the proof of Theorem 25. ■

6. Existence of norms. Let u denote a solution to equation (1). In this section, the following result is proved.

THEOREM 31. *Let u be a solution to equation* (1) *in $\mathcal{S}_{2\gamma,\alpha}$ for $\alpha < 1$. Let*

$$\|u\|_p(t) = \Big(\int_{\mathbb{S}^1} u(t,x)^p\,dx\Big)^{1/p}.$$

Then for each $p < \infty$ and each $\alpha \in (0,1/2)$ and each $T < \infty$ such that the initial condition u_0 satisfies $\int_0^T \|P_t u_0\|_{2p}^\alpha\,dt < \infty$, there is a constant $C(p,\alpha,T,u_0) < \infty$ such that

$$\mathbb{E}\Big[\int_0^T \|u\|_{2p}^\alpha(t)\,dt\Big] < C(p,\alpha,T,u_0).$$

Proof. Let

$$U(s,t;x) = P_t u_0(x) + \int_0^s \int_{\mathbb{S}^1} p_{t-r}(x-y) u(r,y)^\gamma \, W(dy,dr).$$

Then $u(t,x) = U(t,t;x)$. By Itô's formula,

$$\begin{aligned} U(s,t;x)^{2p} &= (P_t u_0(x))^{2p} \\ &\quad + 2p \int_0^s \int_{\mathbb{S}^1} \big(U(r,t;x)^{2p-1} p_{t-r}(x-y)\big) u(r,y)^\gamma \, W(dy,dr) \\ &\quad + p(2p-1) \int_0^s \int_{\mathbb{S}^1} \big(U(r,t;x)^{2p-2} p_{t-r}^2(x-y)\big) u(r,y)^{2\gamma} \, dy \, dr. \end{aligned}$$

Let $\|U(s,t;x)\|_p = (\int_{\mathbb{S}^1} U(s,t;x)^p \, dx)^{1/p}$. Then, using $\int_{\mathbb{S}^1} p_{t-r}(x-y)^{2p} \, dx \le 1 + \frac{c(p)}{(t-r)^{p-(1/2)}}$ for some $c(p)$ and Hölder's inequality, we get

$$\begin{aligned} \|U(s,t)\|_{2p}^{2p} &\le \|P_t u_0\|_{2p}^{2p} \\ &\quad + 2p \int_0^s \int_{\mathbb{S}^1} \Big(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) \, dx \Big) u(r,y)^\gamma \, W(dy,dr) \\ &\quad + p(2p-1) \int_0^s \bigg(1 + \frac{c(p)}{(t-r)^{1-1/(2p)}} \bigg) \|U(r,t)\|_{2p}^{2p-2} \|u(r)\|_{2\gamma}^{2\gamma} \, dr. \end{aligned}$$

It follows, again by Itô's formula, that

$$\begin{aligned} &\|U(s,t)\|_{2p}^{2pq} \le \|P_t u_0\|_{2p}^{2pq} \\ &+ 2pq \int_0^s \|U(r,t)\|_{2p}^{2p(q-1)} \int_{\mathbb{S}^1} \Big(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) \, dx \Big) u(r,y)^\gamma \, W(dy,dr) \\ &+ p(2p-1)q \int_0^s \bigg(1 + \frac{c(p)}{(t-r)^{1-1/(2p)}} \bigg) \|U(r,t)\|_{2p}^{2pq-2} \|u(r)\|_{2\gamma}^{2\gamma} \, dr \\ &+ 2p^2 q(q-1) \int_0^s \|U(r,t)\|_{2p}^{2p(q-2)} \Big(\int_{\mathbb{S}^1} U(r,t;x)^{2p-1} p_{t-r}(x-y) \, dx \Big)^2 \|u(r)\|_{2\gamma}^{2\gamma} \, dr. \end{aligned}$$

For $0 < q < 1$, the last term is negative and so may be disregarded for obtaining an upper bound. It follows by the Burkholder–Davis–Gundy inequality that for $\alpha \in (0,1/2)$ and $q \in (0,1)$, there are constants $c(\alpha,p,q)$

and $c(p)$ such that

$$\mathbb{E}[\|u(t)\|_{2p}^{2pq\alpha}] \le \|P_t u_0\|_{2p}^{2pq\alpha}$$
$$+ c(\alpha,p,q)\mathbb{E}\Big[\Big(\int_0^t\Big(1+\frac{c(p)}{(t-r)^{1-1/(2p)}}\Big)\|U(r,t)\|_{2p}^{4p(q-1)+4p-2}\|u(r)\|_{2\gamma}^{2\gamma}\,dr\Big)^{\alpha/2}\Big]$$
$$+ c(\alpha,p,q)\mathbb{E}\Big[\Big(\int_0^s\Big(1+\frac{c(p)}{(t-r)^{1-1/(2p)}}\Big)\|U(r,t)\|_{2p}^{2pq-2}\|u(r)\|_{2\gamma}^{2\gamma}\,dr\Big)^{\alpha}\Big].$$

Firstly, by Jensen's inequality, for a non-negative function f and $\beta \in (0,1)$,

$$\int_0^T f(s)^\beta\,ds \le T^{1-\beta}\Big(\int_0^T f(s)\,ds\Big)^\beta$$

and, for $r \in [0,T]$,

$$\int_r^T \frac{1}{(t-r)^{1-1/(2p)}}\,dr \le 2pT^{1/(2p)}.$$

Note that for $2p \ge 1$, $\|U(r,t)\|_{2p} \ge U(r)$, from which it follows, with $q = \frac{1}{2p}$ and $T < \infty$, that there is a constant $c(\alpha,p,T) < \infty$ such that

$$\mathbb{E}\Big[\int_0^T \|u(t)\|_{2p}^\alpha\,dt\Big] \le \int_0^T \|P_t u_0\|_{2p}^\alpha\,dt$$
$$+ c(\alpha,p,T)\Big(\mathbb{E}\Big[\Big(\int_0^T \|u(r)\|_{2\gamma}^{2\gamma}\,dr\Big)^{\alpha/2}\Big]\mathbb{E}\Big[\Big(\int_0^T \frac{1}{U(r)}\|u(r)\|_{2\gamma}^{2\gamma}\,dr\Big)^{\alpha}\Big]\Big).$$

By Itô's formula,

$$U(t)\log U(t) + U(t) = 1 + \int_0^t (2+\log U(s))\,dU(s) + \frac{1}{2}\int_0^t \frac{1}{U(s)}\|u\|_{2\gamma}^{2\gamma}(s)\,ds,$$

so that for $\alpha < 1/2$, using Hölder's inequality, there is a $c(\alpha) < \infty$ such that

$$\frac{1}{2^\alpha}\mathbb{E}\Big[\Big(\int_0^T \frac{1}{U(r)}\|u\|_{2\gamma}^{2\gamma}(r)\,dr\Big)^\alpha\Big] \le 1 + \mathbb{E}[|U(T)\log U(T)|^\alpha]$$
$$+ \mathbb{E}[U(T)^\alpha] + c(\alpha)\mathbb{E}\Big[\Big(\int_0^T (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)\,ds\Big)^{\alpha/2}\Big].$$

Again, by Itô's formula,

$$\frac{15}{4}U(t)^2 - \frac{3}{2}U(t)^2\log U(t) + \frac{1}{2}U(t)^2(\log U(t))^2$$
$$= \frac{15}{4} + \int_0^t \big(6U(s) - 2U(s)\log U(s) + U(s)(\log U(s))^2\big)\,dU(s) + \frac{1}{2}\int_0^t (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)\,ds,$$

giving, for $\alpha \in (0,1/2)$,

$$\mathbb{E}\Big[\Big(\int_0^t (2+\log U(s))^2\|u\|_{2\gamma}^{2\gamma}(s)\,ds\Big)^\alpha\Big]$$
$$\le \big(\tfrac{15}{2}\big)^\alpha(1+\mathbb{E}[U(t)^{2\alpha}]) + 3^\alpha\mathbb{E}[U(t)^{2\alpha}|\log U(t)|^\alpha]$$
$$+\mathbb{E}[U(t)^{2\alpha}(\log U(t))^{2\alpha}] + c(\alpha)\mathbb{E}\Big[\Big(\int_0^t (6U(s)-2U(s)\log U(s)+U(s)(\log U(s))^2)^2\|u\|_{2\gamma}^{2\gamma}(s)\Big)^{\alpha/2}\Big]$$
$$\le \big(\tfrac{15}{2}\big)^\alpha\Big(1+\mathbb{E}\Big[\Big(\sup_t U(t)\Big)^{2\alpha}\Big]\Big) + 3^\alpha\mathbb{E}\Big[\Big(\sup_t U(t)|\log U(t)|^{1/2}\Big)^{2\alpha}\Big] + \mathbb{E}\Big[\sup_t (U(t)|\log U(t)|)^{2\alpha}\Big]$$
$$+ c(\alpha)\mathbb{E}\Big[\Big(\sup_t\big(6U(t)+2U(t)|\log U(t)| + U(t)(\log U(t))^2\big)\Big)^{2\alpha}\Big]^{1/2}\mathbb{E}\Big[\Big(\int_0^\infty \|u\|_{2\gamma}^{2\gamma}(s)\,ds\Big)^\alpha\Big]^{1/2}$$
$$< \infty.$$

Theorem 31 follows. ∎

7. Conclusion. In this article, existence and uniqueness of solutions to equation (1) in appropriate spaces was established, thus answering the question posed in Mueller [10], of whether the solution could be continued after explosion of the L^∞ norm; for all time, there exists a unique solution in $L^{2\gamma}$. One conjecture is that the solution is identically 0 after the explosion time; it has not been proved that the solution is strictly positive for all finite time.

Appendix: Comparison of Dirichlet boundary conditions and solution on the circle. The appendix gives a sketch of proof that explosion time of the L^∞ norm for the problem posed on $\mathbb{S}^1$ is less than or equal to that for the problem with Dirichlet boundary conditions. This is stated as a theorem. Only a sketch of the proof is provided.

THEOREM 32. *Let u and v solve the problem*

$$\begin{cases} f_t = \frac{\kappa}{2}f_{xx} + f^\gamma\xi, \\ f(0,x) = u_0(x), \quad 0<x<1, \end{cases}$$

with different boundary conditions; v is the solution with the Dirichlet boundary conditions $v(t,0) = v(t,1) = 0$ for all $t>0$, while u is the solution to the problem on $\mathbb{S}^1$; $\mathbb{S}^1 = [0,1]$ with the identification $0 = 1$. Let $\tau_{u,N} = \inf\{t : \sup_x u(t,x) > N\}$ and $\tau_{v,N} = \inf\{t : \sup_x v(t,x) > N\}$.

Let $\tau_u = \lim_{N\to\infty} \tau_{u,N}$ and let $\tau_v = \lim_{N\to\infty} \tau_{v,N}$. Then, for each $N < \infty$, $\tau_{u,N} \le \tau_{v,N}$ and hence $\tau_u \le \tau_v$.

Sketch of proof. Let $w = u - v$. The basic idea is as follows: in principle, w should satisfy

$$\begin{cases} w_t = \frac{\kappa}{2} w_{xx} + ((v+w)^\gamma - v^\gamma)\xi, \\ w(0,x) = 0, \quad x \in (0,1), \\ w(t,0) = w(t,1) = u(t,0), \quad t \ge 0. \end{cases}$$

This would clearly be the case if u and v were twice differentiable in the space variable. For $t \in (0, \tau_{u,N} \wedge \tau_{v,N})$,

$$F(t,x) := \frac{(v+w)^\gamma - v^\gamma}{w}$$

satisfies $|F(t,x)| \le \gamma N^{\gamma-1}$. If w were twice differentiable in the x variable, it would be straightforward to show that $w \ge 0$ on $(0, \tau_{u,N} \wedge \tau_{v,N})$ for any $N < \infty$ and hence $\tau_{u,N} \le \tau_{v,N}$ for each N, so $\tau_u \le \tau_v$, where $\tau_u = \lim_{N\to\infty} \tau_{u,N}$ and $\tau_v = \lim_{N\to\infty} \tau_{v,N}$.

Therefore, consider approximations $u^{(\epsilon)}$, $v^{(\epsilon)}$ and $w^{(\epsilon)} := u^{(\epsilon)} - v^{(\epsilon)}$ which are infinitely differentiable in the space variable, constructed as follows. Let W denote the Wiener sheet defined on $\mathbb{S}^1 \times \mathbb{R}_+$ (Definition 5). Let $\zeta^{(\epsilon)}$ denote the Gaussian field defined by

$$\zeta^{(\epsilon)}(t,x) = \int_{\mathbb{S}^1} \widetilde{p}(\epsilon; x-y)\, W(dy, [0,t])$$

where $\widetilde{p} : \mathbb{R}_+ \times \mathbb{S}^1 \to \mathbb{R}_+$ solves $\widetilde{p}_t = \frac{1}{2}\widetilde{p}_{xx}$ with initial condition $\widetilde{p}(0,x) = \delta_0(x)$. Note that $\zeta^{(\epsilon)}$ is Gaussian, with

$$\mathbb{E}[\zeta^{(\epsilon)}(t,x)] = 0, \qquad \mathbb{E}[\zeta^{(\epsilon)}(s,x)\zeta^{(\epsilon)}(t,y)] = (s \wedge t) \int_0^1 p(\epsilon; x-z) p(\epsilon; y-z)\, dz$$

and furthermore, for $\epsilon > 0$, $\zeta^{(\epsilon)}$ is infinitely differentiable in the space variable. This may be used on $\mathbb{S}^1$ for the equation for $u^{(\epsilon)}$ and taken on $(0,1)$ for $v^{(\epsilon)}$, the equation on $[0,1]$ with Dirichlet boundary conditions in (60) defined below; let $u^{(\epsilon)}$ and $v^{(\epsilon)}$ solve

$$\begin{cases} \partial_t f^{(\epsilon)} = \frac{\kappa}{2} f^{(\epsilon)}_{xx}\, dt + f^{(\epsilon)\gamma} \partial_t \zeta^{(\epsilon)}, \\ f^{(\epsilon)}(0,x) = u_0(x), \quad 0 < x < 1, \end{cases} \tag{60}$$

where $u^{(\epsilon)}$ is the solution on the circle and $v^{(\epsilon)}$ has the boundary conditions $v^{(\epsilon)}(t,0) = v^{(\epsilon)}(t,1) = 0$ for all $t > 0$. Let

$$\tau^{(\epsilon)}_{u,N} = \inf\{t : \sup_x u^{(\epsilon)}(t,x) > N\}, \qquad \tau^{(\epsilon)}_{v,N} = \inf\{t : \sup_x v^{(\epsilon)}(t,x) > N\}.$$

For $\epsilon > 0$, the time differentials are *stochastic* differentials and the stochastic integral is an Itô integral. By construction of $\zeta^{(\epsilon)}$ (which is infinitely differentiable in the space variable for all $\epsilon > 0$) and standard results (found, for example in Kunita [5]), for $0 < t < \tau_{u,N}^{(\epsilon)}$ and all $N < \infty$, $u^{(\epsilon)}(t, \cdot)$ is infinitely differentiable and for $0 < t < \tau_{v,N}^{(\epsilon)}$ and all $N < \infty$, $v^{(\epsilon)}$ is infinitely differentiable. Therefore, for $x \in (0, 1)$ both $u_{xx}^{(\epsilon)}(t, \cdot)$ and $v_{xx}^{(\epsilon)}(t, \cdot)$ are well defined infinitely differentiable functions (in x) for all $t > 0$.

Let

$$F^{(\epsilon)}(t, x) = \frac{(v^{(\epsilon)} + w^{(\epsilon)})(t, x)^\gamma - v^{(\epsilon)}(t, x)^\gamma}{w^{(\epsilon)}(t, x)},$$

so that, for $0 < t < \tau_{u,N}^{(\epsilon)} \wedge \tau_{v,N}^{(\epsilon)}$, $\sup_x |F^{(\epsilon)}(t, x)| \le \gamma N^{\gamma-1}$. Then $w^{(\epsilon)}$ solves

$$\begin{cases} \partial_t w^{(\epsilon)} = \frac{\kappa}{2} w_{xx}^{(\epsilon)}\, dt + F^{(\epsilon)}(t, x) w^{(\epsilon)} \partial_t \zeta^{(\epsilon)}, \\ w^{(\epsilon)}(0, x) = 0, \qquad 0 < x < 1, \\ w^{(\epsilon)}(t, 0) = w^{(\epsilon)}(t, 1) = u^{(\epsilon)}(t, 0), \quad \forall t > 0, \end{cases}$$

and $w^{(\epsilon)}(t, \cdot) \in C^\infty((0, 1))$ for all $0 < t \wedge \tau_{u,N}^{(\epsilon)} \wedge \tau_{v,N}^{(\epsilon)}$ and all $N < \infty$. It then follows almost directly from the equation, using the fact that the boundary conditions for $t > 0$ are positive, that $w^{(\epsilon)}(t, x) \ge 0$ for all $0 < t < \tau_v^{(\epsilon)} \wedge \tau_u^{(\epsilon)}$ and $x \in [0, 1]$.

Indeed, let $\sigma = \inf \{t : \inf_x w^{(\epsilon)}(t, x) < 0\}$ and $x_{\min}$ the point such that $w^{(\epsilon)}(\sigma, x_{\min}) = 0$. Then $w_{xx}^{(\epsilon)}(\sigma, x_{\min}) > 0$ and hence $\frac{d}{dt} w^{(\epsilon)}(t, x_{\min})|_{t=\sigma} > 0$, contradicting the definition of σ (the notation $\frac{d}{dt}$ may be employed here, since at time σ, the stochastic term is 0).

Hence $w^{(\epsilon)} \ge 0$ on $(0, \tau_{u,N}^{(\epsilon)} \wedge \tau_{v,N}^{(\epsilon)})$, so that $\tau_{u,N}^{(\epsilon)} \le \tau_{v,N}^{(\epsilon)}$ for each $N < \infty$, hence $\tau_u^{(\epsilon)} \le \tau_v^{(\epsilon)}$.

Finally, we have to establish that $u^{(\epsilon)}(\cdot \wedge \tau_{u,N}^{(\epsilon)} \wedge \tau_{u,N}) - u(\cdot \wedge \tau_{u,N}^{(\epsilon)} \wedge \tau_{u,N}, \cdot) \to 0$ and that $v^{(\epsilon)}(\cdot \wedge \tau_{v,N}^{(\epsilon)} \wedge \tau_{v,N}, \cdot) - v(\cdot \wedge \tau_{v,N}^{(\epsilon)} \wedge \tau_{v,N}, \cdot) \to 0$ for each $N < \infty$ in an appropriate sense.

Consider $u^{(\epsilon)} - u$; the arguments for $v^{(\epsilon)} - v$ are similar. Note that, for $0 \le t < \tau_{u,N}^{(\epsilon)}$, we have $u^{(\epsilon)} = \widetilde{u}^{(\epsilon)}$ where $\widetilde{u}^{(\epsilon)}$ satisfies

$$\partial_t \widetilde{u}^{(\epsilon)} = \frac{\kappa}{2} \widetilde{u}_{xx}^{(\epsilon)}\, dt + (\widetilde{u}^{(\epsilon)} \wedge N)^\gamma \partial_t \zeta^{(\epsilon)}$$

(taking on $\mathbb{S}^1$ the same initial condition as for $u^{(\epsilon)}$). A key result, for which details are omitted, is that on $0 < t < M$, for any $M < \infty$ and $p \ge 4$,

$$\mathbb{E}[|\widetilde{u}^{(\epsilon)}(t, x + h) - \widetilde{u}^{(\epsilon)}(t, x)|^{2p}] \le C_{1,2p} h^{p-1},$$
$$\mathbb{E}[|\widetilde{u}^{(\epsilon)}(t + k, x) - \widetilde{u}^{(\epsilon)}(t, x)|^{2p}] \le C_{2,2p} k^{p/2-4},$$

where the constants $C_{1,2p}$ and $C_{2,2p}$ depend on M and N, but (crucially) do not depend on ϵ. This may be established following the lines of proof given in Walsh [18, proof of Corollary 3.4, p. 318].

Let $\widetilde{u}$ satisfy

$$\widetilde{u}_t = \frac{\kappa}{2}\widetilde{u}_{xx} + (\widetilde{u} \wedge N)^\gamma \xi$$

with initial condition $\widetilde{u}(0,x) = u_0(x)$. It is reasonably straightforward to show that there is a continuous function $\mathbf{G}(\epsilon)$ with $\mathbf{G}(0) = 0$ and constants $C_{3,2p} < \infty$ such that for $x \in \mathbb{S}^1$, $0 < t < M$ (where $M < \infty$) and $\epsilon \in (0,1)$,

$$\mathbb{E}[|\widetilde{u}^{(\epsilon)}(t,x) - \widetilde{u}(t,x)|^{2p}] \le C_{3,2p}\mathbf{G}(\epsilon)^p.$$

This is seen as follows: letting $p(t,\cdot)$ denote the heat kernel obtained by solving $p_t = \frac{\kappa}{2}p_{xx}$ with initial condition $p(0,x) = \delta_0(x)$ on $\mathbb{S}^1$,

$$\begin{aligned}
&(\widetilde{u}^{(\epsilon)} - \widetilde{u})(t,x) \\
&\quad = \int_0^t\int_0^1\Big(\int p(t-s,x-z)(\widetilde{u}^{(\epsilon)} \wedge N)(s,z)^\gamma \widetilde{p}(\epsilon,z-y)\,dz - p(t-s,x-y)(\widetilde{u} \wedge N)(s,y)^\gamma\Big) \\
&\qquad\qquad \times W(dy,ds) \\
&\quad = \int_0^t\int_0^1 p(t-s,x-y)\big((\widetilde{u}^{(\epsilon)} \wedge N)(s,y)^\gamma - (\widetilde{u} \wedge N)(s,y)^\gamma\big)\,W(dy,ds) \\
&\qquad + \int_0^t\int_0^1\Big(\int p(t-s,x-y)(\widetilde{u}^{(\epsilon)} \wedge N)(s,y)^\gamma(\widetilde{p}(\epsilon,y-z) - \delta_0(y-z))\,dy\Big)\,W(dz,ds),
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}[(\widetilde{u}^{(\epsilon)}(t,x) - \widetilde{u}(t,x))^{2p}] \\
&\le C(p)\mathbb{E}\Big[\Big(\int_0^t\int p(t-s,x-y)^2\big((N \wedge \widetilde{u}^{(\epsilon)})(s,y)^\gamma - (N \wedge \widetilde{u})(s,y)^\gamma\big)^2\,dy\,ds\Big)^p\Big] \\
&\quad + C(p)N^{2\gamma p}\Big(\int_0^t\int\int p(t-s,x-y_1)p(t-s,x-y_2) \\
&\quad \times \Big(\int \widetilde{p}(\epsilon,y_1-z)\widetilde{p}(\epsilon,y_2-z)\,dz - 2\widetilde{p}(\epsilon,y_1-y_2) + \delta_0(y_1-y_2)\Big)\,dy_1\,dy_2\,ds\Big)^p.
\end{aligned}$$

Now let

$$\begin{aligned}
\mathcal{F}(t,\epsilon) = N^{2\gamma}&\int_0^t\int\int p(t-s,x-y_1)p(t-s,x-y_2) \\
&\times \Big(\int \widetilde{p}(\epsilon,y_1-z)\widetilde{p}(\epsilon,y_2-z)\,dz - 2\widetilde{p}(\epsilon,y_1-y_2) + \delta_0(y_1-y_2)\Big)\,dy_1\,dy_2\,ds.
\end{aligned}$$

Then clearly $\mathcal{F}$ is non-negative, $\mathcal{F}(t,0) = 0$, $\mathcal{F}(0,\epsilon) = 0$, $\mathcal{F}$ is increasing in t and in ϵ. Furthermore, let $\mathcal{M}(t) = \sup_{0\le s\le t}\sup_x \mathbb{E}[(\widetilde{u}^{(\epsilon)} - \widetilde{u})(t,x)^{2p}]$. Then easy applications of Jensen's inequality give

$$\mathcal{M}(t) \leq C(p)\mathcal{F}(t,\epsilon)^p + C(p)\gamma^p N^{p(\gamma-1)}\Big(\int_0^t\int p(t-s,x-y)^2\,dy\,ds\Big)^{p-1} \times \int_0^t \mathcal{M}(s)\Big(\int p(t-s,x-y)^2\,dy\Big)\,ds.$$

It is straightforward to establish existence of non-negative constants $C_1, C_2 < \infty$ such that $\int_{\mathbb{S}^1} p(s,x)^2\,dx \leq C_1 + C_2/\sqrt{s}$ and Gronwall style arguments, of the type found in Walsh [18] (for example: bottom of p. 314, top of p. 315, in the proof of Theorem 3.2) give the results.

The constants $C_{1,2p}, C_{2,2p}, C_{3,2p}$ depend on M and N, but (crucially) do not depend on ϵ. Similarly to the treatment of Walsh, this may be used to establish Kolmogorov continuity in all three variables ϵ, t, x, so that the family $(\widetilde{u}^{(\epsilon)})_{\epsilon\geq 0}$ is *equi-continuous* almost surely. L^p convergence has been established for all $p < \infty$; L^2 convergence together with equi-continuity implies uniform convergence. Therefore, for each $N < \infty$, $\widetilde{u}^{(\epsilon)} \to \widetilde{u}$ uniformly, from which $\tau^{(\epsilon)}_{u,N} \to \tau_{u,N}$ almost surely and $u^{(\epsilon)}(.\wedge\tau^{(\epsilon)}_{u,N},.)$ converges to $u(.\wedge\tau_{u,N},.)$ as required.

The sketch of proof is complete. ■

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