Partial exam. Wednesday 11 May, 12:15–14:00

1. Show that the following problem is \( NP \)-complete.

   Given an alphabet \( A \) and a regular expression \( \alpha \) over \( A \), decide if there is a word generated by \( \alpha \) which contains every letter from \( A \).

   *Hint.* For \( NP \)-hardness, you may reduce \( CNF-SAT \). Note that the alphabet \( A \) is not fixed and may depend on the formula.

2. We consider the words of the form

   \[ w = w_1 w_2 \ldots w_{2^m-1} w_{2^m} \]

   with \( w_i \in \{0, 1\}^m \), for \( i = 1, \ldots, 2^m \). Let the language \( L \) consists of all words \( w \) in the above form, in which the number of different blocks \( w_i \) is even. Show that \( L \) can be accepted by a sequence of circuits \( C_n \) of polynomial size and depth \( O(\log n) \).

3. We consider a grid \( n \times n \) with the nodes colored black or white. (It can be encoded as a word in \( \{0, 1\}^{n^2} \) in an obvious manner.) Show that a deterministic Turing machine can check in logarithmic space whether there is a monochromatic path from the topmost level to the lowest level.

4. Show that the complexity class \( P \) is closed under morphic images w.r.t. non-zero morphisms iff \( P = NP \).

   *Hint.* For the only if direction, use problem \( CNF-SAT \).

   *Reminder.* A morphism is defined by a mapping \( h : \Sigma \rightarrow \Sigma^* \), which is extended to \( \hat{h} : \Sigma \rightarrow \Sigma^* \) by

   \[ \hat{h}(\varepsilon) = \varepsilon \]

   \[ h(v w) = h(v) h(w). \]

   The morphic image of a language \( L \subseteq \Sigma^* \) is \( \{\hat{h}(w) : w \in L\} \). A morphism is non-zero iff \( (\forall \sigma \in \Sigma) h(\sigma) \neq \varepsilon \).

   *Remark.* The necessity of the assumption that the morphism is non-zero was noticed during the exam. This yields an additional question: Why the claim fails without this assumption?