Partial exam. Wednesday 11 May, 12:15–14:00

1. Show that the following problem is *NP*-complete.

   Given an alphabet A and a regular expression α over A, decide if there is a word generated by α which contains every letter from A.

   Hint. For *NP*-hardness, you may reduce *CNF-SAT*. Note that the alphabet A is not fixed and may depend on the formula.

2. We consider the words of the form

   $$w = w_1 w_2 \dots w_{2^m-1} w_{2^m}$$

   with $w_i \in \{0,1\}^m$, for $i = 1, \dots, 2^m$. Let the language L consists of all words w in the above form, in which the number of different blocks w_i is even. Show that L can be accepted by a sequence of circuits C_n of polynomial size and depth $\mathcal{O}(\log n)$.

3. We consider a grid $n \times n$ with the nodes colored black or white. (It can be encoded as a word in $\{0,1\}^{n^2}$ in an obvious manner.) Show that a deterministic Turing machine can check in logarithmic space whether there is a monochromatic path from the topmost level to the lowest level.

4. Show that the complexity class P is closed under morphic images w.r.t. **non-zero** morphisms iff $P = NP$.

   Hint. For the *only if* direction, use problem *CNF-SAT*.

   Reminder. A morphism is defined by a mapping $h : \Sigma \to \Sigma^*$, which is extended to $\hat{h} : \Sigma \to \Sigma^*$ by

   $$\begin{aligned} \hat{h}(\varepsilon) &= \varepsilon \\ h(vw) &= h(v)\, h(w). \end{aligned}$$

   The morphic image of a language $L \subseteq \Sigma^*$ is $\{\hat{h}(w) : w \in L\}$. A morphism is **non-zero** iff $(\forall \sigma \in \Sigma)\, h(\sigma) \neq \varepsilon$.

   Remark. The necessity of the assumption that the morphism is non-zero was noticed during the exam. This yields an additional question: Why the claim fails without this assumption ?