A simple indeterminate infinite game

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1 Introduction

One of the simplest and yet non-trivial results in foundations of mathematics, which can be presented to high school pupils, is Theorem of Zermelo on the determinacy of the game of chess [14]: either Black or White has a winning strategy, or both players have strategies to achieve a draw.

The proof smoothly generalizes the problem to all finite games with *perfect information*[1], and reveals an algorithmic aspect of the result. The message however remains incomplete if we hide from our students that even perfect information games can be indeterminate if we allow the players to make infinite sequences of moves.

Infinite games have a long history started with mathematical puzzles considered by Stefan Banach and Stanisław Mazur in the 1930s; see [10] for concise introduction and [13] for historical overview. Although such games can hardly be played in real life, they are vital for multiple areas of mathematics—in particular descriptive set theory [6, 9], and also for computer science; see [4] for a survey.

Indeterminacy of perfect information two player infinite games was observed in the first published work on such games [3]. It means that, even though an actual choice of players' strategies always determines the result of the game, a winning strategy may not exist for any of the players, which makes the result of the game indeterminate. From teaching perspective, we can note a remote analogy with real life games. Indeed, the result of a chess match is "indeterminate" for us because the players presumably ignore the optimal strategies. But we know from the Zermelo Theorem that such optimal strategies exist. In contrast, for infinite games, the winning (or optimal) strategies may not exist at all.

The existence of indeterminate games can be proved in many ways, which however usually introduce some bits of mathematics unknown to high school pupils. Indeterminate games can be constructed by a diagonal argument using

[1] In a two player perfect information game both players know the current state of the game, and the future depends uniquely on their moves, see, e.g., [10].

transfinite induction (see, e.g., [5]); their existence can be also inferred from a classical result in general topology, namely the Bernstein partition of a Polish space into two parts, none of which contains a perfect set[2] (see, e.g., [10]). Another argument, game-theoretic in flavour, can be used to show indeterminacy of a certain game based on a Fréchet ultrafilter; the authors learned this example from the unpublished lecture notes [3] by Jacques Duparc[2]. This argument uses the *strategy stealing* technique, which is well known for solving finite games, in particular Hex. It is based on the following reasoning:

> If player B had a winning strategy, it could be used to construct a winning strategy for player A.

As finite games are determined, this implies that player A has winning strategy. In case of infinite games however, the same argument implies indeterminacy, assuming that the dependence above holds in both directions (i.e., A $\rightarrow$ B, and B $\rightarrow$ A).

In the present note, we use a similar argument to show the indeterminacy of a conceptually simpler game based on an infinite XOR function, i.e., a function $f : \{0,1\}^\omega \rightarrow \{0,1\}$, such that the change of one bit in an argument changes the value. (From $\{0,1\}^n$ to $\{0,1\}$, there are clearly only two such functions: XOR and $\neg$ XOR.) We took this idea from Michael Sipser [12], who showed that infinite XOR functions cannot be computed by *countable* Boolean circuits. This unpublished work constitutes an interesting link between set-theoretic topology and complexity theory. Recall that the celebrated Furst-Saxe-Sipser Theorem shows that (finitary) XOR function cannot be computed by polynomial-size circuits of bounded depth, and according to [12], the solution to the infinitary problem "proved to be useful to direct the search for the solution to the finitary one".

Bibliographical note. The proof presented in this paper has been invented by the first author for a course held by the second author at the University of Warsaw in 2008–2009. The second author used this proof in his talk at the congress on *Square of Opposition* [11]. We have been aware that the indeterminacy of the infinite XOR games is not surprising *per se*, as the winning criteria are non-measurable here.

After having submitted the first version of this article, we have learned that infinite XOR functions are familiar in descriptive set theory under the name of *flip sets* (more precisely, XOR functions are characteristic functions of flip sets). More importantly, also the idea of our proof was known to the descriptive set theory community before. In particular Yurii Khomskii presented an essentially the same proof (in terms of flip sets) in his *Intensive course on Infinite Games* at Sofia University [7]; the origins of the idea can be traced back further, but are hard to fix at this moment [8]. The idea is so natural that could likely be

[2]Because the set of plays consistent with any strategy (in particular, a hypothetical winning strategy) forms a perfect set.

[3]A similar proof appears in lecture notes by Alessandro Andreta [1].

found by more people independently, although we are not aware of any published source except for the Internet publications mentioned above.

In any case, we do not claim priority here, but hope the example can be useful, in particular in teaching computer science students. Indeed, it uses only a tiny portion of abstract mathematics, and relies on the XOR function well familiar to programmers. And we would like to present it to the volume dedicated to Victor Selivanov, as it touches one of his favorite topics — teaching of logic.

2 Infinite XOR game

Let $B = \{0, 1\}$. For two words $v, w \in B^{\mathfrak{m}}$, where $\mathfrak{m} \leq \omega$, let $\mathrm{hd}(v, w) = |\{i : v_i \neq w_i\}|$ be the Hamming distance between v and w. For $v, w \in B^\omega$, we let $v \sim w$ iff $\mathrm{hd}(v, w) < \omega$.

Definition 2.1 *An* **infinite XOR function** $f : B^\omega \to B$ *is a function with the following property: if* $\mathrm{hd}(w_1, w_2) = 1$ *then* $f(w_1) \neq f(w_2)$.

Theorem 2.2 *There exist* $2^{\mathfrak{c}}$ *infinite XOR functions.*

Proof We use the Axiom of Choice. Let S be a set which contains exactly one element from each equivalence class of $\sim$. For $w \in B^\omega$, let $r(w)$ be the element of S such that $w \sim r(w)$. We define $f(w) = \mathrm{hd}(w, r(w)) \bmod 2$. One easily checks that f is an infinite XOR function.

To produce $2^{\mathfrak{c}}$ such functions, observe first that $|S| = \mathfrak{c}$, as each equivalence class of $\sim$ is countable. Then, for each $\alpha : S \to \{0, 1\}$, we obtain a different infinite XOR function given by $f_\alpha(w) = (f(w) + \alpha(r(w))) \bmod 2$. ■

Definition 2.3 *Let f be an infinite XOR function. The* **infinite XOR game** *G_f is played as follows. Player 0 picks a word $w_0 \in B^+$. Then, Player 1 picks a word $w_1 \in B^+$. Player 0 picks a word $w_2 \in B^+$, Player 1 picks a word $w_3 \in B^+$, and so on. Thus, we obtain a* **play** *which is an infinite sequence of words: $w_0 w_1, w_2, w_3, \ldots$ Player i wins iff $f(w_0 w_1 w_2 w_3 \ldots) = i$.*

Definition 2.4 *A* **strategy for player** *i in G_f is a function*

$$S : \bigcup_{k \in \omega} (B^+)^{2k+i} \to B^+.$$

A play $w_0, w_1, w_2, \ldots$ is **consistent** *with S iff $w_{k+1} = S(w_0, w_1, \ldots, w_k)$, for each suitable k (i.e., each move of player i is given by S). S is* **winning** *iff Player i wins each play consistent with S.*

Note that in the above we view $(B^+)^m$ as a product $B^+ \times B^+ \times \ldots \times B^+$ (m times) rather than concatenation $B^+ B^+ \ldots B^+$ (m times). Such an identification would restrict the set of strategies, but in fact it would not affect our result. Note that, by definition, $(B^+)^0 = \{\emptyset\}$.

We use the strategy stealing argument to show that no player has a winning strategy in the infinite XOR game. Intuitively, whenever our opponent answers our move v with w, we could have instead changed one bit in v to obtain another word v', and play $v'w$ instead of v. This effectively exchanges the roles of the two players, so if our opponent had a winning strategy, we can use it now for ourselves. The precise argument follows.

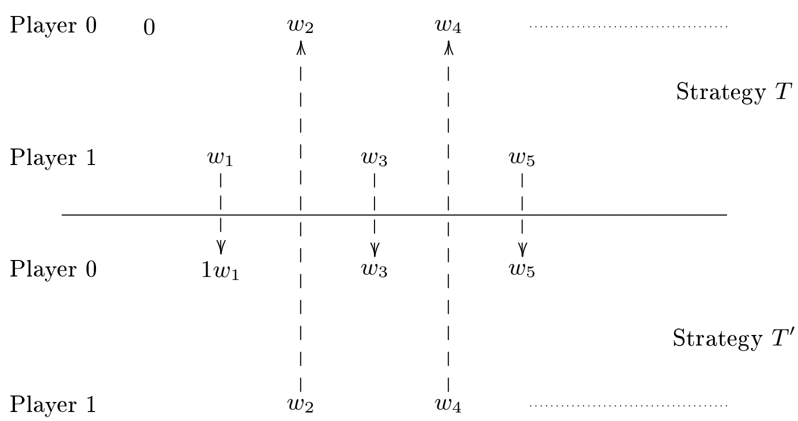
Theorem 2.5 *No player has a winning strategy in an infinite XOR game G_f.*

Proof Let S be a strategy for Player $1-i$. We construct two strategies for Player i, T and T', such that one of them will win at least one play against S.

Consider first $i = 0$, and let the first move of Player 0 (who starts the game) be $T(\emptyset) = 0$. Suppose the answer of Player 1 is $S(0) = w_1$. We let $T'(\emptyset) = 1w_1$. Now, if $S(1w_1) = w_2$ then we let $T(0, w_1) = w_2$, and if $S(0, w_1, w_2) = w_3$, we let $T'(1w_1, w_2) = w_3$, and so on. In symbols, we let

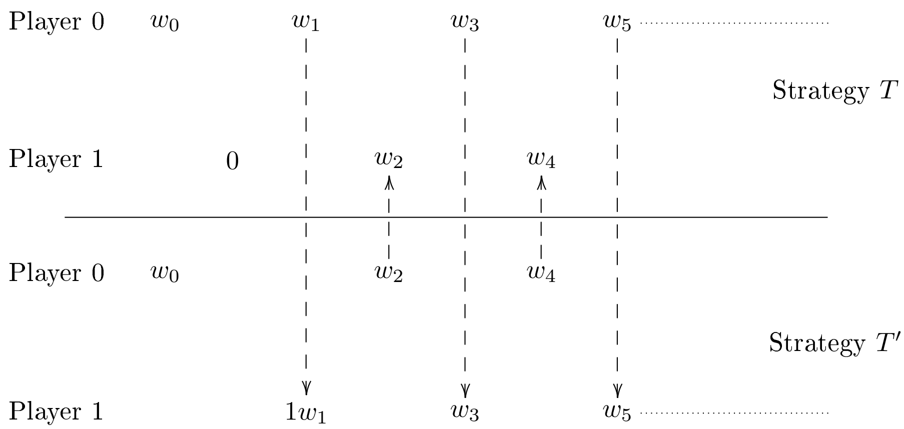
$$\begin{aligned} T'(1w_1, w_2, \ldots, w_{2k}) &= S(0, w_1, \ldots, w_{2k}) \\ T(0, w_1, \ldots, w_{2k+1}) &= S(1w_1, w_2, \ldots, w_{2k+1}). \end{aligned}$$

In the figure below, the dashed arrows indicate the "stealing".



Note that in the two plays above Player 1 uses his strategy S, but the resulting sequences differ exactly in one bit (actually the bit number 0), hence one of the plays is lost by Player 1.

The argument for $i = 1$ is similar. Let the starting move of Player 0 be $S(\emptyset) = w_0$. We let $T(w_0) = 0$. Now suppose $S(w_0, 0) = w_1$. We let $T'(w_0) = 1w_1$. If $S(w_0, 1w_1) = w_2$, we let $T(w_0, 0, w_1) = w_2$, and so on, as represented on the figure below.



Analogically as above, Player 0 uses her strategy S, but the resulting sequences differ exactly in one bit (namely the bit number $|w_0|$), hence this strategy cannot be winning.

Hence the game G_f is indeed indeterminate. ■

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