Trees with decidable theories

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Decidable vs. undecidable

Turing, Church (1936). Arithmetic of natural numbers is undecidable.

All “interesting” mathematical theories are undecidable.

But

- Decidability of mathematical theories is crucial in automatic verification.
- Delimitating decidable fragments of an undecidable theory (e.g., arithmetics) reveals a fine structure of the theory.
Büchi (1960). Monadic second order theory (MSO) of $\langle \omega, \text{succ} \rangle$ is decidable. This subsumes, among others,

Presburger (1929). First order theory of $\langle \omega, + \rangle$ is decidable.

$\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & + \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
\end{array}$
Rabin (1969). MSO theory of $\mathbb{T}_2 = \langle 2^*, \text{succ}_0, \text{succ}_1 \rangle$ is decidable.

This subsumes, among others,

Skolem (1930). First order theory of $\langle \omega, \cdot \rangle$ is decidable.
A great number of decidability results follows from Rabin’s theorem.

An equivalent formalism of tree automata is used for better complexity bounds.

An interpretation of a structure $A \hookrightarrow T_2$ yields decidability of $Th(A)$.

Another construction interprets all models of a formula.

$$\varphi \mapsto \Phi(X)$$
$$A \models \varphi \iff T_2 \models \Phi[A]$$

This yields decidability of the satisfiability problem for numerous logics with the tree model property.

Grädel & Walukiewicz (1999). Guarded first-order logic with fixed points is decidable.
Generalizations of Rabin’s Theorem

Courcelle & Walukiewicz (1997). The MSO theory of the unfolding of a graph reduces to the MSO theory of the original graph.

What about different shapes of trees?

MSO theory of a recursive tree can be $\Pi_1^1$-hard (cf. Thomas 2010).
On positive side

MSO theories of *algebraic* trees are decidable (cf. Courcelle 1995).
Caucał observed (in 1990s) that alternating interpretation and unfolding gives rise to a rich family of trees. This resulted in Caucał’s hierarchy (2002).
Generating trees by 1st order grammars (algebraic)
Generating trees by 2nd order grammars

\[ S \Rightarrow \phi gc \]
\[ \phi \xi x \Rightarrow f(\xi x) (\phi(Copy\xi)x) \]
\[ Copy\xi z \Rightarrow \xi(\xi z) \]
Higher-order tree grammars — definitions

Types $\mathcal{T}$ \quad $\tau ::= 0 \mid \tau \rightarrow \tau$

Nonterminals $N = \{N_\tau\}_{\tau \in \mathcal{T}}$

Variables $X = \{X_\tau\}_{\tau \in \mathcal{T}}$

Signature constants $f, g, c, \ldots : 0^k \rightarrow 0$

Grammar $G = (\Sigma, V, S, E)$

with $\Sigma$ a signature, $V \subseteq \bigcup_{\tau \in \mathcal{T}} N_\tau$, $V \ni S : 0$,

and $E$ a finite set of productions of the form

$$F z_1 \ldots z_m \Rightarrow w$$

with $V \ni F : \tau_1 \rightarrow \tau_2 \cdots \rightarrow \tau_m \rightarrow 0$, $z_i \in X_{\tau_i}$,

and $w$ an applicative term over $\Sigma \cup V \cup \{z_1 \ldots z_m\}$ of type 0.
Derivations

We assume that a grammar $G$ is deterministic, i.e., one production per nonterminal.

Hence there is a unique outermost derivation

$$S = t_0 \rightarrow_G t_1 \rightarrow_G t_2 \rightarrow_G \ldots$$

producing the tree $[G]$ generated by $G$.

Levels

$$\ell(0) = 0, \quad \ell(\tau_1 \rightarrow \tau_2) = \max(1 + \ell(\tau_1), \ell(\tau_2))$$
The model checking problem

Given a grammar $G$ and a formula $\varphi$, decide if $[G] \models \varphi$.

Here, a tree $t : \{1, 2, \ldots, M\}^* \supseteq \text{dom } t \rightarrow \{f, g, c, \ldots\}$ is considered as a logical structure

$$t = \langle \text{dom } t, f^t, g^t, c^t, \ldots, \text{succ}^t_1, \ldots, \text{succ}^t_M \rangle$$

where $f^t(w) \Leftrightarrow t(w) = f$, and $\text{succ}^t_i(w, wi)$, whenever $wi \in \text{dom } t$. 
Reduction of a grammar $\mathcal{G}$ of level $n$ to $\mathcal{G}^{\alpha}$ of level $n-1$

For types, $\tau \mapsto \tau^{\alpha}$,

- $\alpha : 0 \mapsto 0$,
- $\alpha : (0^k \to 0) \mapsto 0$,
- $\alpha : (\tau_1 \to \cdots \to \tau_n) \mapsto (\tau_1^{\alpha} \to \cdots \to \tau_n^{\alpha})$

For terms, $t : \tau \mapsto t^{\alpha} : \tau^{\alpha}$,

- $\alpha : \mathcal{F} \mapsto \mathcal{F}^{\alpha}$,
- $\alpha : z \mapsto z$, for any parameter $z$,
- $\alpha : (ts) \mapsto (t^{\alpha} s^{\alpha})$, whenever $s : \tau$ with $\ell(\tau) \geq 1$,
- $\alpha : (ts) \mapsto (((@t^{\alpha})s^{\alpha})$, whenever $s : 0$ (hence $t^{\alpha}, s^{\alpha} : 0$).
Reduction of grammars cont’d

\[ G = (\Sigma, V, S, E) \mapsto G^\alpha = (\Sigma^\alpha, V^\alpha, S^\alpha, E^\alpha) \]

where

\[ E : \mathcal{F}\phi_1 \ldots \phi_my_1 \ldots y_n \Rightarrow r, \text{ with } y_1 \ldots y_n : 0 \text{ then} \]

\[ E^\alpha : \mathcal{F}^\alpha \phi_1 \ldots \phi_m \Rightarrow \lambda y_1 \ldots \lambda y_n.r^\alpha. \]

Here the \( \lambda y_i \)'s and @ are new constants with \( \lambda y_i : 0 \rightarrow 0 \) and @ : \( 0^2 \rightarrow 0 \).

The tree is a \([G^\alpha]\) is a \( \lambda \)-definition of \([G]\).
Goal: to interpret

\[ \mathcal{G} \] in \[ \mathcal{G}^\alpha \]

\[ f \leftarrow \cdots \rightarrow g \]

\[ g \quad c \]

\[ \cdots \]

\[ @ \quad @ \quad c \]

\[ f \]

\[ g \]

\[ \cdots \]
Reduction level 1 to level 0 – example
Reduction level 2 to level 1 – example

\[
\begin{align*}
S & \Rightarrow \phi gc \\
\phi \xi x & \Rightarrow f(\xi x) (\phi (\text{Copy}\xi)x) \\
\text{Copy}\xi z & \Rightarrow \xi (\xi z)
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
S & \Rightarrow @ (\phi g)c \\
\phi \xi & \Rightarrow \lambda x @ (\lambda f (@f (\xi x))) (\lambda \phi (\text{Copy}\xi)x) \\
\text{Copy}\xi & \Rightarrow \lambda z @ \xi (@\xi z)
\end{align*}
\]
\[ S \Rightarrow \text{@}(\phi g)c \]
\[ \phi \xi \Rightarrow \lambda x \text{@} (\text{@}f(\text{@}\xi x)) \left( \text{@}\phi(Copy\xi)x \right) \]
\[ Copy\xi \Rightarrow \lambda z \text{@}\xi \left( \text{@}\xi z \right) \]
Reduction level 2 to level 1 – example cont’d
A problem may arise with a conflict of binding.
Ambiguity.
Explicit definition of binding leads to **undecidability**.
A term of level $k > 0$ is \textit{unsafe} if it contains an occurrence of a parameter of level strictly less than $k$.

An \textit{occurrence} of an unsafe term $t$ is \textit{unsafe}, unless it is in the context $\ldots(ts)\ldots$

$$
F\phi xy \Rightarrow f(F(F\phi x)x)yy)x
$$

A grammar without such occurrences is \textit{safe}.

\textbf{Note.} If a grammar $G$ is safe, so is $G^\alpha$. 
Lemma. If $\mathcal{G}$ is safe then the MSO theory of the tree $[\mathcal{G}]$ is recursively reducible to the MSO theory of the tree $[\mathcal{G}^\alpha]$.

Note. A grammar $\mathcal{G}$ of level $\leq 1$ is always safe and $[\mathcal{G}]$ has decidable MSO theory.

Theorem (KNU 2002). The MSO theory of the tree generated by a safe grammar of any level is decidable.

Theorem (Caucal 2002). The hierarchy of trees generated by safe grammars of level $n$ coincides with the hierarchy obtained by interpretation $+$ unfolding ($\rightarrow$ Caucal’s hierarchy).
But safety is not the frontier of decidability.

**Theorem (Ong 2006).** The MSO theory of the tree generated by any grammar is decidable.

Preceded by Aehlig, de Miranda and Ong 2005 for level 2, and independently KNUW 2005, *via panic automata* (of level 2).

Further development

Hague, Murawski, Ong and Serre 2008: another proof *via collapsible automata* of any level.

Kobayashi & Ong 2009: another proof *via* a type system.

Salvati & Walukiewicz 2012: another proof *via* Krivine machine.
Language-theoretic characterization of trees

By the complexity of sets of words \( \{ w \in \text{dom} \ t : t(w) = f \} \).

Let \( t = [G] \).

- **level 0**: regular
- **level 1**: deterministic pushdown
- **safe level \( n \)**: deterministic pushdown of level \( n \)
- **level \( n \)**: collapsible automata of level \( n \)

Parys 2012 used these characterizations to separate **safe** from **unsafe** grammars.
Higher order pushdown store

Maslov 1974
At the initial state, the stack is filled with the following elements: a, b, c, and b.

1. **push\(_1(c)\)**: After pushing 'c' onto the stack, the stack's contents become: a, b, c, b.

2. **pop\(_1\)**: After popping the top element, the stack's contents become: a, b, a, b.

The diagram illustrates the sequence of operations and the resulting stack state.
push₂

pop₂
Second-order pushdown stores

A level 1 pushdown store is a non-empty word $a_1 \ldots a_k$ over $\Gamma$.

A level 2 pds is a non-empty sequence of 1-pds’ $[s_1][s_2] \ldots [s_l]$.

Operations:

\[
push_1(a)([s_1][s_2] \ldots [s_l][w]) = [s_1][s_2] \ldots [s_l][wa]
\]
\[
pop_1(\alpha[w\xi]) = \alpha[w]
\]
\[
push_2(\alpha[w]) = \alpha[w][w]
\]
\[
pop_2(\alpha[v][w]) = \alpha[v]
\]
\[\downarrow\]
\[\downarrow a\]
\[\downarrow ab\]
\[\downarrow ab \downarrow ab\]
\[\downarrow ab \downarrow a\]
\[\downarrow ab \downarrow a a\]
\[\downarrow ab \downarrow a a \downarrow a\]
\[\downarrow ab \downarrow a a \downarrow a a\]
\[\downarrow ab \downarrow a a \downarrow a a \downarrow a a\]

\[\downarrow ab \downarrow a a \downarrow a a \downarrow a a\]

\[\downarrow ab \downarrow a a \downarrow a a \downarrow a a\]

\[\downarrow ab \downarrow a a \downarrow a a \downarrow a a\]

\[\downarrow ab \downarrow a a \downarrow a a \downarrow a a\]
Second-order pushdown stores with time stamps

A level 1 pushdown store is a non-empty word $a_1 \ldots a_k$ over $\Gamma \times \omega$.

A level 2 pds is a non-empty sequence of 1-pds’ $[s_1][s_2] \ldots [s_l]$.

Operations ($O_{p2}$):

$$push_1(a)([s_1][s_2] \ldots [s_l][w]) = [s_1][s_2] \ldots [s_l][w(a, l)]$$

$$pop_1(\alpha[w\xi]) = \alpha[w]$$

$$push_2(\alpha[w]) = \alpha[w][w]$$

$$pop_2(\alpha[v][w]) = \alpha[v]$$

$$panic([s_1][s_2] \ldots [s_m] \ldots [s_l][w(a, m)]) = [s_1][s_2] \ldots [s_m]$$
\[ \bot \]
\[ \bot a \]
\[ \bot ab \]
\[ \bot ab \quad \bot ab \]
\[ \bot ab \quad \bot a \]
\[ \bot ab \quad \bot a \quad \bot a \]
\[ \bot ab \quad \bot a \quad \bot a \quad \bot a \]
\[ \bot ab \quad \bot a \quad \bot a \quad \bot a \quad \bot a \]
\[ \bot ab \quad \bot a \quad \bot a \quad \bot a \quad \bot a \quad \bot a \]
\[ \bot ab \quad \bot a \quad \bot a \quad \bot a \quad \bot a \quad \bot a \quad \bot a \]

\[ push_1 \langle a \rangle \]
\[ push_2 \]
\[ pop_1 \]
\[ panic! \]
The model checking problem for level 2.

Given a grammar $G$ and a formula $\varphi$, decide if $[G] \models \varphi$.

Reduces to:

Given a second-order pushdown system with panic $C$, and a parity tree automaton $A$, decide if $A$ accepts the tree $[C]$.

Reduces to:

Given a second-order pushdown systems with panic $C$, and a parity tree automaton $A$, decide if Eve wins a certain parity game $Game(C \times A)$. 
**Parity games**

Eve (○) and Adam (□) move a token on a graph.

Eve wants to visit **even** priorities infinitely often.

Adam wants to visit **odd** priorities infinitely often.

Maximal priority wins.
Reduction of types is implemented by the structure of the game.
But is **safety** a true restriction?

**Example — panic not needed**

Recognize words of the form $w^{n+1}$, where:
- $w$ is a prefix of a correctly parenthesized expression;
- $n = |w|$.

Words like this one: 

```
[ [ [ ] ] [ [ ] ] ] ************
```

Not a context-free language.
Example (Urzyczyn) — panic seems to be needed

Recognize words of the form $uv^{n+1}$, where:

- $u$ is a prefix of a correctly parenthesized expression ending with $[;$
- $v$ is a correctly parenthesized expression;
- $n = |u|$.

Words like this one:

[ [ [ ] ] [ [ ] [ [ ] ] ] * * * * * * * * ]
The example is related to the following grammar (Urzyczyn).

\[
S \Rightarrow D\varphi_{ab}
\]

\[
D\varphi_{xy} \Rightarrow (fD(D\varphi_x)y\bar{y})(f(\varphi_y)x)
\]

Parys (2011, 2012) proved that the above language $U$ cannot be recognized by a deterministic automaton without panic of any level.

The level hierarchy of collapsible pushdown automata is strict Parys & Kartzow 2012.
Hierarchy of trees with decidable MSO theories

4 safe 5
3 safe 4
2 safe 3
safe 2
algebraic
regular
Questions

Is there a Caucal–like hierarchy of unsafe trees?

Does safety admit some decidable characterization?

Are there other reasons for decidability (e.g., low entropy)?