Algorithmic aspects of game theory. Assignments III

Solutions are due to **11 September 2019**.

Solutions to the problems should be sent to Damian Niwiński **niwinski@mimuw.edu.pl** .

They can be written in English or in Polish, but a solution to one problem should not mix up the two languages. For each problem, we indicate the number of **points** one can get for the solution. You may send answers to any selection of problems.

All mathematical claims in your solution that have not been proven during the lectures or tutorials, should come with their proofs.

1 Adding plus and minus ones (1.5 points)

We consider an arena like in mean payoff game, but the weights are only 1 or -1. We assume that Eve wins a play $p_0, p_1, \ldots$ whenever

$$\sum_{i=0}^{\infty} w(p_i, p_{i+1}) \quad = \quad \infty$$

(where $w(x, y)$ denotes the weight of an edge (x, y)), otherwise Adam is the winner. Design a polynomial-time algorithm that for a finite arena computes the set of positions winning for Eve.

2 Coloured territories (1 point)

Two players play on a board consisting of a square grid $n \times n$, with n odd. Eve plays **red** pawns, and Adam **blue** pawns. Initially the board is empty except that the lower left corner is occupied by a a read pawn, and the upper right corner by a blue pawn. Eve starts the game and then the players play in alternation. A move consists of putting a pawn in a square that adjoins[1] a square already occupied by a pawn of the same color. If a player cannot make such a move, she/he skips, but not loses. The game ends if none of the players cannot make a move, and the winner is the one who put more pawns.

Who–if anyone–has a winning strategy in this game?

3 Symmetric zero-sum game (0.5 + 1 points)

We consider a symmetric zero-sum game of two players with the set of strategies $\{1, \ldots, n\}$. (Example is rock–paper–scissor.)

Find an expected outcome of both players in a Nash equilibrium.

Is a profile in which both players choose any strategy with uniform probability always a Nash equilibrium?

4 Repeating game (1.5 points)

We consider an arena $(Pos_\exists, Pos_\forall, Move)$ possibly infinite, but with finite out-degree (for any position, only a finite number of moves is possible). An infinite play is won by Eve if some position repeats in this play infinitely often, otherwise Adam is the winner. Show that whenever Eve has a strategy to win from a position p, she has also a *positional* strategy to win from this position.

[1] We consider two squares as adjoint if they have a common edge.

5 Zebra game (1 + 1 points)

We consider an arena with positions coloured in $\{0, 1\}$; we assume there are no terminal positions. The winning condition for Eve is $(01)^\omega + (10)^\omega$; that is, Eve wins if no label occurs twice in a row, otherwise Adam is the winner.

Show that this game is positionally determined on any arena.

Design a polynomial-time algorithm that, for any *finite* arena, computes a winning strategy for Eve.

6 A different parity game (0.5 + 1 points)

We consider a finite arena like in mean payoff game, with weights in $\mathbb{Z}$ (and no terminal positions). We assume that a play ends as soon as a first loop is closed

$$\pi \quad = \quad (p_0, \ldots, p_m, p_{m+1}, \ldots p_n = p_m)$$

and Eve wins the play if the sum of the weights on this loop

$$\sum_{i=m}^{n-1} w\left(p_i, p_{i+1}\right)$$

is *even*; otherwise Adam is the winner.

Is this game positionally determined (over finite arenas)?

Design an algorithm that computes the winning regions of both players. Estimate from above its running time.