Algorithmic aspects of game theory

Synopsis of course — draft

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Determinacy of games

Zermelo's theorem on chess

In 1913, Ernst Zermelo proved mathematically that in the play of chess one of the following possibilities holds.

- White have a winning strategy,
- Black have a winning strategy,
- both parties have the strategies to achieve at least a draw.

We will show this theorem on a more abstract level, not forgetting its algorithmic aspect.

Arenas

We think of two players who will play by changing "situations" of the game according to some rules. The current situation is always known to both players, and it determines who should play. We name the players Eve and Adam.

An *arena* is a directed graph, consisting of

- the set of *positions* Pos,
- the set of *moves* $Move \subseteq Pos \times Pos$.

The set of positions is partitioned into two disjoint sets $Pos_\exists$ and $Pos_\forall$ of positions of Eve and Adam, respectively, i.e.,

$$\begin{aligned} Pos_\exists \cup Pos_\forall &= Pos \\ Pos_\exists \cap Pos_\forall &= \emptyset \end{aligned}$$

(any of these sets can be empty). Relation $(p, q) \in \mathit{Move}$ is usually written by $p \to q$. A position p, such that $(\forall q)\, p \not\to q$ is called *terminal*, which we also write $p \not\to$.

A *play* is a finite or infinite sequence

$$q_0 \to q_1 \to q_2 \to \ldots \to q_k (\to \ldots)$$

A finite play that ends in a position $q_k \not\to$ is *lost* by the player who owns this position. Thus, a player who cannot move, looses. Note that we allow $k = 0$, i.e., a play can consist of just one position (no move), but an empty sequence ε is not considered as a play.

Note that in our concept of an arena we do not privilege any of the players. All concepts can be defined equally for both players. We will usually present definition for Eve, but they can be adapted for Adam, by symmetry.

Game equation

In symbols, the equation for Eve is

$$X = (E \cap \Diamond X) \cup (A \cap \Box X) \tag{1}$$

and the dual equation for Adam

$$Y = (A \cap \Diamond Y) \cup (E \cap \Box Y). \tag{2}$$

Here, the variables X and Y range over subsets of Pos, $E = Pos_\exists$, $A = Pos_\forall$, $\cup$ and $\cap$ have their usual meaning and, for any $Z \subseteq Pos$,

$$\begin{aligned} \Diamond Z &= \{v : \exists w,\, w \in Z \wedge \mathit{Move}(v, w)\} \\ \Box Z &= \{v : \forall w,\, \mathit{Move}(v, w) \Rightarrow w \in Z\}. \end{aligned}$$

To ensure that the game equations have solutions, we recall a classical result about ordered sets. A *complete lattice* is a partially order set $\langle L, \leq \rangle$, such that each subset $Z \subseteq L$ has the least upper bound $\bigvee Z$, and the greatest lower bound $\bigwedge Z$. In particular, $\bigvee \emptyset$ is the least element denoted $\perp$, and $\bigwedge \emptyset$ is the greatest element denoted $\top$.

Theorem 1 (Knaster-Tarski) *A monotonic mapping over a complete lattice $\langle L, \leq \rangle$ has a* least fixed point

$$\mu x.f(x) \quad = \quad \bigwedge\{z : f(z) \leq z\} \tag{3}$$

and a greatest fixed point

$$\nu x.f(x) \quad = \quad \bigvee\{z : z \leq f(z)\}. \tag{4}$$

Proof. We show the result for the greatest fixed point. Let

$$a \quad = \quad \bigvee \underbrace{\{z : z \leq f(z)\}}_{A}$$

By monotonicity of f, $z \leq a$ implies $f(z) \leq f(a)$. For $z \in A$, this further implies $z \leq f(z) \leq f(a)$. Hence, $f(a)$ is an upper bound of A, which follows $a \leq f(a)$. Using again monotonicity of f, we obtain $f(a) \leq f(f(a))$. Hence $f(a) \in A$, which follows the converse inequality $f(a) \leq a$. $\square$

We will abbreviate

$$\begin{aligned} Eve\,(Z) \quad &= \quad (E \cap \Diamond\, Z) \cup (A \cap \Box\, Z) \\ Adam\,(Z) \quad &= \quad (A \cap \Diamond\, Z) \cup (E \cap \Box\, Z). \end{aligned}$$

Clearly the operators Eve and $Adam$ are monotonic over the complete lattice $\langle \wp(Pos), \subseteq \rangle$. Hence, the game equations have solutions.

Traps and gardens of Eden

A set of positions $Z \subseteq Pos$ is a *trap* for Adam if $Z \subseteq Eve\,(Z)$. Intuitively, Adam cannot go out of there, so the message for him is:

> *You must stay there or loose.*

A set of positions is *gardens of Eden* for Adam if $Adam\,(Z) \subseteq Z$. Now the message for Adam is

> *You cannot get there, unless you are already there.*

Note that these concepts are implicitly present in the formulas of the Knaster-Tarski Theorem. If we apply (3) and (4) to the game equations, we get the following messages:

The *greatest* trap for Adam is a garden of Eden for Eve.

The *least* garden of Eden for Eve is a trap for Adam.

It is worth to see that the above are indeed dual concepts. We use the notation

$$\overline{Z} = Pos - Z.$$

Lemma 1

$$\overline{Eve\,(X)} \quad = \quad Adam\,(\overline{X})$$

Proof. We have

$$\begin{aligned}
\overline{Eve\,(X)} &= \overline{(E \cap \Diamond X) \cup (A \cap \Box X)} \\
&= \left(\overline{E \cap \Diamond X}\right) \cap \left(\overline{A \cap \Box X}\right) \\
&= \left(\overline{E} \cup \overline{\Diamond X}\right) \cap \left(\overline{A} \cup \overline{\Box X}\right) \\
&= \left(A \cup \Box \overline{X}\right) \cap \left(E \cup \Diamond \overline{X}\right) \\
&= \left(A \cap \Diamond \overline{X}\right) \cup \left(E \cap \Box \overline{X}\right) \cup \underbrace{(A \cap E)}_{\emptyset} \cup \left(\Diamond \overline{X} \cap \Box \overline{X}\right) \\
&= Adam\,(\overline{X}).
\end{aligned}$$

(The last summand can be omitted, because it is included in the first two.)
□

The following consequences are immediate.

Corollary 1 *The complement of a trap for Adam is a garden of Eden for him; similarly for Eve.*

Corollary 2

$$\begin{aligned}
\overline{\mu X.Eve\,(X)} &= \nu Y.Adam\,(Y) \\
\overline{\nu X.Eve\,(X)} &= \mu Y.Adam\,(Y)
\end{aligned}$$

Exercise 1 *Show that the union of any family of traps for a player is again a trap for this player.*

Note that by Corollary 1 this implies that the intersection of any family of gardens of Eden for a player is again a garden of Eden for this player.

Which more general property of ordered sets underlines these facts ? (Remember the Knaster-Tarski Theorem.)

Strategies

Intuitively, a strategy for a player, say Eve, tells her how to continue the play, assuming that she has followed this strategy so far. Note that a strategy must "answer" all legal moves of Adam.

A *strategy* for Eve can be represented by a non-empty set of finite plays S, such that

- if $\pi \in S$ and $\mathrm{last}(\pi) \in Pos_\exists$ then $(\exists! q)\, \pi q \in S$,
- if $\pi \in S$ and $\mathrm{last}(\pi) \in Pos_\forall$ then $(\forall q)\, (\mathrm{last}(\pi) \to q) \Rightarrow \pi q \in S$,
- S is closed under initial segments, i.e., if $s_0 s_1 \ldots s_k \in S$ then $s_0 s_1 \ldots s_i \in S$, for $0 \le i \le k$.

Here $\mathrm{last}(\pi)$ denotes the last element of the sequence π.

A play (finite or infinite) π *conforms* with a strategy S if any finite prefix of π belongs to S. Note that a strategy can be visualized as a tree (after adding an auxiliary element as a root). Then the conforming plays form the branches of this tree.

A strategy always contains some single positions – if we identify them as one-element plays. If $p \in S \cap Pos$, we say that p is an *initial* position of S, and S is a strategy *starting from* p. We call a position q *safe* for Eve, if it is initial in some strategy for her. (We emphasize that q itself need not be position of Eve !)

Exercise 2

1. *Show that if a position is safe for Eve then she can play in such a way that she will never loose a play starting from this position (but a play can be infinite).*
2. *Give an example of an arena, where some position is safe for both players.*
3. *Give an example of an arena, where the positions safe for Eve are precisely the positions of Adam, and* vice-versa*.*

A strategy S (for Eve) induces a *strategy function* f_S, which is a partial function over the set of plays. The value $f_S(\pi)$ is defined whenever $\pi \in S$ with $\mathrm{last}(\pi) \in Pos_\exists$, and amounts to the unique q, such that $\pi q \in S$.

A strategy S is *positional* (or *memory-less*) if this function depends only on the current position, i.e.,

$$\mathrm{last}(\pi) = \mathrm{last}(\rho) \quad \Rightarrow \quad f_S(\pi) = f_S(\rho).$$

In this case f_S induces a partial function on Pos_E, that we denote by the same symbol, $f_S : Pos_\exists \supseteq dom\, f_S \to Pos$,

$$f_S(p) = q$$

whenever $f_S(\pi) = q$, for some π with $\mathrm{last}(\pi) = p$.

It is useful to see under which condition a converse construction is possible. Let $f : Pos_\exists \supseteq dom\, f \to Pos$ be a partial function on $Pos_\exists$ which agrees with moves, i.e., $f \subseteq Move$. We say that f is *safe* if there exists a trap Z for Adam, such that $dom\, f = Z \cap Pos_\exists$ and $range\, f \subseteq Z$. This concept is motivated by the following.

Lemma 2 *For f and Z as above, there is a positional strategy S for Eve, such that $Z \subseteq S$. In particular, all positions in $dom\, f$ are safe.*

Proof. We construct this strategy by stages. Let

$$\begin{aligned} S_1 &= Z \\ S_{n+1} &= \{wf(\mathrm{last}(w)) : w \in S_n \wedge \mathrm{last}(w) \in Pos_\exists\} \cup \\ &\quad\ \{wp : w \in S_n \wedge \mathrm{last}(w) \in Pos_\forall\}. \end{aligned}$$

It follows by induction that all paths in S_n remain in Z; in particular $f(\mathrm{last}(w))$ is defined, whenever $\mathrm{last}(w) \in Pos_\exists$. It is straightforward to see that

$$S = \bigcup S_n$$

satisfies the requirements; note that $f_S = f$. $\square$

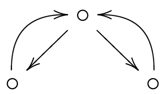
Note that if S is any positional strategy for Eve then the strategy function f_S is safe; we can take as Z the set of *all* positions that occur in some π in S. The strategy produced by Lemma 2 can then be slightly larger that S, as more positions can be initial. In the sequel we usually represent positional strategies by safe functions, so that the actual strategies are always like in Lemma 2 (with some Z).

Remark Let us see the above properties in terms of graphs. For f and Z as in Lemma 2, consider a sub-arena obtained by restricting the set of positions to Z and the set of moves to

$$Move' = Move \cap ((Pos_\forall \cap Z) \times Pos \cup f)\,.$$

That is, we remove all Eve's moves except for those indicated by f. Then no play on this arena can be lost by Eve.

Exercise 3 *Consider an arena*



where all positions belong to Eve. Compute the cardinalities of the sets of

1. *all strategies of Eve,*
2. *all positional strategies of Eve,*
3. *all safe functions of Eve,*
4. *all strategies of Adam,*
5. *all safe functions of Adam.*

In game theory, as well as in game practice, we often construct some new strategies by combining some already existing ones. The simplest operation is that of *sub-strategy*.

Lemma 3 *Let S be a strategy of Eve and $w \in S$. Let*

$$S.w \quad = \quad \{v : wv \in S\}$$

Then $S.w$ is also a strategy of Eve. Moreover, if S is positional, so is $S.w$.

Proof. Straightforward from the definition. $\square$

The following lemma prepares the first important connection between the game equation and strategies that we establish next.

Lemma 4 *If Z is a trap for Adam then all positions in Z are safe for Eve. Moreover, there is a* positional *strategy S for her, such that $Z \subseteq S$.*

Proof. By Lemma 2, it is enough to define a safe function on $Z \cap Pos_\exists$. Since Z is a trap, for any $p \in Z \cap Pos_\exists$, there is some $q \in Z$, such that $p \to q$. The existence of a safe function follows from the axiom of choice. $\square$

Proposition 1 *The set of all safe positions of Eve is the greatest fixed point of the operator Eve . Moreover, there exists a* positional *strategy with this set as initial positions.*

Proof. Let Z_0 be the set of all safe positions of Eve. We first show

$$Z_0 \quad \subseteq \quad Eve\,(Z_0). \tag{5}$$

Let $p \in Z_0$ and let S be a strategy of Eve starting from p. Consider the strategy $S.p$ (c.f. Lemma 3). There are two possibilities. If $p \in Pos_\exists$ then $pq \in S$, for some q. Then $q \in S.p$, hence q is safe. If $p \in Pos_\forall$ then $pq \in S$, for all q, such that $p \to q$. Hence all such q's are safe. In any case, $p \in Eve\,(Z_0)$. (5) and (4) give us $Z_0 \subseteq \nu x.Eve\,(x)$. The converse inequality, as well as the strategy claim, follows directly from Lemma 4. $\square$

Winning in finite time

If a position is not safe for Adam then he has no guarantee that he will not loose. But does it mean that Eve is sure to win ? Note that we have no "guarantee" of winning other than a strategy. Again, the game equation will be useful in answering this question.

A strategy S of Eve is *finitely winning* if any play that conforms with S is finite – and hence lost for Adam. A position p is finitely winning if $p \in S$, for some strategy with this property. (Again recall that p need not to be a position of Eve !)

The following fact is dual to Proposition 1.

Proposition 2 *The set of all positions finitely winning for Eve is the least fixed point of the operator Eve . Moreover, there exists a* positional *finitely winning strategy with this set as initial positions.*

Proof. Let W' be the set of positions that are initial for some *positional* finitely winning strategy. We first show

$$Eve\,(W') \quad \subseteq \quad W'. \tag{6}$$

Suppose $p \in Eve\,(W')$. If $p \in Pos_\exists$ then Eve can make a move to a position q, from which there is a positional finitely winning strategy S. Let f_S be the strategy function induced by S (see page 5). If $p \in dom\, f_S$ then $p \in W'$, and we are done. Otherwise, we can consider the extended function $f_S \cup \{(p,q)\}$. This function is clearly safe, and it is straightforward to see that the strategy constructed in Lemma 2 is (positional and) finitely winning. Hence $p \in W'$.

If $p \in Pos_\forall$ then, for each q, such that $p \to q$, we have $q \in W'$. Then, for each such q, we have some safe function f_q, inducing a positional finitely winning strategy from q. However, the union of these functions need not be a function. To remedy this, we will use the possibility of well-ordering of any set, which is the "great" Theorem by Zermelo. But first, a lemma is in order.

Lemma 5 *Let α be an ordinal, and let S_ξ be a family of positional, finitely winning strategies for Eve, each S_ξ induced by*[1] *a safe function f_ξ. Let f be defined on $\bigcup_{\xi<\alpha} dom\, f_\xi$ by*

$$f(p) \quad = \quad f_\xi(p)$$

where ξ is the least, such that $f_\xi(p)$ is defined. Then f is safe and the induced strategy is finitely winning.

Proof of the lemma. Let $init(S_\xi)$ be the set of initial positions of S_ξ. We have $dom\, f_\xi = S_\xi \cap Pos_\exists$, for each $\xi < \alpha$. Then $I = \bigcup_{\xi<\alpha} S_\xi$ is a trap (see Exercise 1), and moreover $dom\, f = I \cap Pos_\exists$ and $range\, f \subseteq I$. Hence f is safe and induces a positional strategy with the set of initial positions I. Let us see that this strategy is finitely winning. Suppose on the contrary that some infinite play

$$q_0 \to q_1 \to q_2 \to \dots$$

conforms with this strategy. Note that no suffix of this play can conform with any S_ξ, because each S_ξ is finitely winning. Hence, Eve must infinitely many times switch the strategy from S_ξ to some other $S_{\xi'}$. But this may only happen if $\xi' < \xi$. However, we cannot have an infinite decreasing sequence of ordinal numbers, a contradiction. More precisely, for each n, let $ord\,(q_n)$ be the least ξ, such that $q_n \in init(S_\xi)$. Then this sequence of ordinals is non-increasing, but it cannot stabilize, which yields a contradiction. $\square$

We come back to the proof of the proposition. If we well-order the set of successors of p and apply the above lemma, we obtain a positional finitely

[1] That is, S_ξ coincides with Z of Lemma 2.

winning strategy which has all successors of p among its initial positions. Then it easy to see that the position p is finitely winning as well. (If this strategy is induced by a safe function f and some trap Z, we can extend this trap by p.)

This completes the proof of (6).

By the equation (3) of the Knaster-Tarski Theorem, this implies $\mu X.Eve\,(X) \subseteq W'$. But we already know that all positions outside $\mu X.Eve\,(X)$ are in $\nu Y.Adam\,(Y)$ (Corollary 2), and then are safe for Adam (Lemma 4), so they cannot be finitely winning for Eve. Hence W' coincides with the set W of all positions finitely winning for Eve, and

$$W = W' = \mu X.Eve\,(X).$$

For each q in this set, we have a positional finitely winning strategy from q. To obtain a single strategy good for all q's, we can again use Lemma 5. □

Determinacy

We are ready to state the determinacy result which essentially comprises Zermelo's theorem about chess.

Theorem 2 (Zermelo) *For any position p, one of the following possibilities holds:*

1. *p is finitely winning for Eve,*
2. *p is winterly winning for Adam,*
3. *p is safe for both players.*

Proof Let $X_{\min}$ and $X_{\max}$ be the least and greatest fixed points of Eve , and similarly with Y for $Adam$. By Proposition 2, $X_{\min}$ and $Y_{\min}$ are disjoint. (No position can be finitely winning for both players.) Using Corollary 2, we get

$$\overline{X_{\min} \cup Y_{\min}} = X_{\max} \cap Y_{\max}.$$

Hence, if 1 or 2 do not hold, 3 must hold. □

Note that we have proved this theorem without any restriction on the cardinality of arenas. For finite arenas, there is a natural algorithmic question to decide which of the cases of Theorem 2 holds, and to find a suitable strategy.

Exercise 4 *Design an algorithm which, for a finite arena, determines which of the cases of Theorem 2 holds. The algorithm can be made to run in time linear in the size of the arena.*

Next lecture (5.03.2009)

- Determinacy of the zero-sum games in the matrix form (definition).
- Indeterminate games.

If time permits...

References

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[2] Christos P. Papadimitriou, Złożoność obliczeniowa. WNT, 2002.

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