# Algorithmic aspects of game theory assignments II

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Solutions are due to **18 VI 2019**.

Solutions to the problems should be sent to Marcin Przybyłko:
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They can be written in English or in Polish, but a solution to one problem should not mix up the two languages. For each problem, we indicate the number of **points** one can get for the solution. You may send answers to any selection of problems.

**All facts that were not proven during the lectures or the tutorials but were used in your solution have to be proven.**

## 1 Symmetrisation (0.5 + 1 points)

In this exercise we assume that the sets of pure strategies in consideration are finite.

A two-player game is *symmetric* if both players have the same pure strategies, and the *payoff functions* $u_1$ and $u_2$ satisfy: $u_1(x, y) = u_2(y, x)$. An equilibrium $(p, q)$ is *symmetric* if $p = q$.

**Tasks:** Show that any symmetric game admits a symmetric equilibrium.

Show that two-player games admit a *symmetrisation* in the following sense: there is a polynomial time procedure that transforms an arbitrary game $A$ into a symmetric game $S_A$ such that every symmetric equilibrium of $S_A$ can be transformed to an equilibrium in $A$, again in polynomial time.

## 2 Constructive fixed-point (1 + 1 + 1 points)

The unit $n$-dimensional *simplex* $\Delta_n$ is defined as follows:

$$\Delta_n = \{\langle x_1, x_2, \ldots, x_n\rangle \in [0, 1]^n \colon \sum_{i=1}^{n} x_i = 1\}$$

A *k-face* $F$ of $\Delta_n$ is an $(n-k)$-dimensional simplex of the form:

$$F = \{\langle x_1, x_2, \dots, x_n \rangle \in \Delta_n \colon x_{a_1} = x_{a_2} = \dots = x_{a_k} = 0\}$$

for some $1 \leq a_1 < a_2 < \dots < a_k \leq n$.

Consider any *triangulation* $T$ of $\Delta_n$. Denote by $V$ the set of vertices of $T$. Let $c \colon V \to \{0, 1, \dots, n\}$ be a coloring function satisfying th following constraints:

- $c$ restricted to the set of vertices of $\Delta_n$ is a bijection;
- if $v \in V$ belongs to face $F$ then $c(v)$ belongs to the set of the colors of the vertices of $F$.

**Task:** Show that there exists a simplex $S$ in $T$, such that the coloring $c$ restricted to the set of vertices of $S$ is a bijection.

Now consider a continuous function $f \colon \Delta_n \to \Delta_n$ on a unit $n$-dimensional simplex $\Delta_n$. For any $\epsilon > 0$, let us call $x \in \Delta_n$ an $\epsilon$-fixed point of $f$ if:

$$|f(x) - x| \leq \epsilon.$$

**Task:** use the previous task to show that $f$ has an $\epsilon$-fixed point.
**Extra task:** assume that $f$ is Lipschitz with constant $k$, i.e. $\forall_{x,y \in \Delta_n} |f(y) - f(x)| \leq k|y - x|$. Design and implement an algorithm (in any reasonable programming language) that finds $\epsilon$-fixed point of $f$.

# 3 Tom and Jerry (1 + 2 points)

Tom and Jerry move on a rope of length 10. Each of them chooses its initial position, direction of movement, and speed. When a player reaches an end of the rope (and only then), they change the direction retaining the same speed. We let Jerry to move twice as faster as Tom. Specifically, we assume that at the start of the game Tom chooses a pair of reals $(p, v)$ with $p \in [0, 10],\ v \in [-1, 1]$, and Jerry a pair $(q, w)$ with $q \in [0, 10],\ w \in [-2, 2]$; $p, q$ are the initial positions and $v, w$ the initial velocities. The choices are simultaneous and independent. Tom *catches* Jerry if they are in the same position on the rope; if it happens, Jerry dies.

**Task:** What (mixed) strategy can be chosen by Jerry to maximize its life expectancy? What (mixed) strategy can be chosen by Tom to minimize the expected time needed to catch Jerry?

**Extra task** (of open character)**:** Consider a variant of the game, where the rope has a form of *circle* (rather then interval), and the players can modify their velocities during the game (within their respective scopes). You are free to define the conditions of your variant of the game, and perform the analysis.

## 4 Optimal strategy in Mean-Payoff Games (1 point)

Consider a finite mean-payoff game $G$ played by the *maximiser* (Eve) and *minimiser* (Adam). Recall from the course that these games are memoryless determined in the sense that, for every position $u$, there exists a compromise value $v_u$ and memoryless strategies starting from this position, which are optimal for both players (maximisers gains at least $v_u$, whereas minimizer loses at most $v_u$).

**Task:** Show that optimal strategies can be made *global*. That is, for every finite mean-payoff game $G$, there exist positional strategies of maximiser and minimiser respectively, that do not depend on initial positions, but for any position are optimal for both of the players. Estimate the complexity of finding such strategies.

## 5 Concurrent games on graphs (2 points)

The concurrent games on graphs are games played by two players called *maximiser* and *minimiser*.

A finite concurrent game on graphs is a tuple $G = \langle V, E, A, B, s, v_I, L\rangle$ where $V$ is a finite set of vertices, $E \subseteq V \times V$ is a set of edges, $A, B$ are non-empty finite sets of actions, $s : V \times A \times B \to V$ is a successor function, $v_I$ is an initial vertex and $L \subseteq V^\omega$ is a set of infinite words.

The game is played by moving a token, initially placed in $v_I$, along the edges of the graph $\langle V, E\rangle$. To move the token from vertex $v \in V$ both players independently and concurrently choose an action: maximiser chooses an action $a \in A$ and minimiser chooses an action $b \in B$. Then the successor function is applied and the token is moved to the vertex $s(v, a, b)$.

For simplicity and consistency we assume that every vertex has at lest one outgoing edge and every combination of actions results in valid transition, i.e. for all $a \in A, b \in B, v \in V$ if $s(v, a, b) = u$ then $\langle v, u\rangle \in E$.

The play $p \in V^\omega$ of the game is the sequence of vertices visited by the token. After a play is formed, minimiser pays one coin to maximiser if $p \in L$ and maximiser pays one coin to minimiser if $p \notin L$.

**Task:** Show that if the set $L$ is topologically *closed* then the game is determined under mixed strategies. The topology is defined as before.