# Algorithmic aspects of game theory

Synopsis of course — draft

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# Contents

# Chapter 1

# Getting started

I started to compose these notes during my course at the University of Warsaw in 2012. I am currently revising and extending them at the Fall semester of 2015. I am highly indebt for the feedback I have received from students of all periods. The course and consequently the notes are based on the work of many authors, the references at the end are by far not complete. I apologize to all those to whom the credit has not been explicitly given.

## 1.1 Do games need a theory ?

The role of an abstract mathematical theory is twofold: it should help to explain empirical reality like, e.g., in physics and other natural sciences, but it may also help to design/construct some useful real objects, like in engineering. A good example is *Turing machine*. It was invented as a mathematical model of a human being performing computation, and later became a basis for a general purpose computer.

| | | | | |
|---|---|---|---|---|
| empirical reality | $\rightarrow$ | abstraction | $\rightarrow$ | created reality |
| human's computation | $\rightarrow$ | Turing machine | $\rightarrow$ | computer program |

Game theory has two sources, which issue two concepts of a game. *Games in extensive form* is a mathematical abstraction of social games, like board games or card games. Such games can be implemented as computer games played for entertainment, but they can also give an idea to construct some useful algorithm, *e.g.*, for program verification. *Games in strategic form* model rational human behavior, especially decision making in presence of conflicts. They can be used to make prediction, *e.g.*, of behavior of markets, and also to design some frame of economic action (*mechanism design*).

| | | | | |
|---|---|---|---|---|
| social gaming | $\rightarrow$ | extensive games | $\rightarrow$ | verification algorithms,<br>computer games, ... |
| decision making | $\rightarrow$ | strategic games | $\rightarrow$ | prediction,<br>mechanism design, ... |

Games in extensive form can be presented as a special case of strategic games, but this abstraction is not always useful.

**Remark.** In some sense, social games anticipated game theory, as they are often metaphors of human dealing with conflicts, cf. the rules of chess or Go.

### 1.1.1 Social games

We list few examples of games and informally point some features, which can be present (+) or absent (–) in a particular game: randomness, completeness of information, and simultaneous (as opposite to turn-based) moves.

| | # players | randomness | full information | simultaneous moves |
|---|---|---|---|---|
| chess | 2 | – | + | – |
| football | 22 (2 teams) | + | + (?) | + |
| diplomacy | 7 | – | + | + |
| sea battle | 2 | – | – | + |
| Go | 2 | – | + | – |
| solitaire | 1 | – | + | |
| bridge | 4 (2 teams) | + | – | – |
| Yahtzee | $1^+$ | + | + | – |
| minesweeper[a] | 1 | – | – | |
| don't get angry[b] | 2–6 | + | + | – |

[a]Fixed setting.
[b] In Poland called *Chińczyk*.

#### Convention about names

While considering a two player game, we usually name the players: Eve and Adam and, unless stated otherwise, Eve is the player who starts the game.

## 1.2 Game Hex

The game is played on a board consisting of a hexagonal grid $n \times n$ (classically: $n = 11$); for nice pictures we refer the reader to Internet, *e.g.*, Wikipedia.

Eve plays red pawns, and Adam plays blue pawns, say. The players place in turn their pawns on arbitrary free cells of the board. As mentioned above, Eve starts the game. Eve wants to connect the topmost and bottom-most sides of the board by a read path, whereas Adam wants to connect the leftmost and rightmost sides by a blue path. The game ends if there is no free place left. At this moment (if not earlier) one of the players has certainly won, because of the following.

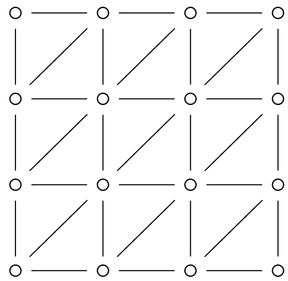


Figure 1.1: Graph representation of a Hex board $3 \times 3$.

**Lemma 1.** *If the whole board is covered by pawns, there is always a read path connecting the topmost and bottom-most sides, or a blue path connecting the leftmost and rightmost sides, but not both. Consequently, when the game ends, there is always a winner (no draw).*

We postpone for a moment the proof of the lemma, and show the following.

**Theorem 1.** *Eve has a winning strategy in the game Hex, for any $n$.*

*Sketch.* We know from the lemma that there cannot be a draw. Suppose to the contrary that Adam has a winning strategy[1]. By symmetry of the board, this readily implies that Eve would win the game if she played as the second player. Let $G$ ba a hypothetical winning strategy for Eve in the modified game. We will show that Eve can use this strategy to win the original game.

Eve plays her first pawn anywhere, and marks it *extra*. When Adam answers by placing his first pawn, Eve starts to play the strategy $G$. She may encounter a problem when this strategy requires to place a pawn on the position already occupied by the *extra* pawn. (She might like to skip her move, but this is not allowed.) Then Eve places her pawn on any free position, and at the same time moves the label *extra* to this new pawn. Note that, in this way, all Eve's pawns except for the *extra* pawn are placed according to the strategy $G$.

Suppose that at some moment Adam forms his winning blue path. Then he could readily form the same path if the pawns marked *extra* were never present on the board. But this would contradict the assumption that strategy $G$ is winning for Eve. (Recall that this strategy works in the modified game, where Eve plays as the second player.) Hence, we conclude that the strategy for Eve described above (using *extra* pawns) is winning in the original game.

The argument used in this proof is known as *strategy stealing*. Indeed, we show that if Adam had a winning strategy, Eve could "steal" it (cf. strategy $G$ above) and with some minor modification (like *extra* pawns) win the game.

Note that this argument is non-constructive. Indeed, no explicit strategy is known for $n > 9$. $\square$

[1] Here we use the fact that finite games are *determined*; we will show this fact in a more general setting.

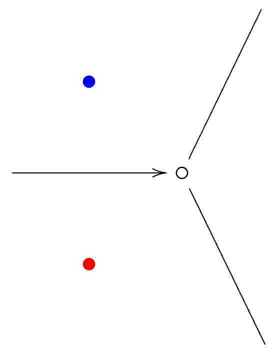


Figure 1.2: Forming a path.

We will now sketch the proof of the lemma. To this end, it will be convenient to use the original, hexagonal, representation of the board. We refer to a cell occupied by a blue or red pawn as blue-colored or red-colored, respectively. We additionally color the area below the bottom-most side (South) red, and the area to the left from the leftmost side (West) blue[2]. (The North and East areas are not colored.) We will form a (directed) path going by the *edges* of hexagon cells. The path starts in the left-bottom corner (South-West). If the left-bottom cell is colored red, we go up, and if it is collared blue, we go right. Note that in either case, because of the additional coloring, we have red on the right-hand side and blue on the left-hand side. This will be an invariant of our construction. The path will terminate if it arrives to the upper side, or to the rightmost side of the board. Note that, in the former case, the cells adjacent to the right of our path are, by the invariant, all colored red, and hence for a sequence wining for Eve. Similarly, in the latter case, the cells adjacent to the left of the path form a winning sequence for Adam. We will now show that if a path does not reach the topmost/rightmost side then a next move is always possible, preserving the invariant. Whenever our path arrives in a node of degree 3 having blue on the left and red on the right, it will turn left or right depending on the color of the third ("new") area, and the invariant will be preserved (see Figure 1.2).

If our path arrives in a node of degree 2, it can be continued in a unique way. But this can happen only on the southern or western border of the board and then, because of the additional coloring, the invariant is preserved. To the end of the proof, it remains to observe that the invariant excludes a possibility of forming a loop (see again Figure 1.2).

## 1.3 Infinite games

The rules of many games do not *a priori* forbid playing them indefinitely although, for obvious reasons, it can be hardly happen in practice. In our course, however, we will give much attention to infinite games, as they have nice mathematical theory and can model

[2]This step, simplifying the argument, has been suggested by a student, Mr. ? during the lecture.

information systems, which are designed to act indefinitely as, *e.g.*, operation systems, or Internet.

### 1.3.1 Choquet game

In this game players select open intervals of the real line; each new interval should be contained in the previous one,

| | | | | | | |
|---|---|---|---|---|---|---|
| Eve | $I_0$ | | $I_2$ | | $I_4$ | ........ |
| Adam | | $I_1$ | | $I_3$ | | ........ |

where $I_{n+1} \subseteq I_n$. Eve wins the game if $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$; otherwise Adam is the winner. Let $I_n = (a_n, b_n)$. At first sight, it may appear that the best strategy for Eve is to copy the choices of Adam because, in this way, she will "save" as many points as possible. But this strategy will *fail* if, in his moves, Adam fixes one end of the interval and makes the other ends to converge to it. For example, if $b_n = b$, for $n \in \mathbb{N}$, and $a_n \to b$ (*i.e.*, $\lim_{n\to\infty} a_n = b$) then $\bigcap_{n\in} I_n \neq \emptyset$, and Adam wins.

But Eve *can* win this game ! To this end, she should narrow the interval from both sides, *i.e.*, $a_n < a_{n+1} < b_{n+1} < b_n$, whenever Eve moves in the $n+1$-th turn. We have $a_n \to a \leq b \leftarrow b_n$, for some $a$ and $b$. Since the sequences $a_n$ and $b_n$ do not stabilize (because of the moves of Eve), we have $a_n < a \leq b < b_n$, for any $n$. Hence $\bigcap_{n\in\mathbb{N}} I_n \supseteq [a, b] \neq \emptyset$.

### 1.3.2 Infinite XOR

A special feature of infinite games is that they can be *indeterminate*: even if any infinite play is won by one of the players (no draws), it may happen that none of the players has a winning strategy. We will again use a strategy stealing argument, but in the special game we are going to construct, either of the players will be able to steal a strategy from his/her opponent.

We need some preliminaries. Let $2 = \{0, 1\}$, and let $2^\omega$ denote the set of infinite words over 2, *i.e.*, mappings $\mathbb{N} \to 2$. For $v, w \in B^{\mathfrak{m}}$, where $\mathfrak{m} \leq \omega$, let $\mathrm{hd}(v, w) = |\{i : v_i \neq w_i\}|$ be the *Hamming distance* between $v$ and $w$. For $v, w \in B^\omega$, we let $v \sim w$ iff $\mathrm{hd}(v, w) < \omega$.

**Definition 1.** *An* **infinite XOR function** $f : 2^\omega \to 2$ *is a function with the following property: if* $\mathrm{hd}(w_1, w_2) = 1$ *then* $f(w_1) \neq f(w_2)$.

**Proposition 1.** *Infinite XOR functions exist.*

*Proof.* We use the Axiom of Choice. Let $S$ be a set which contains exactly one element from each equivalence class of $\sim$. For $w \in 2^\omega$, let $r(w)$ be the element of $S$ such that $w \sim r(w)$. We define $f(w) = \mathrm{hd}(w, r(w)) \bmod 2$. One easily checks that $f$ is an infinite XOR function. $\square$

**Remark.** It follows from a standard set-theoretical argument that there is $2^{\mathfrak{c}}$ infinite XOR functions (where $\mathfrak{c}$ denotes *continuum*, *i.e.*, the cardinality of $2^\omega$). Observe first that each equivalence class of $\sim$ is countable, which follows $|S| = \mathfrak{c}$. Then, for each $\alpha : S \to \{0,1\}$, we obtain a different infinite XOR function given by $f_\alpha(w) = (f(w) + \alpha(r(w))) \bmod 2$.

In what follows, we suspend our convention, and disguise Eve and Adam under the names of Player 0 and Player 1 respectively.

**Definition 2.** *Let $f$ be an infinite XOR function. The* **infinite XOR game** *$G_f$ is played as follows. Player 0 picks a word $w_0 \in B^+$. Then, Player 1 picks a word $w_1 \in B^+$. Player 0 picks a word $w_2 \in B^+$, Player 1 picks a word $w_3 \in B^+$, and so on. Thus, we obtain a* **play** *which is an infinite sequence of words: $w_0w_1, w_2, w_3, \ldots$ Player $i$ wins iff $f(w_0w_1w_2w_3\ldots) = i$.*

**Definition 3.** *A* **strategy for player** *$i$ in $G_f$ is a function*

$$S : \bigcup_{k\in\omega} (2^+)^{2k+i} \to 2^+.$$

*A play $w_0, w_1, w_2, \ldots$ is* **consistent** *with $S$ iff $w_{k+1} = S(w_0, w_1, \ldots, w_k)$, for each suitable $k$ (i.e., each move of player $i$ is given by $S$). $S$ is* **winning** *iff Player $i$ wins each play consistent with $S$.*

Note that in the above we view $(2^+)^m$ as a product $2^+ \times 2^+ \times \ldots \times 2^+$ ($m$ times) rather than concatenation $2^+2^+ \ldots 2^+$ ($m$ times). Such an identification would restrict the set of strategies, but in fact it would not affect our result. Note that, by definition, $(2^+)^0 = \{\emptyset\}$.

We use the strategy stealing argument to show that no player has a winning strategy in the infinite XOR game. Intuitively, whenever our opponent answers our move $v$ with $w$, we could have instead changed one bit in $v$ to obtain another word $v'$, and play $v'w$ instead of $v$. This effectively exchanges the roles of the two players, so if our opponent had a winning strategy, we can use it now for ourselves. The precise argument follows.

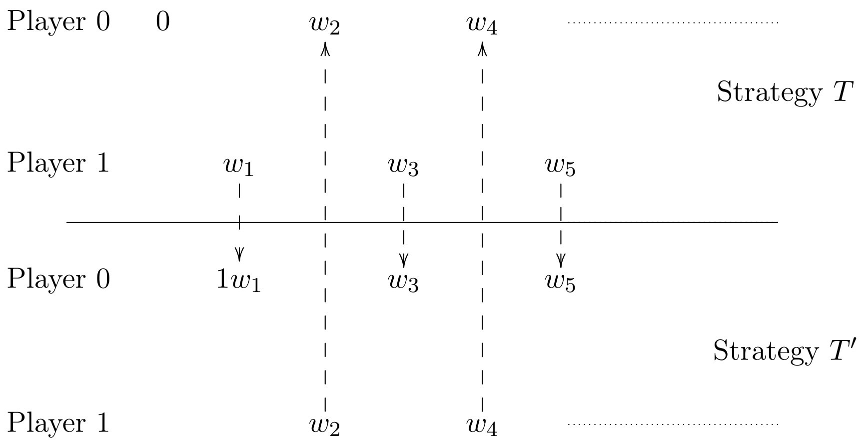
**Theorem 2.** *No player has a winning strategy in an infinite XOR game $G_f$.*

*Proof.* Let $S$ be a strategy for Player $1-i$. We construct two strategies for Player $i$, $T$ and $T'$, such that one of them will win at least one play against $S$.

Consider first $i = 0$, and let the first move of Player 0 (who starts the game) be $T(\emptyset) = 0$. Suppose the answer of Player 1 is $S(0) = w_1$. We let $T'(\emptyset) = 1w_1$. Now, if $S(1w_1) = w_2$ then we let $T(0, w_1) = w_2$, and if $S(0, w_1, w_2) = w_3$, we let $T'(1w_1, w_2) = w_3$, and so on. In symbols, we let

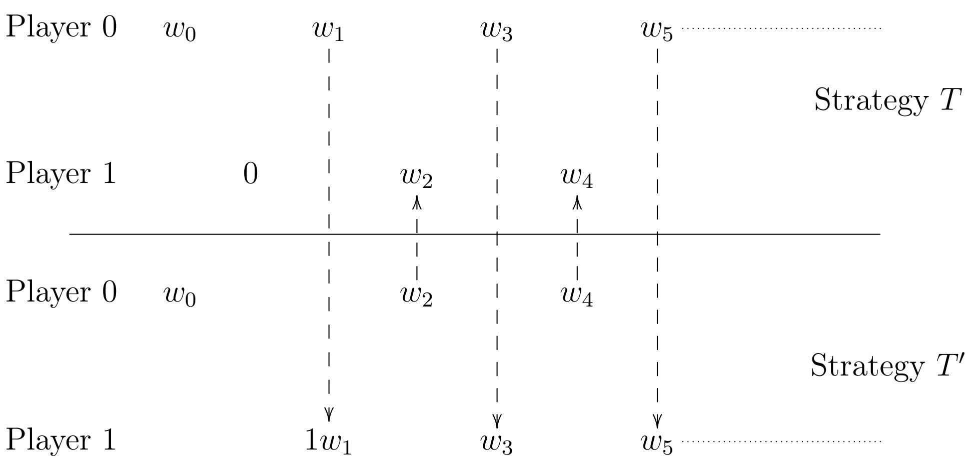
$$\begin{aligned} T'(1w_1, w_2, \ldots, w_{2k}) &= S(0, w_1, \ldots, w_{2k}) \\ T(0, w_1, \ldots, w_{2k+1}) &= S(1w_1, w_2, \ldots, w_{2k+1}). \end{aligned}$$

In the figure below, the dashed arrows indicate the "stealing".




Note that in the two plays above Player 1 uses his strategy $S$, but the resulting sequences differ exactly in one bit (actually the bit number 0), hence one of the plays is lost by Player 1.

The argument for $i = 1$ is similar. Let the starting move of Player 0 be $S(\emptyset) = w_0$. We let $T(w_0) = 0$. Now suppose $S(w_0, 0) = w_1$. We let $T'(w_0) = 1w_1$. If $S(w_0, 1w_1) = w_2$, we let $T(w_0, 0, w_1) = w_2$, and so on, as represented on the figure below.




Analogically as above, Player 0 uses her strategy $S$, but the resulting sequences differ exactly in one bit (namely the bit number $|w_0|$), hence this strategy cannot be winning.

Hence the game $G_f$ is indeed indeterminate. □

# Chapter 2

# Determinacy of chess

**Zermelo's theorem on chess**

In 1913, Ernst Zermelo proved mathematically that in the play of chess one of the following possibilities holds.

- White have a winning strategy,
- Black have a winning strategy,
- both parties have the strategies to achieve at least a draw.

We will show this theorem on a more abstract level, not forgetting its algorithmic aspect.

## 2.1 Arenas

We think of two players who will play by changing "situations" of the game according to some rules. The current situation is always known to both players, and it determines who should play. We name the players Eve and Adam.

An *arena* is a directed graph, consisting of

- the set of *positions* $Pos$,
- the set of *moves* $Move \subseteq Pos \times Pos$.

The set of positions is partitioned into two disjoint sets $Pos_{\exists}$ and $Pos_{\forall}$ of positions of Eve and Adam, respectively, i.e.,

$$\begin{aligned} Pos_{\exists} \cup Pos_{\forall} &= Pos \\ Pos_{\exists} \cap Pos_{\forall} &= \emptyset \end{aligned}$$

(any of these sets can be empty). Relation $(p, q) \in Move$ is usually written by $p \to q$. A position $p$, such that $(\forall q)\, p \not\to q$ is called *terminal*, which we also write $p \not\to$.

A *play* is a finite or infinite sequence

$$q_0 \to q_1 \to q_2 \to \ldots \to q_k (\to \ldots)$$

A finite play that ends in a position $q_k \not\to$ is *lost* by the player who owns this position. Thus, a player who cannot move, loses. Note that we allow $k = 0$, i.e., a play can consist of just one position (no move), but an empty sequence $\varepsilon$ is not considered as a play.

### 2.1.1 Game equation

In symbols, the equation for Eve is

$$X \;=\; (E \cap \Diamond X) \cup (A \cap \Box X) \tag{2.1}$$

and the dual equation for Adam

$$Y \;=\; (A \cap \Diamond Y) \cup (E \cap \Box Y). \tag{2.2}$$

Here, the variables $X$ and $Y$ range over subsets of $Pos$, $E = Pos_{\exists}$, $A = Pos_{\forall}$, $\cup$ and $\cap$ have their usual meaning and, for any $Z \subseteq Pos$,

$$\begin{aligned} \Diamond Z &= \{v : \exists w,\, w \in Z \wedge Move(v,w)\} \\ \Box Z &= \{v : \forall w,\, Move(v,w) \Rightarrow w \in Z\}. \end{aligned}$$

To ensure that the game equations have solutions, we recall a classical result about fixed-point equations in ordered sets. A *complete lattice* is a partially order set $\langle L, \leq \rangle$, such that each subset $Z \subseteq L$ has the least upper bound $\bigvee Z$, and the greatest lower bound $\bigwedge Z$. In particular, $\bigvee \emptyset$ is the least element denoted $\bot$, and $\bigwedge \emptyset$ is the greatest element denoted $\top$.

**Theorem 3** (Knaster-Tarski)**.** *A monotonic mapping $f$ over a complete lattice $\langle L, \leq \rangle$ has a* least fixed point

$$\mu x.f(x) \;=\; \bigwedge \{z : f(z) \leq z\} \tag{2.3}$$

*and a greatest fixed point*

$$\nu x.f(x) \;=\; \bigvee \{z : z \leq f(z)\}. \tag{2.4}$$

*Proof.* We show the result for the greatest fixed point. Let

$$a \;=\; \bigvee \underbrace{\{z : z \leq f(z)\}}_{A}$$

By monotonicity of $f$, $z \leq a$ implies $f(z) \leq f(a)$. For $z \in A$, this further implies $z \leq f(z) \leq f(a)$. Hence, $f(a)$ is an upper bound of $A$, which follows $a \leq f(a)$. Using again monotonicity of $f$, we obtain $f(a) \leq f(f(a))$. Hence $f(a) \in A$, which follows the converse inequality $f(a) \leq a$. $\square$

We consider the mappings *Eve* and *Adam* defined in the complete lattice $\langle \wp(Pos), \subseteq \rangle$ by

$$\begin{aligned} Eve\,(Z) &= (E \cap \Diamond Z) \cup (A \cap \Box Z) \\ Adam\,(Z) &= (A \cap \Diamond Z) \cup (E \cap \Box Z). \end{aligned}$$

Clearly these mappings are monotonic, which follows that the game equations (2.1) and (2.2) do have solutions.

### Traps and gardens of Eden

A set of positions $Z \subseteq Pos$ is a *trap* for Adam if $Z \subseteq Eve\,(Z)$. Intuitively, Adam cannot go out of there, so the message for him is:

*No exit.*

A set of positions is *gardens of Eden* for Adam if $Adam\,(Z) \subseteq Z$. Now the message for Adam is

*No entrance.*

Combining these concepts with the formulas (2.3) and (2.4) of the Knaster-Tarski Theorem, we can note the following.

The *greatest* trap for Adam is a garden of Eden for Eve.

The *least* garden of Eden for Eve is a trap for Adam.

These concepts are indeed dual. We use notation

$$\overline{Z} = Pos - Z.$$

**Lemma 2.**

$$\overline{Eve\,(X)} = Adam\,(\overline{X})$$

*Proof.* We have

$$\begin{aligned} \overline{Eve\,(X)} &= \overline{(E \cap \Diamond X) \cup (A \cap \Box X)} \\ &= \left(\overline{E \cap \Diamond X}\right) \cap \left(\overline{A \cap \Box X}\right) \\ &= \left(\overline{E} \cup \overline{\Diamond X}\right) \cap \left(\overline{A} \cup \overline{\Box X}\right) \\ &= \left(A \cup \Box \overline{X}\right) \cap \left(E \cup \Diamond \overline{X}\right) \\ &= \left(A \cap \Diamond \overline{X}\right) \cup \left(E \cap \Box \overline{X}\right) \cup \underbrace{(A \cap E)}_{\emptyset} \cup \left(\Diamond \overline{X} \cap \Box \overline{X}\right) \\ &= Adam\,(\overline{X}). \end{aligned}$$

(The last summand can be omitted, because it is included in the first two.) $\Box$

We note the following consequences.

**Proposition 2.** *The complement of a trap for Adam is a garden of Eden for him; similarly for Eve.*

*Proof.* Immediate from Lemma 2 and the definitions. □

**Proposition 3.**

$$\begin{aligned} \overline{\mu X.\mathit{Eve}\,(X)} &= \nu Y.\mathit{Adam}\,(Y) \\ \overline{\nu X.\mathit{Eve}\,(X)} &= \mu Y.\mathit{Adam}\,(Y) \end{aligned}$$

*Proof.* It follows from Lemma 2 that $X$ is a fixed point of $\mathit{Eve}$ if and only if its complement $\overline{X}$ is a fixed point of $\mathit{Adam}$. But the smaller is $X$, the bigger is its complement, and *vice versa*. □

**Corollary 1.** *The set of all positions of an arena can be partitioned into three disjoint sets (possibly empty):*

$$\begin{aligned} \mu X.\mathit{Eve}\,(X), \\ \mu X.\mathit{Adam}\,(X), \\ \nu X.\mathit{Eve}\,(X) \cap \nu X.\mathit{Adam}\,(X). \end{aligned}$$

We will see in the next section that these sets correspond precisely to the regions, where Eve or Adam can win in finite time, or where both players have strategies, which guarantee them to *survive*.

**Exercise 1.** *Show that the union of any family of traps for a player is again a trap for this player. Note that by Corollary 2 this implies that the intersection of any family of gardens of Eden for a player is again a garden of Eden for this player.*
*Which more general property of ordered sets underlines these facts ? (Remember the Knaster-Tarski Theorem.)*

## 2.2 Strategies

Intuitively, a strategy for a player, say Eve, advises her how to continue the play, assuming that she has followed the advices given so far. Note that a strategy must "answer" all legal moves of Adam.

A *strategy* for Eve can be represented by a non-empty set of finite plays $S$, such that

- if $\pi \in S$ and $\mathrm{last}(\pi) \in \mathit{Pos}_\exists$ then $(\exists! q)\, \pi q \in S$,
- if $\pi \in S$ and $\mathrm{last}(\pi) \in \mathit{Pos}_\forall$ then $(\forall q)\, (\mathrm{last}(\pi) \to q) \Rightarrow \pi q \in S$,

- $S$ is closed under initial segments, i.e., if $s_0 s_1 \ldots s_k \in S$ then $s_0 s_1 \ldots s_i \in S$, for $0 \leq i \leq k$.

Here $\mathrm{last}(\pi)$ denotes the last element of the sequence $\pi$.

A play (finite or infinite) $\pi$ *conforms* with a strategy $S$ if any finite prefix of $\pi$ belongs to $S$. Note that a strategy can be visualized as a tree (after adding an auxiliary element as a root). Then the conforming plays form the branches of this tree.

### 2.2.1 Strategy functions

Given a strategy $S$, the plays of length one contained in $S$ can be identified with positions; we call them the *initial positions* of the strategy $S$, and denote

$$start\,(S) \quad = \quad S \cap Pos.$$

A position $q$ is *safe* for Eve, whenever it is initial in some strategy for her, i.e., $q \in start\,(S)$, for some $S$. (We emphasize that $q$ itself need not be position of Eve !)

**Exercise 2.** *1. Show that if a position is safe for Eve then she can play in such a way that she will never lose a play starting from this position (but a play can be infinite).*

*2. Give an example of an arena, where some position is safe for both players.*

*3. Give an example of an arena, where the positions safe for Eve are precisely the positions of Adam, and* vice-versa.

A strategy $S$ (for Eve) induces a *strategy function* $f_S$, which is a partial function over the set of plays[1]. The function $f_S$ maps any $\pi \in S$ with $\mathrm{last}(\pi) \in Pos_\exists$ to the unique $q$, such that $\pi q \in S$, and is undefined otherwise.

A strategy $S$ is *positional* (or *memory-less*) if this function depends only on the current position, i.e.,

$$\mathrm{last}(\pi) = \mathrm{last}(\rho) \quad \Rightarrow \quad f_S(\pi) = f_S(\rho).$$

In this case, $f_S$ induces a partial function $\hat{f}_S$ on $Pos_E$, with

$$dom\, \hat{f}_S \quad = \quad \{\mathrm{last}(\pi) : \pi \in dom\, f_S\},$$

defined by

$$\hat{f}_S(\mathrm{last}(\pi)) \quad = \quad f_S(\pi).$$

In the sequel, we will omit the hat and identify $\hat{f}_S$ with the strategy function of the positional strategy $S$.

[1]Recall that for a partial function $f : X \to Y$, we denote $dom\, f = \{x : f(x) \text{ is defined }\}$ and $range\, f = \{f(x) : x dom\, f\}$.

We now characterize those partial functions $f$, which represent some positional strategies for Eve. Let $Z$ be a trap for Adam (i.e., $Z \subseteq \mathit{Eve}\,(Z)$, cf. page 11). A function $f$ is a *witnesses for the trap* $Z$ if $\mathit{dom}\, f = Z \cap \mathit{Pos}_\exists$ and, for all $p$ in $\mathit{dom}\, f$, $f(p) \in Z$. Note that, by Axiom of Choice, such function exists for any trap (it may be empty if $Z \subseteq \mathit{Pos}_\forall$). We call a function *safe for Eve* if it is a witness for some trap for Adam.

**Proposition 4.** *A function* $f : \mathit{Pos}_\exists \supseteq \mathit{dom}\, f \to \mathit{Pos}$ *is a strategy function for some positional strategy for Eve, iff* $f$ *is safe for Eve.*

*Moreover, if* $f$ *is a witness for a trap (for Adam)* $Z$ *then* $f = f_S$*, for some positional strategy* $S$*, such that* $\mathit{start}\,(S) = Z$*.*

*Proof.* Let $f$ be a safe function, and suppose it is a witness for some trap for Adam, $Z$. We construct a positional strategy $S$ by stages. Let

$$S_1 \;=\; Z$$

(so the additional claim is satisfied). Next, let

$$\begin{aligned} S_{n+1} \;=\; & \{wf(\mathrm{last}(w)) : w \in S_n \wedge \mathrm{last}(w) \in \mathit{Pos}_\exists\} \,\cup \\ & \{wp : w \in S_n \wedge \mathrm{last}(w) \in \mathit{Pos}_\forall\}. \end{aligned}$$

It follows by induction that any path $w$ in $S_n$ has all its positions in $Z$; hence $f(\mathrm{last}(w))$ is defined, whenever $\mathrm{last}(w) \in \mathit{Pos}_\exists$. It is straightforward to see that

$$S \;=\; \bigcup S_n$$

is a positional strategy for $\mathit{Eve}$, and $f_S = f$.

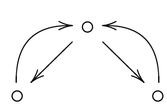
Conversely, if $S$ is a positional strategy and $f = f_S$, we can take $Z = \{\mathrm{last}(\pi) : \pi \in S\}$. $\square$

**Remark** Let us see the above properties in terms of graphs. For $f$ and $Z$ as in Proposition 4, consider a sub-arena obtained by restricting the set of positions to $Z$ and the set of moves to

$$\mathit{Move}' = \mathit{Move} \cap ((\mathit{Pos}_\forall \cap Z) \times \mathit{Pos} \,\cup\, f)\,.$$

That is, we remove all Eve's moves except for those indicated by $f$. Then no play on this arena can be lost by Eve.

**Exercise 3.** *Consider an arena*



*where all positions belong to Eve. Compute the cardinalities of the sets of*

1. *all strategies of Eve,*

2. *all positional strategies of Eve,*

3. *all safe functions of Eve,*

4. *all strategies of Adam,*

5. *all safe functions of Adam.*

## 2.3 Determinacy

In game theory (and practice), we often construct new strategies from some previous ones. The simplest operation is that of *sub-strategy*.

**Lemma 3.** *Let $S$ be a strategy of Eve and $w \in S$. Let*

$$S.w \quad = \quad \{v : wv \in S\}$$

*Then $S.w$ is also a strategy of Eve. Moreover, if $S$ is positional, so is $S.w$.*

*Proof.* Straightforward from the definition. □

### 2.3.1 Safe positions

The following lemma prepares the first important connection between the game equation (2.1) and strategies.

**Lemma 4.** *If $Z$ is a trap for Adam then all positions in $Z$ are safe for Eve. Moreover, there is a* positional *strategy $S$ for her, such that $Z \subseteq S$.*

*Proof.* By Proposition 4, it is enough to define a safe function on $Z \cap Pos_\exists$. Since $Z$ is a trap, for any $p \in Z \cap Pos_\exists$, there is some $q \in Z$, such that $p \to q$. The existence of a safe function follows from the axiom of choice. □

**Proposition 5.** *The set of all safe positions of Eve is the* **greatest** *fixed point of the operator $Eve$. Moreover, there exists a* **positional** *strategy for Eve, whose set of initial positions coincides with this set.*

*Proof.* Let $Z_0$ be the set of all safe positions of Eve. We first show

$$Z_0 \quad \subseteq \quad Eve\,(Z_0). \tag{2.5}$$

Let $p \in Z_0$ and let $S$ be a strategy of Eve starting from $p$. Consider the strategy $S.p$ (cf. Lemma 3). There are two possibilities. If $p \in Pos_\exists$ then $pq \in S$, for some $q$. Then $q \in S.p$, hence $q$ is safe. If $p \in Pos_\forall$ then $pq \in S$, for all $q$, such that $p \to q$. Hence all such $q$'s are safe. In any case, $p \in Eve\,(Z_0)$. (2.5) and (2.4) give us $Z_0 \subseteq \nu x.Eve\,(x)$. The converse inequality, as well as the strategy claim, follows directly from Lemma 4. □

### 2.3.2 Winning in finite time

Recall that a player loses the game if a (finite) play arrives in a position from which this player has no move (cf. page 10); this is like *checkmate* in chess. In such situation, the other player *wins in finite time*.

If a position is not safe for Adam then he has no guarantee that he will not lose. But is Eve sure to win ? The answer is not completely obvious, since we have seen in Section 1.3.2 that games can in general be indeterminate. Again, the game equation will be helpful.

A strategy $S$ of Eve is *finitely winning* if any play that conforms with $S$ is finite – and hence lost for Adam. A position $p$ is finitely winning if $p \in S$, for some strategy with this property. (Again recall that $p$ need not to be a position of Eve !)

The following fact is analogous to Proposition 5.

**Proposition 6.** *The set of all positions finitely winning for Eve is the* **least** *fixed point of the operator Eve . Moreover, there exists a* **positional** *finitely winning strategy, whose set of initial positions coincides with this set.*

*Proof.* Let $W'$ be the set of positions that are initial for some *positional* finitely winning strategy. We first show

$$Eve\,(W') \quad \subseteq \quad W'. \tag{2.6}$$

Suppose $p \in Eve\,(W')$. If $p \in Pos_\exists$ then Eve can make a move to a position $q$, from which there is a positional finitely winning strategy $S$. Let $f_S$ be the strategy function induced by $S$ (see page 13). If $p \in dom\, f_S$ then $p \in W'$, and we are done. Otherwise, we can consider the extended function $f_S \cup \{(p, q)\}$. This function is clearly safe, and it is straightforward to see that the strategy constructed in Proposition 4 is positional and finitely winning. Hence $p \in W'$.

If $p \in Pos_\forall$ then, for each $q$, such that $p \to q$, we have $q \in W'$. Then, for each such $q$, we have some safe function $f_q$, inducing a positional finitely winning strategy from $q$. However, the union of these functions need not be a function. To remedy this, we will use the possibility of well-ordering of any set, which is the "great" Theorem by Zermelo. But first, a lemma is in order.

**Lemma 5.** *Let $\alpha$ be an ordinal, and let $S_\xi$ be a family of positional, finitely winning strategies for Eve, each $S_\xi$ induced by*[2] *a safe function $f_\xi$. Let $f$ be defined on $\bigcup_{\xi<\alpha} dom\, f_\xi$ by*

$$f(p) \quad = \quad f_\xi(p)$$

*where $\xi$ is the least, such that $f_\xi(p)$ is defined. Then $f$ is safe and the induced strategy is finitely winning.*

*Proof.* (Of the lemma.) Let $init(S_\xi)$ be the set of initial positions of $S_\xi$. We have $dom\, f_\xi = S_\xi \cap Pos_\exists$, for each $\xi < \alpha$. Then $I = \bigcup_{\xi<\alpha} S_\xi$ is a trap (see Exercise 1), and moreover

[2] That is, $S_\xi$ coincides with $Z$ of Proposition 4.

$dom\, f = I \cap Pos_\exists$ and $range\, f \subseteq I$. Hence $f$ is safe and induces a positional strategy with the set of initial positions $I$. Let us see that this strategy is finitely winning. Suppose on the contrary that some infinite play

$$q_0 \to q_1 \to q_2 \to \dots$$

conforms with this strategy. Note that no suffix of this play can conform with any $S_\xi$, because each $S_\xi$ is finitely winning. Hence, Eve must infinitely many times switch the strategy from $S_\xi$ to some other $S_{\xi'}$. But this may only happen if $\xi' < \xi$. However, we cannot have an infinite decreasing sequence of ordinal numbers, a contradiction. More precisely, for each $n$, let $ord\,(q_n)$ be the least $\xi$, such that $q_n \in init(S_\xi)$. Then this sequence of ordinals is non-increasing, but it cannot stabilize, which yields a contradiction. □

We come back to the proof of Proposition 6. If we well-order the set of successors of $p$ and apply the above lemma, we obtain a positional finitely winning strategy which has all successors of $p$ among its initial positions. Then it easy to see that the position $p$ is finitely winning as well. (If this strategy is induced by a safe function $f$ and some trap $Z$, we can extend this trap by $p$.)

This completes the proof of ( 2.6).

By the equation ( 2.3) of the Knaster-Tarski Theorem, this implies $\mu X.Eve\,(X) \subseteq W'$. But we already know that all positions outside $\mu X.Eve\,(X)$ are in $\nu Y.Adam\,(Y)$ (Corollary 1), and then are safe for Adam (Lemma 4), so they cannot be finitely winning for Eve. Hence $W'$ coincides with the set $W$ of all positions finitely winning for Eve, and

$$W = W' = \mu X.Eve\,(X).$$

For each $q$ in this set, we have a positional finitely winning strategy from $q$. To obtain a single strategy good for all $q$'s, we can again use Lemma 5. □

### 2.3.3 The Zermelo theorem

We are ready to state the determinacy result which essentially comprises Zermelo's theorem about chess.

**Theorem 4** (Zermelo). *For any position $p$, one of the following possibilities holds:*

1. *$p$ is finitely winning for Eve,*
2. *$p$ is finitely winning for Adam,*
3. *$p$ is safe for both players.*

*Proof.* Let $X_{\min}$ and $X_{\max}$ be the least and greatest fixed points of $Eve$, and similarly with $Y$ for $Adam$. By Proposition 6, $X_{\min}$ and $Y_{\min}$ are disjoint. (No position can be finitely winning for both players.) Using Corollary 1, we get

$$\overline{X_{\min} \cup Y_{\min}} = X_{\max} \cap Y_{\max}.$$

Hence, if 1 or 2 do not hold, 3 must hold. □

**Remark** We leave to the reader an adaptation of Theorem 4 to the game of chess, which was the original focus of Zermelo. Clearly, in social games, infinite plays do not really happen (to our knowledge). The rules of chess specify the situations, when a draw can occur. However, we can artificially expand such situations to infinite loops, so that the above theorem for the modified game induces the claim of Zermelo for chess.

## 2.4 Algorithms to compute strategies

An algorithmic aspect of Theorem 4 is twofold:

- compute the winning/safe regions of Eve and Adam,
- compute respective strategies, i.e.,
  - a winning strategy for Eve in region 1,
  - a winning strategy for Adam in region 2,
  - the draw strategies for both players in region 3.

These questions make sense whenever an arena of the game is finite, or at least finitely presented. In this subsection, we restrict ourself to *finite* arenas.

To compute the winning regions, it is clearly enough to solve the following question.

Given an arena and a position $p \in Pos$. Is this position finitely winning for Eve ?

We can see at first glance that this question is in $NP$, as we can guess a positional strategy and verify that it is finitely winning winning. More specifically, a strategy can be represented by partial function $f : Pos_{\exists} \to Pos$. To verify that $f$ represents a strategy, it is enough to check if it is safe in the sense of Proposition 4. To verify if the induced strategy is finitely winning, it is enough to complete the function $f$ (understood as a set of edges) by the moves of Adam, and then check if the subgraph reachable from $p$ contains infinite paths. All these questions can be easily checked in polynomial time.

A similar argument shows the membership in $NP$ for other regions, in particular for the safe positions of Adam, which constitute the complement of the above set. Therefore, the question is in fact in $NP \cap$ co-$NP$.

However, this estimation is far too weak, as indeed the problem can be solved in polynomial time. A natural argument comes from the fixed point representations given in Propositions 5 and 6. We recall a useful property of fixed points.

Let $\langle L, \leq \rangle$ be a complete lattice with the least and greatest elements denoted by $\perp$ and $\top$, respectively. For a monotonic mapping $f : L \to L$, define a transfinite sequence $f^{\xi}$ by

$$\begin{aligned} f^{\xi+1}(\perp) &= f\left(f^{\xi}(\perp)\right) \\ f^{\eta}(\perp) &= \bigvee_{\xi<\eta} f^{\xi}(\perp) \text{ for limit } \eta. \end{aligned}$$

By monotonicity of $f$, this sequence is non-decreasing, and therefore stabilizes for some $\xi$ with the value [3] $\bigvee f^{\xi}(\bot)$.

The sequence $f(\top)$ is defined similarly with $\bigwedge$ instead of $\bigvee$. We have the following representations of the least and greatest fixed points, alternative to those given in Theorem 3.

**Lemma 6.** *For a monotonic mapping $f$ over a complete lattice $\langle L, \leq \rangle$,*

$$\mu x.f(x) = \bigvee f^{\xi}(\bot) \tag{2.7}$$

$$\nu x.f(x) = \bigwedge f^{\xi}(\top). \tag{2.8}$$

*Proof.* We have $\mu x.f(x) \geq \bot = f^0(\bot)$, and this inequality continues to hold for all $\xi$, because the application of $f$ does not change the left-hand side. If $f^{\xi+1}(\bot) = f^{\xi}(\bot)$ then $f^{\xi}(\bot)$ as a fixed point of $f$, hence the converse inequality also holds.

The argument for (2.8) is similar. □

In our case, if we wish to compute the least/greatest fixed points of the operators *Eve*/*Adam* over a finite arena, we need at most $|Pos|+1$ steps of the iteration, which roughly gives the time $\mathcal{O}(|Pos| \cdot |Move|)$. Moreover, the positional strategies can be computed at the same time.

For example, the following program computes a finitely winning strategy for the player Eve.

$\mathrm{Win}_E := \mathrm{Stra}_E := \emptyset$

**Repeat**

$\mathrm{Old} := \mathrm{Win}_E$

$\mathrm{Win}_E := \mathit{Eve}\,(\mathrm{Win}_E)$

$\Delta := \mathit{Eve}\,(\mathrm{Win}_E) - \mathrm{Old}$

$\mathrm{Stra}_E :=$ for each $p \in \Delta \cap Pos_{\exists}$, add to $\mathrm{Stra}_E$ some move $(p, q)$, with $q \in \mathrm{Win}_E$
{ By definition of *Eve*, this step is well defined. }

**until** $\Delta = \emptyset$.

However, this algorithm is still not optimal. With some care, we may ensure that each move is examined only once.

**Exercise 4.** *Design an algorithm, which computes the winning regions of Theorem 4 and respective positional strategies in time linear with respect to the size of an arena.*

[3] Here, $\xi$ ranges over ordinals less than the first ordinal above $|L|$ or, with a slight abuse of terminology, over all ordinals.

# Chapter 3

# Parity games

In this section, we continue to study perfect information games on graph arenas, but we now allow more winning scenarios. Either of the players will be able to win in infinity, depending on some winning criteria. For example, Eve may wish to visit three distinct nodes infinitely often. It is convenient to specify the winning criteria independently from particular arenas, and to this end we will use colors. We usually put colors on positions, and specify the the winner by the sequence of colors encountered in an infinite play. Alternatively, we may put colors on moves; this option is not very different, and will be also sometimes in use.

## 3.1 Games on colored graphs

**Definition 4.** *A graph game is specified by the following items:*

$$G = \langle Pos_\exists, Pos_\forall, Move, C, rank, W_\exists, W_\forall \rangle, \tag{3.1}$$

*where $Pos_\exists, Pos_\forall, Move$ constitute an arena in the sense of Section 2.1, $C$ is a set of* **colors***, $rank : Pos \to C$ is a coloring (or ranking) function, and $W_\exists, W_\forall \subseteq C^\omega$ are the* **winning criteria** *for Eve and Adam, respectively, satisfying $W_\exists \cap W_\forall = \emptyset$.*

*The rules of winning/loosing finite plays are as before. An infinite play $q_0, q_1, q_2, \ldots$ is won by Eve if the sequence*

$$rank(q_0), rank(q_1), rank(q_2), \ldots$$

*belongs to $W_\exists$, and by Adam if this sequence belongs to $W_\forall$; otherwise there is a draw.*

We usually assume $W_\exists \cup W_\forall = C^\omega$ (no draw). In this case, it is enough to specify one of the sets $W_\exists, W_\forall$; by convenience we specify the criterion for Eve.

The concept of strategy, as well as positional strategy, is defined exactly as in Section 2.2. An Eve's strategy $S$ is *winning*, if every play conforming with $S$ is won by Eve. It is *safe*, if every play conforming with $S$ is either won by Eve or a draw. The analogous concepts for Adam are defined similarly.

**Definition 5.** *A position $p$ is* **winning for Eve** *if it belongs*[1] *to some strategy winning for Eve. It is* **safe for Eve** *if it belongs to some strategy safe for Eve. The analogous concepts for Adam are defined similarly. (Note that a position winning for a player need not belong to this player.)*

*A graph game $G$ is* **determined** *if every position is winning for Eve, or winning for Adam, or safe for both players.*

Although determinacy depends on the whole game, we will see in the sequel that some winning criteria guarantee determinacy for all arenas.

**Example 1.** *The game considered in Section 2, can be presented in the above setting with one (dummy) color and $W_\exists = W_\forall = \emptyset$. Theorem 4 shows that all such games are determined. If we change the rules so that any infinite game is won by Eve, we can model it by $W_\exists = \{c^\omega\}$, $W_\forall = \emptyset$. Again, all such games are determined. We call this last game a* **survival for Eve**.

### Isomorphism of games

Intuitively, two games are isomorphic if one can be obtained from the other by possibly interchanging everything: positions, colors, and even players.

More specifically, let

$$\begin{aligned} G &= \langle Pos_\exists, Pos_\forall, Move, C, rank, W_\exists, W_\forall \rangle, \\ G' &= \langle Pos'_\exists, Pos'_\forall, Move', C', rank', W'_\exists, W'_\forall \rangle \end{aligned}$$

be two games as in Definition 4. An isomorphism $G \to G'$ consists of three bijections

$$\begin{array}{lcl} \gamma_{players} & : & \{\exists, \forall\} \to \{\exists, \forall\} \\ \gamma_{positions} & : & Pos \to Pos' \\ \gamma_{colors} & : & C \to C' \end{array}$$

As we may assume that the domains of these mappings are disjoint, we will omit the subscripts in the sequel. We require the following conditions, for any $p, q \in Pos$, $c_0, c_1, \ldots \in C$, and $X \in \{\exists, \forall\}$,

$$\begin{aligned} Move(p, q) &\Leftrightarrow Move'(\gamma(p), \gamma(q)) \\ (c_0, c_1, \ldots) \in W_X &\Leftrightarrow (\gamma(c_0), \gamma(c_1), \ldots) \in W'_{\gamma(X)} \end{aligned}$$

## 3.2 Parity games

In these games, colors form a finite segment of natural numbers, and we usually call them *ranks*. Intuitively, Eve likes even ranks and Adam odd ones. The winner of an infinite play is determined by the highest rank, which repeats infinitely often.

[1] Recall that we identify a position with a one-element play.

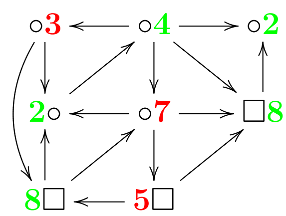


Figure 3.1: A parity game.

More specifically, for $i \le k$, let

$$\begin{aligned} C_{i,k} &= \{i, i+1, \ldots, k\} \\ M_{i,k} &= \{u \in C_{i,k}^{\omega} : \limsup_{n\to\infty} u_n \text{ is } \textbf{even} \} \\ \overline{M_{i,k}} &= C_{i,k}^{\omega} - M_{i,k}. \end{aligned}$$

A parity game of **index** $(i, k)$ is a game of the form

$$G = \langle Pos_{\exists}, Pos_{\forall}, Move, C_{i,k}, rank, M_{i,k}, \overline{M_{i,k}} \rangle$$

That is, in a parity game, an infinite play is won by Eve if the highest rank repeating infinitely often is even, and by Adam if it is odd (Figure 3.1).

It is easy to see that a parity game of index $(i, k)$ is isomorphic to a game of index $(i+2, k+2)$ over the same arena and with the same players, where the isomorphism affects only colors

$$\gamma(j) = j+2, \text{ for } i \le j \le k$$

Therefore, it is enough to consider indices $(i, k)$ with $i = 0$ or $i = 1$.

The indices form a hierarchy, presented on Figure 3.2, where a game of index $(i, k)$ can be represented as a game of any index index $(i', k')$ lying above, *via* an isomorphism, which affects only colors. For example, a game of index $(0, 1)$ is a special case of a game of index $(1, 3)$ *via* an isomorphism $\gamma(j) = j + 2$.

The indices lying on the same level, i.e., $(0, n)$ and $(1, n + 1)$ are called *dual*.

**Proposition 7.** *Any parity game of index* $(0, n)$ *is isomorphic to some parity game of index* $(1, n)$*, and* vice versa.

*Proof.* We obtain a game of index $(1, n)$ from a game of index $(0, n)$, by interchanging the players and shifting the ranks by 1. The inverse isomorphism yields the second claim. $\square$

### 3.2.1 Positional determinacy of parity games

Parity games have very good properties. Not only are they determined (which can be inferred from a more general theorem on Borel games), but they enjoy *positional determinacy*: the

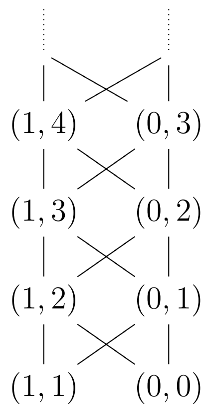



Figure 3.2: Index hierarchy.

winner can always use a positional strategy. Moreover, this strategy is *global*, i.e., does not depend on initial position. The result was first proved by Emerson and Jutla [3] and independently by A.W.Mostowski [10].

**Theorem 5.** *Let $G$ be a parity game of index $(i, k)$. There exist* **positional** *strategies $S_E$ and $S_A$, for Eve and Adam, respectively, such that*

$$Pos \quad \subseteq \quad S_E \cup S_A.$$

*Proof.* See [12]. □

### 3.2.2 Muller games

We first introduce a construction, which adds some information to the positions of a game, which can make it easier to solve. the game easier to solve.

**Definition 6** (Cylindrification)**.** *Let $G$ be a graph game in the sense of Definition 4, with the set of colors $C$. Let $M$ be a set, which we call* **memory***, and let* **up** *be a partial function from $C \times M$ to $M$, which we call* **update***. We define a new arena with the set of positions $Pos \times M$, where the owner of a position $(p, m)$ is the same as for $p$. There is a move from $(p, m)$ to $(q, n)$, whenever $p \to q$ is a move in the original arena and* **up** $(rank(p), m) = n$*. Any game using this new arena is a* **cylindrification** *of $G$ by $M$ and* **up** *.*

Note that, whenever **up** $(rank(p), m)$ is undefined, there is no move from a position $(p, m)$, even if there were some moves in the original game.

For an infinite word $u \in C^\omega$, let $Inf\,(u)$ be the set of colors that repeat infinitely often in $u$, in symbols

$$Inf\,(u) \quad = \quad \{c : |u^{-1}(\{c\})| = \omega\} \tag{3.2}$$

If $C$ is finite then $Inf\,(u)$ is non-empty, for any $u$. For a family of sets $\mathcal{F} \subseteq \wp(C)$, let

$$M_{\mathcal{F}} \;=\; \{u \in C^{\omega} : Inf\,(u) \in \mathcal{F}\} \tag{3.3}$$

Note that

$$\overline{M_{\mathcal{F}}} \;=\; M_{\overline{\mathcal{F}}} \tag{3.4}$$

(where we complement in $C^{\omega}$ and in $\wp(C)$, correspondingly).

**Definition 7.** *A* **Muller game** *over a set of colors $C$ is a game with the winning criteria*

$$W_{\exists} \;=\; M_{\mathcal{F}} \qquad W_{\forall} \;=\; \overline{M_{\mathcal{F}}}$$

*for some family $\mathcal{F} \subseteq \wp(C)$.*

Note that a parity game of index $(i,k)$ can be presented as a Muller game, where $\mathcal{F}$ consists of subsets of $\{i, i+1, \ldots, k\}$ with even maximum. The Muller games form a richer class, in particular, they do not, in general enjoy positional determinacy. We will see, however, that any Muller game can be cylindrified to a parity game. This will yield the determinacy of Muller games as well as an algorithm for solving them (for finite arenas).

We fix a set of colors $C$. The memory set will consists of the so-called *latest appearance records*, in short **LAR**s; the concept first introduced (with a slight difference) by Gurevich and Harrington.

A LAR is a finite word over the alphabet $C \cup \{\natural\}$, where each symbol occurs *at most* once, and $\natural$ must appear.

The update function, for a LAR $\alpha$ and a color $c$, is defined as follows. If $c$ does not occur in $\alpha$, we let $\mathbf{up}\,(\alpha, c) = \alpha c$. If $c$ occurs in $\alpha$, we append it to $\alpha$ as well, but the previous occurrence of $c$ is replaced by the natural ($\natural$), which is moved from its previous place. To help the intuition, we may view this operation as an e-mail policy: keep only the last e-mail from a given sender.

**Lemma 7.** *Let $u \in C^{\omega}$. Define an infinite sequence of LARs by*

$$\begin{aligned} \alpha_0 &= \natural \\ \alpha_{n+1} &= \mathbf{up}\,(\alpha_n, u_n) \end{aligned}$$

*Let $\alpha_i = v_i \natural w_i$, for $i \in \mathbb{N}$, and let $set\,(w)$ denote the set of symbols occurring in a word $w$. Then there exists $m_0$, such that,*

- *for all $m \geq m_0$, $set\,(w_m) \subseteq Inf\,(u)$, and,*
- *for infinitely many $m$'s, $set\,(w_m) = Inf\,(u)$.*

*Proof.* Clearly, there is some $\ell_0$, such that $(\forall m \geq \ell_0)\ u_m \in \mathit{Inf}\,(u)$. Note that there may be some $d \notin \mathit{Inf}\,(u)$ occurring in $w_{\ell_0}$. Let $\ell_1 > \ell_0$ be such that all $c \in \mathit{Inf}\,(u)$ occur in the sequence $u_{\ell_0}, u_{\ell_0+1}, \ldots, u_{\ell_1}$. There may still be some $d \notin \mathit{Inf}\,(u)$ occurring in $w_{\ell_1}$, but all $c \in \mathit{Inf}\,(u)$ occur to the right of this $d$ (because that have appeared later than $d$). Therefore, for $\ell_2 = \ell_1 + 1$, we have $\mathit{set}\,(w_{\ell_2}) \subseteq \mathit{Inf}\,(u)$. We claim that $m_0 = \ell_2$ satisfies the conditions of the lemma. Clearly no $d \notin \mathit{Inf}\,(u)$ can appear anymore in $w_t$. To see that $\mathit{set}\,(w_t) = \mathit{Inf}\,(u)$, for infinitely many $t$'s, observe first that, for any $t \geq \ell_2$, the LAR $\alpha_t$ has the form

$$v\, c_1 \ldots c_i \natural\, c_{i+1} \cdots c_k$$

for some $v$, where $\{c_1, \ldots, c_k\} = \mathit{Inf}\,(u)$. (Starting from $\ell_1$, all symbols in $\mathit{Inf}\,(u)$ are situated to the right from all the other symbols.) Let $t' \geq t$ be the least, such that $u_{t'} = c_1$. Note that before it happens, all the symbols $c_2, \ldots, c_k$ remain to the right of $c_1$. Then $\mathit{set}\,(v_{t'+1}) = \mathit{Inf}\,(u)$.

This remark completes the proof. $\square$

We now extend the cylindrification of $G$ to a parity game of index $(0, 2|C|)$, by setting the ranks

$$\mathit{rank}(p, v\natural a_1 \ldots a_\ell) \;=\; \begin{cases} 2\ell & \text{if } \{a_1, \ldots, a_\ell\} \in \mathcal{F} \\ 2\ell + 1 & \text{otherwise} \end{cases}$$

Let us call the resulting parity game $G'$.

**Theorem 6.** *A position $p$ is winning for Eve in the Muller game $G$ iff $(p, \natural)$ is winning for Eve in the parity game $G'$.*

*Similarly, a position $p$ is winning for Adam in the Muller game $G$ iff $(p, \natural)$ is winning for Adam in the parity game $G'$.*

*Consequently, Muller games are determined.*

*Proof.* As we know already that parity games are determined (Theorem 5), it is enough to show that if a position[2] $(p, \natural)$ is winning for Eve (Adam) in $G'$ then the position $p$ is winning for Eve (resp. Adam) in $G$.

Suppose Eve has a winning strategy in $G'$ from a position $(p, \natural)$. Eve uses this strategy while playing in $G$. More specifically, any path $p_0, p_1, \ldots, p_k$ (with $p_0 = p$) in the arena of $G$ induces a unique path $(p_0, \alpha_0), (p_1, \alpha_1), \ldots, (p_k, \alpha_k)$ in $G'$, such that $\alpha_0 = \natural$, and $\alpha_{i+1} = \mathbf{up}\ (\mathit{rank}(p_i), \alpha_i)$, for $i < k$. Suppose that the last position belongs to Eve and the strategy indicates a move $(p_k, \alpha_k) \to (q, \beta)$. (Note that by the definition of cylindrification, it must be the case that $\beta = \mathbf{up}\ (\mathit{rank}(p_k), \alpha_k)$.) Then, in $G$, Eve makes a move $p_k \to q$.

We will show that this strategy is winning for Eve in $G$. Note that Eve cannot lose in finite time, since otherwise the original strategy would not be winning in $G'$. (Recall that the function **up** is everywhere defined.) Consider an infinite path $\pi = (p_0, p_1, \ldots)$ conforming

[2]The claim holds in fact for any $(p, \alpha)$.

with the defined strategy. The induces path $\pi' = ((p_0, \alpha_0), (p_1, \alpha_1), \ldots)$ has been conforming with the original strategy in $G'$. Let $\alpha_i = v_i \natural w_i$, for $i \in \mathbb{N}$. From Lemma 7, there is some $m_0$ that, for all $t \geq m_0$, $\mathit{set}\,(w_t)$ is a subset of $\mathit{Inf}\,(\pi)$, and $\mathit{Inf}\,(\pi)$ occurs infinitely often. Note that the rank of a position $(p_t, \alpha_t)$ (for $t \geq m_0$) is the highest when $\mathit{set}\,(w_t) = \mathit{Inf}\,(\pi)$, and it is even if $\mathit{Inf}\,(\pi) \in \mathcal{F}$, and odd otherwise. Since the path $\pi'$ is winning for Eve in $G'$, we conclude that $\mathit{Inf}\,(\pi) \in \mathcal{F}$, and hence the path $\pi$ is winning for Eve in $G$.

An analogous argument shows that if Adam has a winning strategy from a position $(p, \natural)$ in $G'$ then he has a winning strategy in $G$ from the position $p$. □

## 3.3 Algorithms

In this section, we consider the algorithmic problems related to parity games.

**Problem 1.** Given a parity game $G$ and a position $pPos$. Is this position winning for Eve ?

**Problem 2.** Given a parity game $G$. Compute positional winning strategies $S_E$ and $S_A$, for Eve and Adam, respectively, satisfying the requirement of Theorem 5.

We start with two complexity-theoretic observations.

### 3.3.1 Upper bound

**Proposition 8.** *Problem 1 is in* NP $\cap$ *co*-NP.

*Proof.* We first show that the problem is in *NP*. (For complexity-theoretic notions, see, e.g. citePapa or [13]. ) Let $S_f$ denote a positional strategy induced by a safe partial function $f : Pos_\exists \to Pos$, according to Proposition 4. Consider the relation

$$\{((G, p), f) : S_f \text{ is winning for Eve and } p \in S_f\} \tag{3.5}$$

where $G, p$, and $f$ are encoded in a standard way. Let us verify that this relation is polynomial. We can view $S_f$ as a subgraph of the arena of $G$, where we leave all edges sorting from $q$ if it is a position of Adam, and precisely one edge, if it is a a position of Eve, namely $(q, f(q))$. The strategy $S_f$ is winning iff any infinite path in this subgraph satisfies the parity criterion, and this amounts to the property that any circle in $S_f$ has the highest rank is even. To verify this last property, we proceed in iterative manner. For each non-trivial strongly connective component, select all nodes with the highest rank. If this rank is odd, stop and reject. Otherwise, eliminate the selected nodes, and ask the question again. Continue this process until there is no non-trivial SSC in the graph.

Clearly, if there is a circle with the highest rank odd, it will be detected at the stage when all nodes with higher ranks have been eliminated; otherwise the verification will terminate successfully.

The number of times we call the loop does not exceed the number of even ranks. In each loop, we need to compute the SCC's, which is polynomial, and hence the whole procedure is polynomial as well.

□

# Chapter 4

# Meanpayoff games

Our presentation follows closely the original paper [4].

**Bibliographical note.**

These lecture notes are based on numerous sources. The bibliography is under construction. We wish to apologize to all the authors whom we have not credited properly.

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