Introduction to Combinatorics
Graphs 2 – Problems

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1. Let (V, E, c) be a network and f be an $s - t$ flow such that $|f| \geq 0.$. Prove that f can be decomposed to $s - t$ paths and cycles, that is, there exist functions $f_1, \ldots, f_k$ such that $f = f_1 + \ldots + f_k$ and for each i we have that f_i is a valid $s - t$ flow and edges e such that $f_i(e) > 0$ form either a simple $s - t$ path or a simple cycle.

2. Let $G = (A \uplus B, E)$ be a bipartite graph where $|A| = |B|$ and all vertices have the same nonzero degree. Prove that there exists a perfect matching in G.

3. For a graph $G = (V, E)$, let $\chi'(G)$ (called *chromatic index*) be a lowest number k such that it is possible to color edges of G in k colors such that whenever two edges share a vertex, they have different colors. Prove that if G is bipartite then $\chi'(G) = \Delta(G)$ (where $\Delta(G)$ denotes the biggest degree of a vertex in G).

4. We are given an $n \times n$ board and integers $r_1, \ldots, r_n, c_1, \ldots, c_n$ such that $0 \leq r_i, c_i \leq n$ and $r_1 + \ldots + r_n = c_1 + \ldots + c_n$. Prove that we can color subset of board cells in black such that for every i there are r_i black cells in i-th row and there are c_i black cells in i-th column if and only if there are no sets $X, Y \subseteq [n]$ such that $\sum_{x \in X} r_x > \sum_{y \in Y} c_y + |X|(n - |Y|)$.

5. Let $G = (A \uplus B, E)$ be a bipartite graph where $|A| = |B|, E \subseteq A \times B$ and let $M \subseteq E$ be an arbitrary perfect matching of G. Let us create a following directed graph H on the same set of vertices where we put following edges. For every $(a, b) \in E$ we put edge from a to b in H. Moreover for every $(a, b) \in M$ we put edge from b to a in H (so edges from matching are put in both directions). Prove that:

   1) edge (a, b) belongs to some perfect matching of G if and only if a and b belong to the same strongly connected component of H (and equivalently – edge (a, b) does not belong to any perfect matching of G if and only if a and b belong to different strongly connected components)
   2) edge (a, b) belongs to every perfect matching of G if and only if a and b form a two-element strongly connected component

   Conclude that in every bipartite graph containing a perfect matching there is a vertex such that all edges incident to it belong to some perfect matching.

   (For a definition of strongly connected component refer to this link).

6. We say that an equilateral triangle is *pointed upwards* if and only if one of its sides is parallel to OX axis and its third vertex is above this side. We are given an equilateral triangle pointed upwards whose sides have length n, which is divided into n^2 equilateral unit triangles with sides of length 1. Somebody removed n of these unit triangles and all removed triangles were pointed upwards. Prove that we can divide all remaining unit triangles into diamonds (where a diamond is a pair of unit triangles sharing a side) if and only if there is no triangle pointed upwards with sides of length k containing more than k removed triangles.