# Introduction to Combinatorics
## Graphs 2 – Solutions

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1. Firstly, let us assume that $f$ has value 0 and we will prove that such flow can be decomposed into cycles. Such flows are called *circulations* – maybe the fact that they can be expressed as a sum of cycles explains the name. In that case there is no point in distinguishing source and sink. Let us create a directed graph $H$ such that $(u, v) \in E(H) \Leftrightarrow f(u, v) > 0$. If this graph does not contain any edges then it means $f$ is zero everywhere and we are done, so let us assume it has some edges. Every vertex which is not isolated has at least one ingoing edge and at least one outgoing edge (what is a consequence of Kirchhoff's law).

   *Claim:* $H$ contains a cycle.

   *Proof.* That fact follows from an argument that if we start from a vertex with some outgoing edge and keep walking along edges, we can always find outgoing edge from current vertex (since it has an ingoing edge which we used to get to it), so we can move indefinitely, which means we need to repeat some vertices on our walk. So, it means we can find a directed cycle $C$ on edges with positive value of pushed flow. □

   Let $m$ be the minimum value of pushed flow through all edges on cycle $C$. We can now create a flow $f_1$ such that its values on edges of that cycle are $m$, and 0 elsewhere. Let us now consider flow $f - f_1$. It is a valid flow that has value 0 as well and has at least one nonzero edge less than $f$ (all edges that had value $m$ on this cycle disappeared). That is, we can use induction on the number of nonzero edges to wrap this argument up.

   (*Remark*: Reasoning in that case is very similar to reasoning in Eulerian digraphs. It seems we skipped topic of Eulerian digraphs in that course, however what we have just done can be considered as a generalization of fact that graphs such that every vertex has equal indegree and outdegree can be decomposed into cycles. From that point on, it is a simple step to prove that every connected directed graph with indegrees equal to outdegrees has Eulerian cycle.)

   Now we need to reduce the general case with $|f| \geq 0$ to the case when $|f| = 0$. Let us create graph $H$ in the same way as before. If it contains a cycle then let us subtract it from $f$ the same way as before. From that point on we can assume that $H$ has no cycles. Let us start a walk from $s$ and move to an arbitrary out-neighbour as long as we have one. Since $H$ has no cycle such walk must be finite. Moreover, $t$ is the only vertex when it can terminate (since all other vertices that have at least ingoing edge have at least one outgoing as well). It means that our walk is a simple $s - t$ path. And again, we can take minimum value $m$ of flow on any edge of this path and create a flow that has value $m$ on this path and 0 elsewhere and subtract it from $f$ and use induction argument.

   *Remark:* Note that if $f$ takes integer values then all $f_i$ produced by this construction take integer values as well. Moreover, note that with knowledge that flow decomposes into paths and cycles, it is straightforward to prove Menger's theorem from lecture notes.

2. It suffices to prove that Hall's condition is satisfied for this graph. Let $d$ denote degree of all vertices. Let us take some $C \subseteq A$ and we will prove that $|N(C)| \geq |C|$. Vertices in $C$ have sum of degrees $|C|d$. However each of these $|C|d$ edges have its other end in $N(C)$, what means that sum of degrees of vertices in $N(C)$ has to be at least $|C|d$. On the other hand each vertex in $N(C)$ has degree exactly $d$, so sum of their degrees is $|N(C)|d$. We conclude that $|N(C)|d \geq |C|d \Rightarrow |N(C)| \geq |C|$, so Hall's condition is satisfied for any subset of $A$, hence it contains a perfect matching.

3. Firstly, let us prove that if both sides of our bipartite graph $G = (A \uplus B, E)$ have equal sizes and all vertices have equal degrees. Intuitively it seems like this should be "the hardest" case since if we fix $\Delta(G)$, then, the more edges we have, the harder it should be to color them with the same number of colors. For such case it can be easily seen that such coloring of edges is in fact partitioning edges into $\Delta(G)$ perfect matchings (since every vertex has to have exactly one incident edge in every color). However from previous exercise we know that such graph contains a perfect matching. We can remove it from our graph and what remains is a regular bipartite graph as well, where its maximum degree was decreased by 1. Hence, we can use induction to wrap up this reasoning.

   Now we should prove our intuition that regular case is the hardest one, or more formally, that we can reduce general case to the regular one. Firstly, if one side of our bipartite graph is bigger than the other one, let us put some isolated vertices to the smaller part so that they both have the same number of vertices. Now, let us consider a pair of vertices $a \in A, b \in B$. If both $a$ and $b$ have degrees smaller than $\Delta(G)$ and $ab \notin E$ then let us consider graph $G' = (A \uplus B, E \cup \{(a, b)\})$. Based on our assumption it is clear that $\Delta(G) = \Delta(G')$, so if we know that we can find coloring of edges $G'$ admitting our condition then we can find coloring of edges of $G$, since the same coloring with edge $ab$ removed will be good. Note that proof of the statement for regular graphs actually works without any change if we allow $G$ to be a multigraph, so if we treat $E(G)$ and $E(G')$ to be multisets, we do not need to require that $ab \notin E$ for this auxiliary statement to work. As long as there is any vertex $a \in A$ with degree smaller than $\Delta(G)$, there is a vertex $b \in B$ with degree smaller than $\Delta(G)$ as well (since both sides have equal sizes and sums of degrees on both sides are equal) and we can put edge $ab$ without changing $\Delta(G)$. When we can no longer find such vertex $a$, we must be left with a regular graph that admits required coloring, so we are done. In short, we put vertices and edges in our graph to make it regular while not increasing $\Delta(G)$, color its edges using partitioning of them into $\Delta(G)$ matchings using previous exercise and remove artificially added edges.

4. This is a bit unusual exercise since it really uses full expressive power of max-flow min-cut theorem as opposed to many other problems where we are just interested in matchings and Hall condition.

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Let us start by proving easy implication, that is, if there exists such coloring then no such pair can be found. However, actual solution tackles the whole equivalence at once, so you may skip this part, but it may bring some intuition behind the meaning of inequality from the statement.

Let us assume that there is some subset of cells that can be colored black so that there are appropriate numbers of black cells within each row and column. Let $A, B \subseteq [n]$ and by slight abuse of notation let us say that $A$ is a subset of rows and $B$ is a subset of columns. Board can be now partitioned into cells of four types:

(a) In $A$ and in $B$ (there are $|A||B|$ cells of that type)

(b) In $A$ but not in $B$ (of size $|A|(n - |B|)$)

(c) Not in $A$ but in $B$ (of size $(n - |A|)|B|$)

(d) Not in $A$ and not in $B$ (of size $(n - |A|)(n - |B|)$)

Let $d_{AB}, d_A, d_B$ and $d$ denote numbers of black cells in these four regions respectively. It is clear that $\sum_{a \in A} r_a = d_{AB} + d_A$ and $\sum_{b \in B} c_b = d_{AB} + d_B$. Because of that $\sum_{a \in A} r_a - \sum_{b \in B} c_b = d_A - d_B \leq d_A \leq |A|(n - |B|)$ what proves claimed implication, so let us focus now on the reverse implication.

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We have already seen some problems where we have 2-dimensional boards and some constraints on rows and columns, for example problem about pawns on a board from previous class. Our experience shows that it is a good idea to create a bipartite graph with vertices $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ corresponding to rows and columns. Let us denote $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Set of black cells is good if and only if corresponding set of edges (black cell $(i, j)$ corresponds to edge between $a_i$ and $b_j$) is such that vertex $a_i$ has degree $r_i$ and vertex $b_i$ has degree $c_i$. It can be viewed as follows: if water flows from $A$ to $B$ then vertices $a_i$ want to have outflux $r_i$ of water and vertices $b_i$ want to have influx $c_i$ of water where each pair of vertices from different sides are connected with an edge with capacity one in direction from $A$ to $B$. We can model having many sources and many sinks by adding vertex $s$ and connecting it to $a_i$ with an edge with capacity $r_i$ (in direction from $s$ to $a_i$) and adding vertex $t$ and connecting it to $b_i$ with an edge with capacity $c_i$ (in direction from $b_i$ to $t$) for every $1 \leq i \leq n$. From the construction, it is clear that appropriate set of black cells can be found if and only if there exists an $s-t$ flow in this network with value $F := \sum r_i (= \sum c_i)$ – edges between $A$ and $B$ that this flow will use will correspond to cells we want to make black (let us remember that if network has integer capacities then there exists a maximum flow with integer values only).

Instead of determining value of maximum flow in this network, we are going to determine value of minimum $s - t$ cut, since we know that these two are equal. Hence, let us inspect a structure of cuts in this network. For sure, if we cut either all edges adjacent to $s$ or all edges adjacent to $t$ we will pay $F$ for that, but what about other cuts? Let $C$ be some $s - t$ cut. It consists of some edges adjacent to $s$, some edges adjacent to $t$ and some edges between $A$ and $B$. Let $S \subseteq [n]$ be the set of indices $i$ such that $(s, a_i)$ belongs to $C$ and let

$T \subseteq [n]$ be the set of indices $i$ such that $(b_i, t)$ belongs to $C$. In order to kill all paths from $s$ to $t$ we need to cut all edges $(a_i, b_j)$ such that $i \notin S$ and $j \notin T$. The total capacity of all mentioned edges is $\sum_{i \in S} r_i + \sum_{i \in T} c_i + (n - |S|)(n - |T|)$. If for some $S, T$ this value is less than $F$ then we know we cannot find appropriate set of black cells to fulfill all conditions, however if such $S$ and $T$ do not exist then we know that we can find it. If we denote $[n] \setminus S$ by $X$ we get that $\sum_{x \in X} c_x = \sum_{x \in [n]} c_x - \sum_{x \in S} c_x = F - \sum_{x \in S} c_x$ and $|X| = n - |S|$, so $\sum_{i \in S} r_i + \sum_{i \in T} c_i + (n - |S|)(n - |T|)$ can be rewritten as $(F - \sum_{i \in X} r_i) + \sum_{i \in T} c_i + |X|(n - |T|)$. This expression is smaller than $F$ if and only if $\sum_{i \in X} r_i > \sum_{i \in T} c_i + |X|(n - |T|)$, which is exactly the inequality from statement, so we are done.

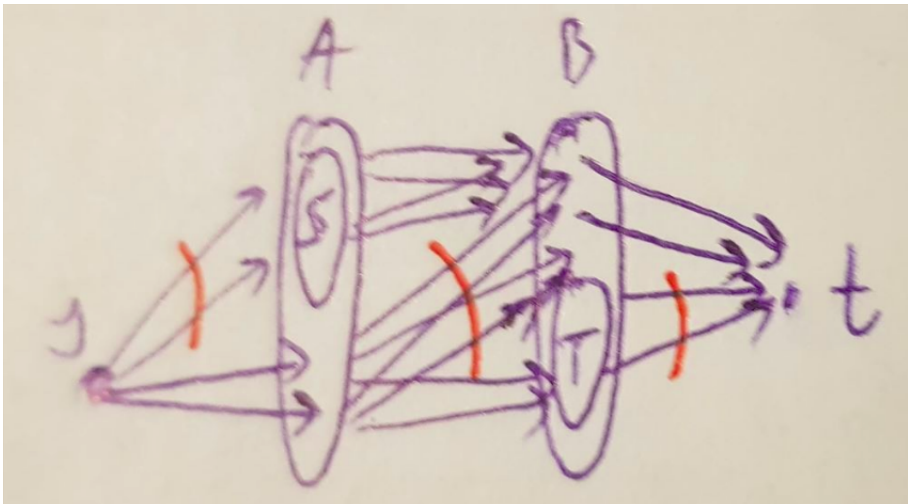


Figure 1: Low effort depiction of cut determined by the choice of $S$ and $T$ (edges between $S$ and $T$ are not drawn for clarity)

5. Let us first prove 1). Let us consider some perfect matching $T \subseteq E$ in $G$ and let $R$ be a set of edges corresponding to $T$ in $E(H)$ going from $A$ to $B$. Moreover, let $L$ be a set of edges corresponding to $M$ in $E(H)$ going from $B$ to $A$. Every vertex has exactly one ingoing and one outgoing edge in $L \cup R$, so $L \cup R$ forms a spanning set of cycles ("spanning" means that these cycles cover every vertex). If an edge $(x, y)$ belongs to some cycle in that cycle cover then $x$ and $y$ lie in the same strongly connected component (we will abbreviate strongly connected component as SCC), which means that we have just proved left to right implication of 1), because that cycle cover consists of edges from two perfect matchings.

   Proof of implication from right to left follows a similar fashion. Let $(x, y) \in E(G)$ be an edge such that $x$ and $y$ belong to the same SCC of $H$, where $x \in A$ and $y \in B$. In $H$ there is an edge from $x$ to $y$ and there exists a simple path from $y$ to $x$. Let this path be $(b_0, a_0, b_1, a_1, \ldots, b_k, a_k)$, where $b_0 = y, a_k = x$, $a_i \in A$ and $b_i \in B$. All edges from $B$ to $A$ correspond to edges from $M$, so $(b_i, a_i) \in M$ for every $0 \leq i \leq k$. Because of that, this path, along with an edge $(x, y)$, forms an *alternating cycle* and we can now "alternate it". That is, consider $N = (M \setminus \bigcup_{0 \leq i \leq k}(b_i, a_i)) \cup \bigcup_{0 \leq i \leq k-1}(a_i, b_{i+1}) \cup \{(a_k, b_0)\}$. It can be easily seen that $N$ is a matching as well and it contains edge $(x, y) = (a_k, b_0)$, what concludes proof of 1).

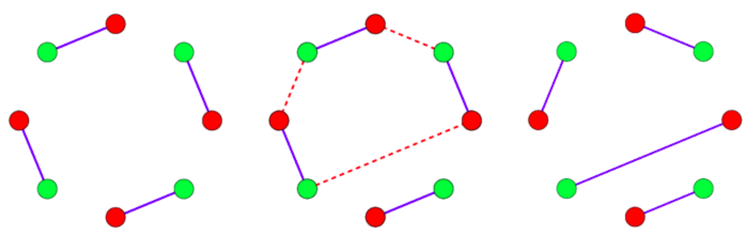

Figure 2: Example of a matching alternation along an alternating cycle

   We will now show that 1) implies 2). Based on 1) we can remove all edges $(a, b)$ from $G$ such that $a$ and $b$ are in different SCCs in $H$ and that will not affect set of perfect matchings of $G$, so let us do that and denote resulting graph by $G'$. We can also remove set of edges corresponding to them in $H$ and call resulting graph $H'$, which will become a set of SCCs without any other connections between them (SCCs of $H$ are exactly the same as SCCs of $H'$). Moreover, any edge that is in $H'$ corresponds to edge in $G$ that is a part of some perfect matching. If we have some edge $(a, b) \in E(G)$ such that $\{a, b\}$ forms a two-element SCC in $H$, then they are not connected with any other vertices of $H'$, so they need to be connected with each other in any perfect matching of $G$, what concludes proof of implication from right to left.

   Let us now assume that it is the case that either $a$ or $b$ is connected with some other vertex in $H'$ in any direction (which means SCC of $a$ and $b$ contains more vertices than just these two) and we will prove that $(a, b)$ does not belong to some perfect matching of $G$. Assume that there is $c \in V(G)$ such that either $(a, c)$ or $(c, a)$ is an edge of $H'$. Then, based on mentioned property of $H'$, we know that $(a, c)$ is a part of some perfect matching of $G$, so edge $(a, b)$ cannot be a part of this matching as well. If $c$ is a neighbour of $b$ then argument goes in the same way. Hence $(a, b)$ does not belong to every perfect matching of $G$ and we are done with proving 2).

We should now conclude that in every bipartite graph containing a perfect matching there is a vertex such that all edges incident to it belong to some perfect matching. For such graph $G$, construct a graph $H$ as we did before.

*Claim*: There exists a "terminal" SCC in $H$, that is a SCC $C \subseteq V(H)$ such that there are no edges $(x, y) \in E(H)$ such that $x \in C, y \notin C$.

*Proof.* Let us consider a graph of SCCs, that is, we create a "supergraph", where we treat each SCC as one vertex and there is an edge from SCC $C_1$ to SCC $C_2$ whenever there is an edge $(c_1, c_2) \in E(H)$ such that $c_1 \in C_1$, $c_2 \in C_2$. Such supergraph is acyclic – a cycle in it would actually mean that SCCs on this cycle should be merged into one. As we proved in first exercise, every directed acyclic graph (such graphs are abbreviated as DAGs) contains a vertex with no outgoing edges which corresponds to a "terminal" SCC. □

Let $a$ be any vertex from $A$ from that terminal SCC (it is obvious that every SCC contains vertices from both $A$ and $B$ since for every edge from $M$ its two ends are in the same SCC). For every edge incident to $a$ in $G$ there is an edge outgoing from $a$ in $H$. However, since we chose $a$ from a terminal SCC then it means all edges outgoing from $a$ need to be within the same SCC as SCC of $a$. Based on 1) it follows that all of them belong to some perfect matching, so we are done.

6. First crucial observation in this task is to note that partition into diamonds is in fact a perfect matching between unremoved unit triangles pointed upwards (denote their set by $U$) and unit triangles pointed downwards (denote their set by $D$). In other words, we create a bipartite graph $G$, where every vertex on one side corresponds to unremoved unit triangle pointed upwards and on the other side every vertex corresponds to unit triangle pointed downwards and there is an edge between two vertices if their corresponding triangles share a side (so they can form a diamond). Before removal of $n$ unit triangles pointed upwards, there were $n$ more of them than the ones pointed downwards, so after that removal, number of unremoved triangles in these two types are equal.

   Perfect matching in this graph can be achieved if and only if Hall's condition is true for all subsets of $D$. However condition from the problem statement translates to the fact that Hall's condition is true for any set of all triangles from $D$ within some (not unit) triangle pointed upwards. So, we need to prove that if there is any such subset of $D$ breaking Hall's condition, then there is at least one subset of $D$ breaking Hall's condition of that form.

   Let us create an auxiliary graph $H$ on triangles from $D$ only, where vertices corresponding to two triangles from $D$ are connected with an edge if and only if these triangles share a vertex. We say that set $S \subseteq D$ is connected if corresponding set of vertices in $H$ induces a connected subgraph.

   Let us now note that if two triangles from $D$ are not connected in $H$ then they have disjoint sets of neighbours in $G$. It means that if we have a subset $S \subseteq D$ which induces many connected components in $H$, then all these components have disjoint sets of neighbourhoods in $G$. It can be concluded that if $S \subseteq D$ breaks Hall's condition (that is $|N_G(S)| < |S|$) then at least one connected component of $S$ breaks Hall's condition, so if there is any set $S$ breaking Hall's condition, then there is a connected set breaking Hall's condition.

   Let $S \subseteq D$ be the biggest connected set breaking Hall's condition. Let us assume that there is a point which is a vertex of exactly two triangles from $S$. Let $t$ be the third triangle from $D$ that has its vertex in that point (it always exists). Then $|N_G(t) \setminus N_G(S)| \leq 1$. That is true because $t$ has three neighbours, but two of them are already neighbours of the other two triangles having vertex in that point, which are already in $S$. Moreover let us not forget that some of these triangles from $U$ may have been removed, but that does not change the argument. So $|N_G(S \cup \{t\})| = |N_G(S)| + |N_G(t) \setminus N_G(S)| \leq |N_G(S)| + 1 < |S| + 1 = |S \cup \{t\}|$, so $S \cup \{t\}$ breaks Hall's condition as well. Moreover, if $S$ is connected then $S \cup \{t\}$ is connected as well. However, we assumed that $S$ was the biggest connected set breaking Hall's condition, so it must be true that there does not exist a point which is a vertex of exactly two triangles from $S$. One can check that this fact, joined with the fact that $S$ is connected, implies that $S$ is a set of all unit triangles pointed downwards within some triangle pointed upwards (let me omit a formal proof of that, but that should be rather intuitive), so we are done.