Liga zadaniowa 2018/2019

Seria V, 17/12/2018

Pytania proszę kierować do Piotra Nayara na adres nayar@mimuw.edu.pl .

Wszelkie informacje o lidze, w tym zadania i ewentualne korekty treści zadań, rozwiązania zadań i punktacja, będą pojawiały się na stronie www.mimuw.edu.pl/∼nayar.

Problem 1. (A) Let $(a_n)_{n\geq 1}$ be a non-increasing sequence of non-negative real numbers. Suppose $\varepsilon_i \in \{-1, 1\}$, $i \geq 1$ are such that $\sum_{i=1}^{\infty} \varepsilon_i a_i$ is convergent. Prove that

$$\lim_{n\to\infty} (\varepsilon_1 + \ldots + \varepsilon_n) a_n = 0.$$

Problem 2. (G) Let $m \geq 2$ and $n \geq 1$. Suppose we are give unit vectors $v_1, \ldots, v_{n+1}$ in $\mathbb{R}^n$. Show that

$$\max_{i\neq j} |\langle v_i, v_j \rangle| \ \geq \ \frac{1}{n}.$$

Problem 3. (A) For which $c > 1$ is the sequence

$$c, \quad c^c, \quad c^{c^c}, \quad c^{c^{c^c}}, \quad \ldots$$

convergent?

Problem 4. (K) Suppose we cover a unit disk $\{x \in \mathbb{R}^2 : \ |x| \leq 1\}$ with n closed strips[1] of widths $w_1, \ldots, w_n$. Prove that $w_1 + \ldots + w_n \geq 2$.

Problem 5. (A+K) A bijection $\sigma : \mathbb{N} \to \mathbb{N}$ is called *good* if it satisfies the following two properties:

(a) For any sequence of real numbers $(a_n)_{n\geq 1}$, if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then also the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent.

(b) There exists a sequence of real numbers $(b_n)_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty} b_n$ is divergent, but the series $\sum_{n=1}^{\infty} b_{\sigma(n)}$ is convergent.

Show an example of a good bijection, or prove that such a bijection does not exist.

Problem 6. (K) Let A be a finite subset of integers. Prove or disprove the inequality

$$\#\{k + l : \ k, l \in A\} \leq \#\{k - l : \ k, l \in A\}.$$

Problem 7. (G+K) Let m_n be the number of all singular $n \times n$ matrices having ± 1 entries. Show that $\lim_{n\to\infty} m_n/2^{n^2} = 0$.

Problem 8. (A+K) Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{[n\sqrt{2}]}}{n}.$$

[1] A strip is a part of the plane between two parallel lines and the width of the strip is the distance between these two lines.