Liga zadaniowa 2018/2019

Seria IV, 03/12/2018

Pytania proszę kierować do Piotra Nayara na adres nayar@mimuw.edu.pl .

Wszelkie informacje o lidze, w tym zadania i ewentualne korekty treści zadań, rozwiązania zadań i punktacja, będą pojawiały się na stronie www.mimuw.edu.pl/∼nayar.

Problem 1. (A) Is it true that for any bijection $f : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$ is divergent?

Problem 2. (K) Prove that in a simple graph G with an even number of vertices there exist two distinct vertices with an even number of common neighbors.

Problem 3. (A) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that the sequence $(2a_{n+1} + \sin a_n)_{n\geq 1}$ is convergent. Does this imply the convergence of the sequence $(a_n)_{n\geq 1}$?

Problem 4. (A+K) Let $A_1, \ldots, A_n$ be subsets of a given finite set Ω. Prove that for any real numbers $x_1, \ldots, x_n$ we have

$$\sum_{i,j=1}^{n} \#(A_i \cap A_j) \cdot x_i x_j \geq 0.$$

Problem 5. (G+K) Let $A = (a_{ij})_{i,j}$ be an $n \times n$ matrix with entries belonging to the set $\{0, 1\}$. Assume that[1] $\mathrm{tr}(A^k) = 0$ for every positive integer k . Show that there exists a permutation σ of $\{1, \ldots, n\}$ such that the $n \times n$ matrix $(a_{\sigma(i)\sigma(j)})_{i,j}$ is upper triangular.

Problem 6. (A+G+K) Let n be a positive integer. Inside a convex polygon of perimeter 1 a finite number of line segments is drawn such that their total length is strictly greater than n. Show that there exists a line that intersects at least $2n + 1$ of the segments.

[1]Here tr is the trace of a matrix, that is the sum of its diagonal entries.