Liga zadaniowa 2018/2019

Seria III, 19/11/2018

Pytania proszę kierować do Piotra Nayara na adres nayar@mimuw.edu.pl .

Wszelkie informacje o lidze, w tym zadania i ewentualne korekty treści zadań, rozwiązania zadań i punktacja, będą pojawiały się na stronie www.mimuw.edu.pl/∼nayar.

Problem 1. (A) For $n \geq 1$ let a_n be a unique real number such that $e^{a_n} + na_n = 2$. Prove that $\lim_{n\to\infty} n(1 - na_n) = 1$.

Problem 2. (G) Let $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be real numbers. Prove that the rank of the $n \times n$ matrix with entries $a_{ij} = x_i + y_j$ does not exceed 2.

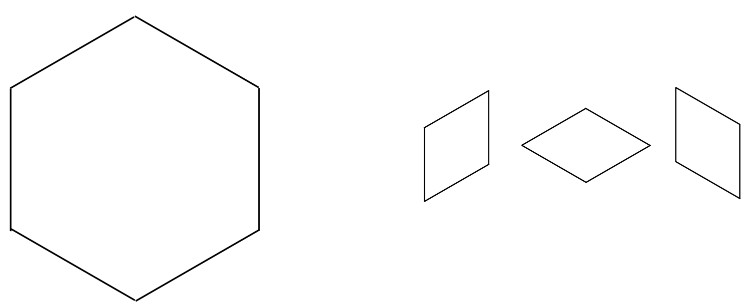
Problem 3. (G) Let $x_1 \ldots x_n \in \mathbb{R}^2$ be vertices of a convex polygon inscribable in a circle. Consider a matrix $A = (a_{ij})_{i,j=1}^n$ whose entries are given by

$$a_{ij} = \begin{cases} |x_i - x_j| & \text{if } \ i \geq j \\ -|x_i - x_j| & \text{if } \ i < j \end{cases} .$$

Prove that A has rank 2.

Problem 4. (A) For a real number a_0 define a sequence $(a_n)_{n\geq 0}$ via the recurrence relation $a_{n+1} = \frac{2a_n}{1-a_n^2}$, $n \geq 0$ (when $|a_n| = 1$ for some n, then the rest of the sequence remains undefined). Find $a_0 \in (0, \frac{\pi}{2})$ such that $a_0 < a_1 < \ldots < a_{2018}$ and $a_{2018} = 1$.

Problem 5. (K) Suppose we tile a regular hexagon (below on the left) of side length n with $3n^2$ rhombi (of side lengths 1) of three different types (below on the right). Prove that we have to use precisely n^2 tiles of each type.



Rysunek 1: The hexagon and the three possible types of tiles.

Problem 6. (K) Is it possible to cover the plane with closed discs in such a way that every two discs intersect in at most one point?