Liga zadaniowa 2018/2019

Seria II, 5/11/2018

Pytania proszę kierować do Piotra Nayara na adres nayar@mimuw.edu.pl .

Wszelkie informacje o lidze, w tym zadania i ewentualne korekty treści zadań, rozwiązania zadań i punktacja, będą pojawiały się na stronie www.mimuw.edu.pl/$\sim$nayar.

Problem 1. (A) Suppose $(a_n)_{n\geq 1}$ is a bounded sequence of nonzero real numbers. Does there always exist a subsequence $(b_n)_{n\geq 1}$ of $(a_n)_{n\geq 1}$ such that $(b_{n+1}/b_n)_{n\geq 1}$ converges?

Problem 2. (K) Suppose $n \geq 1$ is even. Prove that there exist vectors $v_1, \ldots, v_n \in \{-1,1\}^n$ with the following property: for any vector $u \in \{-1,1\}^n$ we can find $i \in \{1,\ldots,n\}$ such that $\langle v_i, u\rangle = 0$. Here $\langle \cdot,\cdot\rangle$ stands for the standard Euclidean scalar product, namely $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$.

Problem 3. (A) Let $a_1 = a_2 = 1$ and $a_{n+1} = \frac{1}{a_n} + \frac{1}{a_{n-1}}$ for $n \geq 2$. Is the sequence $(a_n)_{n\geq 1}$ convergent?

Problem 4. (G) Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$. Consider the following two conditions:

(a) The functions f_i are linearly independent over $\mathbb{R}$.

(b) There exist real numbers $x_1, \ldots, x_n$ such that the vectors

$$u_k = (f_1(x_k), \ldots, f_n(x_k)), \qquad k = 1, \ldots, n$$

are linearly independent.

Does (b) imply (a)? Does (a) imply (b)?

Problem 5. (K) Suppose we are given a collection of $2n+1$ real numbers with the following property: if we remove any of the $2n+1$ numbers, the remaining $2n$ numbers can be split into two groups of n numbers with the same sum of elements. Does it follow that all the $2n+1$ numbers are equal?

Problem 6. (A+G+K) Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} f(n) = \infty$ such that the following property is satisfied: for any set $A \subset \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq 1\}$ of cardinality n there exists a strip $S \subset \mathbb{R}^2$ of width $1/n$ (a strip is a region between two parallel lines and the width of a strip is the distance between these two lines) such that $\#(S \cap A) \geq f(n)$?