Liga zadaniowa 2018/2019

Seria I, 22/10/2018

Pytania proszę kierować do Piotra Nayara na adres nayar@mimuw.edu.pl .

Wszelkie informacje o lidze, w tym zadania i ewentualne korekty treści zadań, rozwiązania zadań i punktacja, będą pojawiały się na stronie www.mimuw.edu.pl/∼nayar.

Problem 1. (G) Let A be a real square matrix. Does there always exist a complex matrix S such that $A = S^2$?

Problem 2. (G+K) Let (a_{kl}) be an $n \times N$ real matrix such that for every $i \neq j$ we have $\max_{1 \leq k \leq n} |a_{ki} - a_{kj}| = 1$. Prove that $N \leq 2^n$.

Problem 3. (K) A group of people meets in a house with two rooms. Some people know each other, some not. Show that it is always possible to place them is these two rooms in such a way that every person is in a room with no more than half of his friends.

Problem 4. (A) Let α be an irrational real number. Prove that the set $Z_\alpha = \{k + \alpha l,\ k, l \in \mathbb{Z}\}$ is dense in the real line, that is, every open interval contains an element of Z_α.

Problem 5. (K) Let $S = \{0,1\}^n$. For $a, b \in S$ we define $d(a,b) = \#\{1 \leq i \leq n :\ a_i \neq b_i\}$. Show that for any $k \geq 1$ there exists a set $T \subset S$ such that $d(a,b) \geq k$ for all distinct $a, b \in T$ and such that

$$\#T \geq \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \ldots \binom{n}{k-1}}.$$

Problem 6. (A) For $\varepsilon > 0$ we define $S_\varepsilon = \bigcup_{k \in \mathbb{Z}} (k - \varepsilon, k + \varepsilon)$. Is it true that for every $\varepsilon > 0$ the real line $\mathbb{R}$ can be covered by finitely many sets of the form $aS_\varepsilon = \{ax :\ x \in S_\varepsilon\}$, $a \in \mathbb{R}$?