On the Khintchine inequality and its relatives

Piotr Nayar*

July 1, 2023

1 Classical Khintchine inequality

1.1 Warm up – $C_{4,2}$

Suppose we are given a sequence $\varepsilon_1, \ldots, \varepsilon_n$ of i.d.d. symmetric Bernoulli random variables, that is random variables satisfying $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$ and a sequence of real numbers $a_1, \ldots, a_n$. Let us define $S = \sum_{i=1}^n a_i \varepsilon_i$. The classical Khintchine inequality deals with moments of S, namely $\|S\|_p = (\mathbb{E}|S|^p)^{1/p}$. Khintchine proved that for any $p \geq q > 0$ there exists a constant $C_{p,q}$ depending only on p, q (that is, not depending on n and on the sequence (a_i)), such that

$$\|S\|_p \leq C_{p,q} \|S\|_q \tag{1}$$

We shall assume that $C_{p,q}$ denotes the best constant in this inequality. The main goal of the first part of these notes is to give an overview of the known techniques leading to the derivation of $C_{p,q}$ in the case, when the constant is known. For historical remarks we refer the reader to Section 1.8.

Let us first observe that the second moment of S is particularly nice, namely

$$\mathbb{E}|S|^2 = \mathbb{E}\left(\sum_{i=1}^n a_i \varepsilon_i\right)^2 = \mathbb{E}\left(\sum_{i,j=1}^n a_i a_j \varepsilon_i \varepsilon_j\right) = \sum_{i,j=1}^n a_i a_j \mathbb{E}\varepsilon_i \varepsilon_j = \sum_{i=1}^n a_i^2,$$

since due to independence for $i \neq j$ one has $\mathbb{E}\varepsilon_i\varepsilon_j = \mathbb{E}\varepsilon_i \mathbb{E}\varepsilon_j = 0$ and $\mathbb{E}\varepsilon_i^2 = 1$. Let us now try to compute the fourth moment,

$$\mathbb{E}|S|^4 = \mathbb{E}\left(\sum_{i=1}^n a_i \varepsilon_i\right)^4 = \mathbb{E}\left(\sum_{i,j,k,l=1}^n a_i a_j a_k a_l \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l\right) = \sum_{i,j,k,l=1}^n a_i a_j a_k a_l \mathbb{E}\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l.$$

Observe that in order for the expectation $\mathbb{E}\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l$ to be nonzero, every index has to occur an even number of times. Indeed, in general one has

$$\mathbb{E}\varepsilon_1^{j_1}\varepsilon_2^{j_2} \cdot \ldots \cdot \varepsilon_n^{j_n} = \mathbb{E}\varepsilon_1^{j_1}\mathbb{E}\varepsilon_2^{j_2} \cdot \ldots \cdot \mathbb{E}\varepsilon_n^{j_n}$$

and $\mathbb{E}\varepsilon_i^{j_i} = 1$ when j_i is even and $\mathbb{E}\varepsilon_i^{j_i} = \mathbb{E}\varepsilon_i = 0$ when j_i is odd. Therefore

$$\mathbb{E}\varepsilon_1^{j_1}\varepsilon_2^{j_2} \cdot \ldots \cdot \varepsilon_n^{j_n} = \begin{cases} 1 & 2|j_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Thus, either $i = j = k = l$, which contributes the sum $\sum_{i=1}^n a_i^4$, or the indexes form two pairs and are equal in these pairs. For example, in front of the term $a_1^2 a_2^2$ we are going to have the coefficient equal to the number of choices of (i, j, k, l) such that two of these indexes are equal to 1 and the other

*Piotr Nayar, University of Warsaw, email: nayar@mimuw.edu.pl

two equal to 2. The number of way of choosing such indexes is $\binom{4}{2} = 6$, since we just have to declare which two indexes among these four will be equal to 1. As a consequence one gets

$$\mathbb{E}|S|^4 = \sum_{i=1}^n a_i^4 + 6\sum_{i<j} a_i^2 a_j^2.$$

Observe that by homogeneity the inequality (1) does not change when we rescale all the a_i by some fixed number $\lambda \neq 0$, that is consider λa_i instead of a_i. Thus we can always assume that $\sum_{i=1}^n a_i^2 = 1$. In this case

$$1 = \left(\sum_{i=1}^n a_i^2\right)^2 = \sum_{i=1}^n a_i^4 + 2\sum_{i<j} a_i^2 a_j^2 \qquad \text{which implies} \qquad 6\sum_{i<j} a_i^2 a_j^2 = 3 - 3\sum_{i=1}^n a_i^4.$$

Therefore

$$\mathbb{E}|S|^4 = 3 - 2\sum_{i=1}^n a_i^4 \leq 3.$$

The constant $C_{4,2} = \sqrt[4]{3}$ is optimal, as can be seen by taking $a_1 = \ldots = a_n = n^{-1/2}$ in which case one gets $\mathbb{E}|S|^4 = 3 - \frac{2}{n} \to 3$ when $n \to \infty$. In fact due to the inequality between means one has

$$\left(\frac{|a_1|^p + \ldots + |a_n|^p}{n}\right)^{1/p} \leq \left(\frac{|a_1|^q + \ldots + |a_n|^q}{n}\right)^{1/q}, \qquad q > p > 0$$

with equality for $a_1 = \ldots = a_n$. In particular $\sum_{i=1}^n a_i^4 \geq \frac{1}{n}\left(\sum_{i=1}^n a_i^2\right)^2 = \frac{1}{n}$ with equality for $a_1 = \ldots = a_n = n^{-1/2}$. This is fact shows that

$$\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^4 \leq \mathbb{E}\left|\sum_{i=1}^n \frac{1}{\sqrt{n}}\varepsilon_i\right|^4, \qquad \sum_{i=1}^n a_i^2 = 1.$$

In other words, the quantity $C_{4,2}^{(n)} = \|\sum_{i=1}^n n^{-1/2}\varepsilon_i\|_4$ is the best n-dependent constant in (1).

1.2 Constants $C_{2k,2}$

Let us now compute the $2k$-th moment, where $k \geq 1$ is an integer. Recall the multinomial identity: for a positive integer p one has

$$(x_1 + \ldots + x_n)^p = \sum_{j_1+\ldots+j_n=p} \binom{p}{j_1,\ldots,j_n} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}, \qquad \text{where } \binom{p}{j_1,\ldots,j_n} = \frac{p!}{j_1!\ldots j_n!}.$$

Here the sum runs over all integers $k_i \geq 0$ with $\sum_{i=1}^n j_i = p$. Applying this identity one gets

$$\begin{aligned}\mathbb{E}|S|^{2k} = \mathbb{E}\left(\sum_{i=1}^n a_i\varepsilon_i\right)^{2k} &= \sum_{j_1+\ldots+j_n=2k} \binom{2k}{j_1,\ldots,j_n} a_1^{j_1}\ldots a_n^{j_n}\ \mathbb{E}\varepsilon_1^{j_1}\ldots\varepsilon_n^{j_n} \\ &= \sum_{k_1+\ldots+k_n=k} \binom{2k}{2k_1,\ldots,2k_n} a_1^{2k_1}\ldots a_n^{2k_n}.\end{aligned}$$

where we have used (2).

We shall now present Khintchine's derivation of the optimal constants $C_{2k,2}$. We can again assume that $\sum_{i=1}^n a_i^2 = 1$. Let $n!$ denote the product of all positive integers not exceeding n of the same parity as n. Note also that $(2n)!! = 2^n n!$. We claim that

$$\binom{2k}{2k_1,\ldots,2k_n} \leq (2k-1)!!\binom{k}{k_1,\ldots,k_n}.$$

Indeed, under $k_1 + \ldots + k_n = k$ we have

$$\binom{2k}{2k_1, \ldots, 2k_n} = \frac{(2k)!}{(2k_1)! \ldots (2k_n)!} = \frac{(2k-1)!!2^k k!}{(2k_1)! \ldots (2k_n)!} = \frac{(2k-1)!!2^{k_1} \cdot \ldots \cdot 2^{k_n} k!}{(2k_1)! \ldots (2k_n)!}$$
$$\le (2k-1)!! \binom{k}{k_1, \ldots, k_n},$$

which follows from $(2k_i)! \ge (2k_i)!! = 2_i^k k_i!$. As a consequence one gets

$$\mathbb{E}|S|^{2k} \le (2k-1)!! \sum_{k_1+\ldots+k_n=k} \binom{k}{k_1, \ldots, k_n} a_1^{2k_1} \ldots a_n^{2k_n} = (2k-1)!!(a_1^2 + \ldots + a_n^2)^k = (2k-1)!!.$$

Note that if G is a standard Gaussian random variable $\mathcal{N}(0,1)$, namely G has density $\varphi(t) = (2\pi)^{-1/2} e^{-t^2/2}$, then $\mathbb{E}G^{2k} = (2k-1)!!$. Indeed, integrating by parts and using $\varphi'(t) = -t\varphi(t)$ we get for $k \ge 1$

$$\mathbb{E}G^{2k} = \int t^{2k} \varphi(t) \mathrm{d}t = -\int t^{2k-1} \varphi'(t) \mathrm{d}t = (2k-1) \int t^{2k-2} \varphi(t) \mathrm{d}t = (2k-1)\mathbb{E}G^{2k-2}.$$

Iterating gives the desired identity. We have established the bound $\mathbb{E}\|S\|^{2k} \le \mathbb{E}|G|^{2k}$, which is $\|S\|_{2k} \le \|G\|_{2k} \|S\|_2$. This means that $C_{2k,2} = \|G\|_{2k}$ and the optimality can be seen by taking $a_1 = \ldots = a_n = n^{-1/2}$ and taking $n \to \infty$, as $\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i$ converges in distribution to G due to the central limit theorem. This implies convergence of moments as $\sup_n \| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \|_{2k+2} \le \|G\|_{2k+2}$ as in general convergence $X_n \to X$ in distribution together with $\sup_n \|X_n\|_{p+\varepsilon} < \infty$ for some $p, \varepsilon > 0$ imply $\|X_n\|_p \to \|X\|_p$.

Remark 1. Let $p \ge 1$ and assume that $\sum_{i=1}^n a_i^2 = 1$. Then

$$\mathbb{E}|G|^p = \mathbb{E}\left|\sum_{i=1}^n a_i G_i\right|^p \ge \mathbb{E}\left|\sum_{i=1}^n a_i \varepsilon_i |G_i|\right|^p \ge \mathbb{E}\left|\sum_{i=1}^n a_i \varepsilon_i \mathbb{E}|G_i|\right|^p = \left(\sqrt{\frac{2}{\pi}}\right)^p \mathbb{E}\left|\sum_{i=1}^n a_i \varepsilon_i\right|^p.$$

This shows that $C_{p,2}$ are finite for $p \ge 2$.

1.3 Hölder boosting

In this subsection we are going to show that the constants $C_{p,q}$ are finite for every $p, q > 0$. We need the following lemma.

Lemma 1. *Suppose S is a real random variable. Then the function $t \mapsto \mathbb{E}|S|^t$ is log-convex. In other words, for every $p < q < r$ one has*

$$\|S\|_q^{q(r-p)} \le \|S\|_p^{p(r-q)} \|S\|_r^{r(q-p)}. \tag{3}$$

Proof. By Hölder's inequality we have $\mathbb{E}|XY| \le (\mathbb{E}|X|^p)^{1/p} (\mathbb{E}|Y|^q)^{1/q}$ with $p, q \ge 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Using this with $X = |S|^{\lambda s}$, $Y = |S|^{(1-\lambda)t}$ and $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$, where $\lambda \in [0,1]$, we get

$$\mathbb{E}|S|^{\lambda s + (1-\lambda)t} \le (\mathbb{E}|S|^s)^\lambda (\mathbb{E}|S|^t)^{1-\lambda},$$

which is the desired log-concavity. To prove the second part it suffices to use this inequality with $\lambda = \frac{r-q}{r-p}$ and rewrite it in terms of moments of S. □

Remark 2. Note that for $p < q$ the well-known inequality $\|S\|_p \le \|S\|_q$ is a consequence of the convexity of $t \mapsto \mathbb{E}|S|^t$. Indeed, the slopes

$$\log\left((\mathbb{E}|S|^p)^{\frac{1}{p}}\right) = \frac{\log \mathbb{E}|S|^p}{p} = \frac{\log \mathbb{E}|S|^p - \log \mathbb{E}|S|^0}{p - 0}$$

Let us notice that in order to prove an inequality of the form $\|S\|_p \le C_{p,q}\|S\|_q$ with $S = \sum_{i=1}^n a_i\varepsilon_i$, by homogeneity we can always assume that $\|S\|_2 = 1$. To show the existence of $C_{p,q}$, it is enough to prove that under $\|S\|_2 = 1$ one has $A_p \le \|S\|_p \le B_p$ for some positive constants A_p, B_p. Then we get $\|S\|_p \le B_p \le \frac{B_p}{A_q}\|S\|_q$, which implies $C_{p,q} \le \frac{B_p}{A_q}$.

Now, we always have, say, $\|S\|_p \le \|S\|_{2\lceil p\rceil} \le (2\lceil p\rceil - 1)!!$. To prove the lower bound we first observe that if $p \ge 2$ then $\|S\|_p \ge \|S\|_2 = 1$. If $p < 2$ we use Lemma 1 with $q = 2$ and $r = 4$, obtaining

$$1 = \|S\|_2^{2(4-p)} \le \|S\|_p^{2p}\|S\|_4^{4(2-p)} \le \|S\|_p^{2p}3^{2-p},$$

which gives $\|S\|_p \ge 3^{\frac{p-2}{2p}}$.

1.4 Constants $C_{p,2}$ for $p \ge 3$

Let us prove the following theorem established by Pinelis in [71] and independently by Figiel, Hitczenko, Johnson, Schechtman and Zinn in [28].

Theorem 2. *For even functions $\Phi : \mathbb{R} \to \mathbb{R}$ such that Φ'' convex one has*

$$\mathbb{E}\Phi\left(\sum_{i=1}^n a_i\varepsilon_i\right) \le \mathbb{E}\Phi\left(\sum_{i=1}^n a_iX_i\right)$$

for any symmetric variance one independent random variables X_i.

We shall need the following lemma.

Lemma 3. *Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is an even function such that Φ'' is convex. Then for every real number s the function $\psi(x) = \Phi(\sqrt{x} + s) + \Phi(\sqrt{x} - s)$ is convex on $\mathbb{R}_+$.*

Proof. We want to show that $\psi'(x) = \frac{\Phi'(\sqrt{x}+s)+\Phi'(\sqrt{x}-s)}{2\sqrt{x}}$ is non-decreasing. Equivalently, we would like to show that $\psi_1(t) = \frac{\Phi'(t+s)+\Phi'(t-s)}{t}$ is non-decreasing. This would follow from the monotonicity on $\mathbb{R}_+$ of slopes for $\psi_2(t) = \Phi'(t+s) + \Phi'(t-s)$, since $\psi_2(0) = 0$ as Φ' is odd. Thus it is enough to show that ψ_2 is convex or, equivalently, that $\psi_3(t) = \Phi''(t+s) + \Phi''(t-s)$ is non-decreasing on $\mathbb{R}_+$, which is obvious, since ψ_3 is an even convex function, as Φ'' is even and convex. □

Proof of Theorem 2. Let us take $S = \sum_{i=2}^n a_iX_i$. It is enough to show that $\mathbb{E}\Phi(a_1\varepsilon_1+S) \le \mathbb{E}\Phi(a_1X_1+S)$ for any random variable S and use this fact to exchange X_i for ε_i one by one. We can condition on the values s of S. Thus, we are left with proving that $\mathbb{E}\Phi(a_1\varepsilon_1 + s) \le \mathbb{E}\Phi(a_1X_1 + s)$. Let us consider the function ψ from the above lemma. Now, observe that due to symmetry X_1 has the same distribution as $\varepsilon_1|X_1|$, where ε_1 is independent of X_1. Moreover, again by symmetry of X_1 one can assume that $a_1 > 0$. Thus by Jensen's inequality one has

$$\begin{aligned}\mathbb{E}\Phi(a_1X_1 + s) = \mathbb{E}\Phi(a_1\varepsilon_1|X_1| + s) = \mathbb{E}\Phi\left(\varepsilon_1\sqrt{a_1^2|X_1|^2} + s\right) \ge \mathbb{E}\psi(a_1^2X_1^2)\\ \ge \psi(a_1^2\mathbb{E}X_1^2) = \psi(a_1^2) = \mathbb{E}\Phi(a_1\varepsilon_1 + s).\end{aligned}$$

□

We are now ready to show that under $\sum_{i=1}^n a_i^2 = 1$ one has $\mathbb{E}|S|^p \le \mathbb{E}|G|^p$ for $p \ge 3$ and thus $C_{p,2} = \|G\|_p$. The function $\Phi(x) = |x|^p$ satisfies assumptions of Theorem 2 as $\Phi''(x) = p(p-1)|x|^{p-2}$ is convex. Let $X_i = G_i$ be i.i.d. $\mathcal{N}(0,1)$ random variables. Note that $\sum_{i=1}^n a_iG_i$ is also an $\mathcal{N}(0,1)$ random variable and thus

$$\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p \le \mathbb{E}\left|\sum_{i=1}^n a_iG_i\right|^p = \mathbb{E}|G|^p.$$

Let T be an independent copy of S, namely $T = \sum_{i=1}^n a_i\varepsilon_i'$. Let $X_i = \frac{\varepsilon_i+\varepsilon_i'}{\sqrt{2}}$. Then

$$\frac{S+T}{\sqrt{2}} = \sum_{i=1}^n a_i\frac{\varepsilon_i+\varepsilon_i'}{\sqrt{2}} = \sum_{i=1}^n a_iX_i.$$

Thus Theorem 2 implies $\mathbb{E}\left|\frac{S+T}{\sqrt{2}}\right|^p \geq \mathbb{E}|S|^p$. The following conjecture was suggested by Zinn and popularized by Pinelis (mathoverflow.net/questions/208349/).

Conjecture 1. Let $S = \sum_{i=1}^n a_i\varepsilon_i$ and let T be an independent copy of S. Then for $p \in (2,3)$ one has $\mathbb{E}\left|\frac{S+T}{\sqrt{2}}\right|^p \geq \mathbb{E}|S|^p$

By iterating this inequality and using central limit theorem one would easily get $\mathbb{E}|S|^p \leq \mathbb{E}|G|^p$. The latter is a known inequality due to Haagerup, however all known proofs are technical.

1.5 Schur monotonicity of $\|S\|_p$ for $p \geq 3$

Let us introduce some notation.

- For a vector $x = (x_1, \ldots, x_n)$ jest $x_1^*, \ldots, x_n^*$ be the nonincreasing rearrangement of coordinates of x.
- By T-transformation we mean any linear function of the form

$$T_{jk}(x) = (x_1, \ldots, x_{j-1}, (1-\lambda)x_j + \lambda x_k, x_{j+1}, \ldots, x_{k-1}, \lambda x_j + (1-\lambda)x_k, x_{k+1}, \ldots, x_n),$$

  where $\lambda \in [0,1]$.
- A matrix $P = (p_{ij})_{i,j=1}^n$ is called *doubly stochastic* if $p_{ij} \geq 0$ and the sums of elements in each column and row of P is equal to 1.
- The set of permutations of $\{1, \ldots, n\}$ will be denoted by S_n. For $\sigma \in S_n$ and $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ we also define $x_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
- A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be permutation symmetric if $F(x_\sigma) = F(\sigma)$ for every $\sigma \in S_n$.
- A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur convex if $x \prec y$ implies $F(x) \leq F(y)$. Moreover, F is Schur concave if the reverse inequality holds.

The following proposition gives equivalent conditions to the so-called Schur order.

Proposition 4 (Schur order)**.** *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. The following conditions are equivalent:*

(a) We have $\sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

(b) There exists a doubly stochastic matrix P such that $x = Py$.

(c) Vector x is a convex combination of vectors $y_\sigma = (y_{\sigma(1)}, \ldots, y_{\sigma(n)})$, where $\sigma \in S_n$

(d) Vector x is an image of y under composition of finitely many T-transformations.

If one of these conditions holds, we shall write $x \prec y$ and say that x majorizes y.

For the proof we refer the reader to the Appendix. In the sequel we are going to use the following fundamental lemma.

Lemma 5. *Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is convex and permutation symmetric. Then F is Schur convex, that is $x \prec y$ implies $F(x) \leq F(y)$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is convex, then*

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n) \implies \sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i).$$

Proof. From Proposition 4(c) there exists numbers $\lambda_\sigma \geq 0$ summing up to one, such that $x = \sum_{\sigma \in S_n} \lambda_\sigma y_\sigma$. Thus

$$F(x) = F\left(\sum_{\sigma \in S_n} \lambda_\sigma y_\sigma\right) \leq \sum_{\sigma \in S_n} \lambda_\sigma F(y_\sigma) = \sum_{\sigma \in S_n} \lambda_\sigma F(y) = F(y).$$

The second part follows by observing that $F(x_1, \ldots, x_n) = \sum_{i=1}^n f(x_i)$ is convex and permutation symmetric. $\square$

Remark 3. On the simplex $\{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$ one has the relations

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (x_1, \ldots, x_n) \prec (1, 0, \ldots, 0).$$

To prove this, one can check e.g. condition (c), namely $x = \sum_{i=1}^n x_i e_i$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the ith coordinate. This shows the right inequality. To prove the left comparison, let us observe that

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \frac{1}{n}(x_1, \ldots, x_n) + \frac{1}{n}(x_2, \ldots, x_n, x_1) \ldots + \frac{1}{n}(x_n, x_1, \ldots, x_{n-1}).$$

Remark 4. A function $F : \mathbb{R}^n \to \mathbb{R}$ is Schur convex if and only if it is Schur convex with respect to any pair of coordinates. This follows from the fact that $x \prec y$ implies that x is an image of y under composition of finitely many T-transformations. Indeed, note that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ satisfy $y \prec x$ if and only if y is a T-transformation of x, namely

$$(y_1, y_2) = ((1-\lambda)x_1 + \lambda x_2, \lambda x_1 + (1-\lambda)x_2) = (1-\lambda)(x_1, x_2) + \lambda(x_2, x_1),$$

which follows from Proposition 4(c). If we denote $z_\lambda = (1-\lambda)(x_1, x_2) + \lambda(x_2, x_1)$ then note that to check Schur convexity of a function G of two variables we want to verify $F(z_\lambda) \leq F(z_0)$. Observe that $z_{1-\lambda}$ is obtained by transposing coordinates of z_λ and thus if G is permutation symmetric, it is enough to show $F(z_\lambda) \leq F(z_0)$ only for $\lambda \in [0, \frac{1}{2}]$. In fact for $0 \leq \lambda \leq \mu \leq \frac{1}{2}$ one has $z_\mu \prec z_\lambda$. Indeed, to see this take the interval $[z_0, z_1]$ and observe that it is symmetric with respect to the line $\ell = \{x = y\}$. As λ increases, the point z_λ gets closer to ℓ and finally $z_{1/2} \in \ell$. It is clear that z_μ is in the interval $[z_\lambda, z_{1-\lambda}]$, see below.

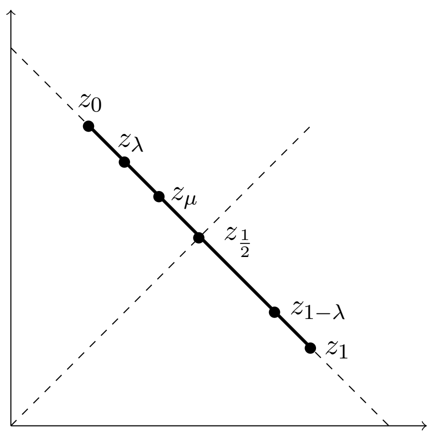


Figure 1: The point z_μ is in $[z_\lambda, z_{1-\lambda}]$.

As a consequence, in order to check Schur convexity of a function with two variables, it is enough to prove monotonicity of $[0, \frac{1}{2}] \ni \lambda \mapsto G(z_\lambda)$. In fact if $G : \mathbb{R}_+^2 \to \mathbb{R}$, we can always extend the interval $[z_0, z_1]$ on which the monotonicity is established to $[(x_1 + x_2, 0), (0, x_1 + x_2)]$. Thus, it is enough to check that $[0, \frac{1}{2}] \ni \lambda \mapsto G((1-\lambda)x, \lambda x)$ is nonincreasing for any given $x > 0$.

Let us prove the following theorem, which is essentially due to Eaton [22] and Komorowski [45].

Theorem 6. *For even functions $\Phi : \mathbb{R} \to \mathbb{R}$ such that Φ'' is convex one has*

$$(a_1^2, \ldots, a_n^2) \prec (b_1^2, \ldots, b_n^2) \qquad \Longrightarrow \qquad \mathbb{E}\Phi\left(\sum_{i=1}^n b_i\varepsilon_i\right) \leq \mathbb{E}\Phi\left(\sum_{i=1}^n a_i\varepsilon_i\right)$$

The following corollary is immediate.

Corollary 7. *For $p \geq 3$ one has*

$$1 = \mathbb{E}|\varepsilon_1|^p \leq \mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p \leq \mathbb{E}\left|\sum_{i=1}^n \frac{1}{\sqrt{n}}\varepsilon_i\right|^p, \qquad \sum_{i=1}^n a_i^2 = 1.$$

Proof of Theorem 6. By symmetry of ε_i we can assume that a_i and b_i are nonnegative. By Remark 4 and by conditioning, it is enough to check monotonicity of the function

$$[0, 1/2] \ni \lambda \longmapsto h(\lambda) = \mathbb{E}\Phi(\varepsilon_1\sqrt{(1-\lambda)x} + \varepsilon_2\sqrt{\lambda x} + s).$$

Recall that by Lemma 3 the function $\psi(t) = \frac{1}{2}\Phi(-\sqrt{x} + s) + \frac{1}{2}\Phi(\sqrt{x} + s)$ is convex. We have

$$\begin{aligned} h(\lambda) &= \mathbb{E}\Phi(\varepsilon_3|\varepsilon_1\sqrt{(1-\lambda)x} + \varepsilon_2\sqrt{\lambda x}| + s) = \mathbb{E}\Phi\left(\varepsilon_3\sqrt{x + 2\varepsilon_1\varepsilon_2 x\sqrt{\lambda(1-\lambda)}} + s\right) \\ &= \mathbb{E}\psi(x + 2\varepsilon_1\varepsilon_2 x\sqrt{\lambda(1-\lambda)}) = \frac{1}{2}\psi(x + 2x\sqrt{\lambda(1-\lambda)}) + \frac{1}{2}\psi(x - 2x\sqrt{\lambda(1-\lambda)}). \end{aligned}$$

Since $2\sqrt{\lambda(1-\lambda)}$ increases from 0 to 1, it is enough to check that the function $\psi(x+t) + \psi(x-t)$ is non-decreasing in $t > 0$, which is obvious, since it is a convex and symmetric function of t. $\square$

1.6 $C_{p,q}$ for p, q even integers

1.6.1 Symmetric functions

Definition 1. A sequence $(a_k)_{k\geq 0}$ is called weakly log-concave if the inequality $a_k^2 \geq a_{k+1}a_{k-1}$ is satisfied for all $k \geq 1$. A sequence $(a_k)_{k\geq 0}$ is called log-concave if in addition the numbers a_k are nonnegative and $\{k : a_k > 0\}$ is a discrete interval.

For real numbers $c_1, c_2, \ldots,$ we define symmetric polynomials $\sigma_k^{(n)}$ and symmetric functions σ_k via

$$\sigma_k^{(n)} = \sum_{S\subseteq[n], |S|=k} \prod_{i\in S} c_i, \qquad \sigma_k = \sum_{S\subseteq\mathbb{N}, |S|=k} \prod_{i\in S} c_i$$

We also define $\sigma_0^{(n)} = \sigma_0 = 1$.

Proposition 8 (Newton inequalities)**.** *Suppose $\sigma_k^{(n)}$ and σ_k are symmetric polynomials and functions associated with real numbers $c_1, c_2, \ldots$. Then*

(a) the sequence $\left(\sigma_k^{(n)}/\binom{n}{k}\right)$ is weakly log-concave for $k \geq 0$;

(b) if c_k are positive, then the sequence $(k!\sigma_k)$ is log-concave for $k \geq 0$.

Remark 5. The condition $c_k > 0$ is not essential. It allows to easily deduce that $\sigma_k^{(n)} \to \sigma_k$ when $n \to \infty$. The numbers σ_k might not be finite, but they will be finite if $\sum_{k=1}^\infty c_k < \infty$, which will be the case in our applications.

Proof. Point (b) follows from (a) by taking the limit $n \to \infty$. To see this observe that from point (a) we have

$$\left(\frac{\sigma_k^{(n)}}{\binom{n}{k}}\right)^2 \geq \frac{\sigma_{k+1}^{(n)}}{\binom{n}{k+1}} \cdot \frac{\sigma_{k-1}^{(n)}}{\binom{n}{k-1}}$$

This can be written as

$$(k!\sigma_k^{(n)})^2 \geq (k+1)!\sigma_{k+1}^{(n)} \cdot (k-1)!\sigma_{k-1}^{(n)} \cdot \frac{n-k}{n-k+1}.$$

Taking the limit $n \to \infty$ finishes the proof.

To prove point (a) let us assume without loss of generality that the numbers c_k are nonzero and take the real rooted polynomial

$$P(x) = (1+c_1x)\dots(1+c_nx) = \sum_{k=0}^{n} \sigma_k^{(n)} x^k.$$

Operations $P(x) \to P^{(l)}(x)$ and $P(x) \to x^n P(x^{-1})$ preserve real-rootedness. Thus

$$Q(x) = P^{(j-1)}(x) = \sum_{k=j-1}^{n} \sigma_k^{(n)} \frac{k!}{(k-j+1)!} x^{k-j+1}$$

is real rooted of degree $n-j+1$. Next

$$R(x) = x^{n-j+1}Q(x^{-1}) = \sum_{k=j-1}^{n} \frac{\sigma_k^{(n)} k!}{(k-j+1)!} x^{n-k}$$

is also real rooted of degree $n-j+1$. Finally,

$$R^{(n-j-1)}(x) = \sum_{k=j-1}^{j+1} \frac{\sigma_k^{(n)} k!(n-k)! x^{j-k+1}}{(k-j+1)!(j-k+1)!} = \frac{1}{2}\tau_{j-1}x^2 + \tau_j x + \frac{1}{2}\tau_{j+1},$$

where $\tau_j = \sigma_j^{(n)} j!(n-j)! = \frac{\sigma_j^{(n)}}{\binom{n}{j}} \cdot n!$, is a real rooted quadratic polynomial. The discriminant Δ of this polynomial must therefore be nonnegative, which leads to $\tau_j^2 \geq \tau_{j-1}\tau_{j+1}$ and finishes the proof. □

1.6.2 Best constants via Hadamard factorization

We are going to show that for even integers $p > q$ one has $C_{p,q} = \frac{(\mathbb{E}|G|^p)^{1/p}}{(\mathbb{E}|G|^q)^{1/q}}$. The proof presented here can be found in [35]. For $S = \sum_{i=1}^n a_i\varepsilon_i$ we want to show

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}} (\mathbb{E}|S|^q)^{\frac{1}{q}}, \qquad p, q - \text{even integers}$$

Equivalently

$$\frac{(\mathbb{E}|S|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^p)^{\frac{1}{p}}} \leq \frac{(\mathbb{E}|S|^q)^{\frac{1}{q}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}$$

In other words, we want to show that the sequence $b_k = \frac{\mathbb{E}|S|^{2k}}{\mathbb{E}|G|^{2k}}$ is such that $b_k^{1/k}$ is non-increasing. Let us prove the following simple lemma.

Lemma 9. *Let $(b_k)_{k\geq 0}$ be a log-concave sequence of positive real numbers with $b_0 = 1$. Then the sequence $(b_k)^{1/k}$ is non-increasing for $k \geq 1$.*

Proof. Take $a_k = \log b_k$. The goal is to prove that $\frac{a_k}{k}$ is non-increasing. Then $a_{k+1} + a_{k-1} \leq 2a_k$ for $k \geq 1$. In other words $\delta_k = a_k - a_{k-1}$ is non-increasing in k. We have $\frac{a_k}{k} = \frac{\delta_1 + \ldots + \delta_k}{k}$, where we have used the fact that $a_0 = 0$. We can see that all we have to show is that consecutive arithmetic means of a non-increasing sequence are non-increasing. The inequality $\frac{\delta_1+\ldots+\delta_{k+1}}{k+1} \geq \frac{\delta_1+\ldots+\delta_k}{k}$ reduces to $k\delta_{k+1} \leq \delta_1 + \ldots + \delta_k$ which is true since $\delta_{k+1} \leq \delta_i$ for $i = 1, \ldots, k$. □

Now, observe that for $x > 0$

$$\mathbb{E}e^{\sqrt{2x}S} = \sum_{l\geq 0} \frac{\sqrt{2x}^l}{l!}\mathbb{E}S^l = \sum_{k\geq 0} \frac{\sqrt{2x}^{2k}}{(2k)!}\mathbb{E}S^{2k}$$
$$= \sum_{k\geq 0} \frac{2^k x^k}{(2k-1)!!2^k k!}\mathbb{E}S^{2k} = \sum_{k\geq 0} \frac{x^k}{k!}\cdot\frac{\mathbb{E}S^{2k}}{\mathbb{E}G^{2k}} = \sum_{k\geq 0} b_k\frac{x^k}{k!}$$

On the other hand

$$\mathbb{E}e^{\sqrt{2x}S} = \mathbb{E}\prod_{i=1}^n e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \mathbb{E}e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \cosh\left(\sqrt{2x}a_i\right)$$

Crucially

$$\cosh(z) = \prod_{l=1}^\infty \left(1 + \frac{4z^2}{\pi^2(2l-1)^2}\right).$$

This gives

$$\mathbb{E}e^{\sqrt{2x}S} = \prod_{i=1}^n\prod_{l=1}^\infty \left(1 + \frac{8a_i^2}{\pi^2(2l-1)^2}x\right) = \prod_i(1+c_i x).$$

Let σ_k be the k-th symmetric function of (c_i). We obtained

$$\sum_{k\geq 0} b_k\frac{x^k}{k!} = \mathbb{E}e^{\sqrt{2x}S} = \prod_i(1+c_i x) = \sum_{k\geq 0}\sigma_k x^k$$

Therefore $b_k = k!\sigma_k$ and thus this sequence is log-concave by Lemma 8.

1.6.3 Best constants via binomial convolutions

In the previous subsection we proved that the sequence $b_k = \frac{\mathbb{E}|S|^{2k}}{\mathbb{E}|G|^{2k}}$ is log-concave, which gave us best constants $C_{p,q}$ for even p, q. It tuns out that this property is closed under taking independent sums. We introduce the following multidimensional definition.

Definition 2. We say that random vector X on $\mathbb{R}^n$ is ultra sub-Gaussian if it is rotation invariant and the sequence $\left(\frac{\mathbb{E}|X|^{2k}}{\mathbb{E}|\mathbf{G}|^{2k}}\right)_{k\geq 0}$ is log-concave with $\mathbf{G} \sim \mathcal{N}(0, I_n)$, where I_n is the $n\times n$ identity matrix.

Therefore we have seen that $S = \sum_{i=1}^n a_i\varepsilon_i$ is ultra sub-Gaussian. An alternative proof of this fact is based on the following theorem.

Theorem 10. *Suppose X, Y are independent ultra sub-Gaussian random vectors. Then $X+Y$ is also ultra sub-Gaussian.*

Proof. Let X_1 be the first coordinate of X and let G_1 be the first coordinate of $\mathbf{G}$. Let θ be a uniform vector on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ and let θ_1 be its first coordinate. We have $\mathbf{G} \sim \theta|\mathbf{G}|$ and since X is rotation invariant we also have $X \sim \theta|X|$, where the factors are independent. Thus by projecting $G_1 \sim \theta_1|G|$ and $X_1 \sim \theta_1|X|$. In particular $\mathbb{E}|G_1|^p = \mathbb{E}|\theta_1|^p\mathbb{E}|G|^p$ and $\mathbb{E}|X_1|^p = \mathbb{E}|\theta_1|^p\mathbb{E}|X|^p$, which gives $\frac{\mathbb{E}|X|^p}{\mathbb{E}|\mathbf{G}|^p} = \frac{\mathbb{E}|X_1|^p}{\mathbb{E}|G_1|^p}$. Since Y and $X+Y$ are also rotation invariant, we have

$$a_k := \frac{\mathbb{E}|X|^{2k}}{\mathbb{E}|\mathbf{G}|^{2k}} = \frac{\mathbb{E}X_1^{2k}}{\mathbb{E}G_1^{2k}}, \qquad b_k := \frac{\mathbb{E}|Y|^{2k}}{\mathbb{E}|\mathbf{G}|^{2k}} = \frac{\mathbb{E}Y_1^{2k}}{\mathbb{E}G_1^{2k}}, \qquad c_k = \frac{\mathbb{E}|X+Y|^{2k}}{\mathbb{E}|\mathbf{G}|^{2k}} = \frac{\mathbb{E}(X_1+Y_1)^{2k}}{\mathbb{E}G_1^{2k}}.$$

Then by symmetry of X and Y we have

$$c_n = \frac{1}{(2n-1)!!}\sum_{k=0}^n \binom{2n}{2k}\mathbb{E}X_1^{2k}\mathbb{E}Y_1^{2n-2k} = \frac{1}{(2n-1)!!}\sum_{k=0}^n \binom{2n}{2k} a_k(2k-1)!!\cdot b_{n-k}(2n-2k-1)!!$$
$$= \frac{1}{(2n-1)!!}\sum_{k=0}^n \frac{2^n n!(2n-1)!!}{2^k k!(2k-1)!!2^{n-k}(n-k)!(2n-2k-1)!!}a_k(2k-1)!!\cdot b_{n-k}(2n-2k-1)!!$$
$$= \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}.$$

Therefore, it is enough to prove the following lemma, which we shall not do here.

Lemma 11 (Walkup, [82])**.** *If $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are log-concave, then it binomial convolution $(c_n)_{n\geq 0}$ defined as*

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

is also log-concave.

For alternative proofs see [52, 32, 65, 58]. □

It is therefore enough to show that symmetric ± 1 random variable ε is ultra sub-Gaussian. It is enough to show that ε is a Gaussian divisor.

Definition 3. A real random vector X in $\mathbb{R}^n$ is a Gaussian divisor if $\mathbf{G}$ has the same distribution as RX for some positive random variable R independent of X.

Lemma 12. *Suppose X is a Gaussian divisor. Then X is ultra sub-Gaussian.*

Proof. We have

$$\frac{\mathbb{E}|X|^p}{\mathbb{E}|\mathbf{G}|^p} = \frac{\mathbb{E}|X|^p \mathbb{E}|R|^p}{\mathbb{E}|\mathbf{G}|^p \mathbb{E}|R|^p} = \frac{\mathbb{E}|RX|^p}{\mathbb{E}|\mathbf{G}|^p \mathbb{E}|R|^p} = \frac{\mathbb{E}|\mathbf{G}|^p}{\mathbb{E}|\mathbf{G}|^p \mathbb{E}|R|^p} = \frac{1}{\mathbb{E}|R|^p}.$$

Since $p \mapsto \mathbb{E}|R|^p$ is log-convex, its reciprocal is log-concave. □

Note that θ is a Gaussian divisor, since $\mathbf{G} = \theta|\mathbf{G}|$ and $\theta, \mathbf{G}$ are independent. Moreover, a uniform random variable U on the unit Euclidean ball is also a Gaussian divisor. In fact we have the following lemma.

Lemma 13. *If X is a rotation invariant random vector in $\mathbb{R}^n$ with radially decreasing density $g(|x|)$, then X has the same distribution as RU, where R has density $-v_n r^n r g'(r)$ on $(0,\infty)$ and U is uniform on the unit Euclidean ball, where v_n stands for the volume of the unit Euclidean ball.*

Proof. Let $u_r(x) = v_n^{-1} r^{-n} \mathbf{1}_{|x|\leq r}$ be the density of rU. We have

$$g(|x|) = \int_{|x|}^{\infty} (-g'(r))\mathrm{d}r = \int_0^{\infty} \mathbf{1}_{\{|x|\leq r\}}(-g'(r))\mathrm{d}r = \int_0^{\infty} u_r(x)(-v_n r^n g'(r))\mathrm{d}r.$$

□

In fact we have proved the following theorem.

We have the following theorem.

Theorem 14. *Let $X_1, \ldots, X_n$ be independent ultra sub-Gaussian random vectors (e.g. random vectors uniform of centered Euclidean spheres or centered Euclidean balls). Then $S = X_1 + \ldots + X_n$ satisfies*

$$(\mathbb{E}|S|^p)^{1/p} \leq \frac{(\mathbb{E}\|\mathbf{G}\|^p)^{1/p}}{(\mathbb{E}\|\mathbf{G}\|^q)^{1/q}} (\mathbb{E}|S|^q)^{1/q}, \qquad p > q \quad \text{even integers.}$$

1.7 Haagerup's work

Here we are going to present basic ideas from Haagerup's paper [33]. Unfortunately, certain technical parts are too complicated to present them here.

1.7.1 Fourier transform

A simple change of variables $s = xt$ together with computation of certain explicit integrals leads to the formulas

$$|x|^p = \begin{cases} C_p \int_0^\infty \frac{1-\cos(xt)}{t^{p+1}} \mathrm{d}t & p \in (0,2) \\ (-C_p) \int_0^\infty \frac{\cos(xt)-1+\frac{1}{2}x^2t^2}{t^{p+1}} \mathrm{d}t & p \in (2,4) \end{cases}, \qquad C_p = \frac{2}{\pi}\sin\left(\frac{p\pi}{2}\right)\Gamma(p+1)$$

Note that the expression changes as we go with p above 2 in order to make the integral convergent. If X is a symmetric real random variable then its characteristic function is

$$\phi_X(t) = \mathbb{E}e^{itX} = \frac{1}{2}\mathbb{E}e^{itX} + \frac{1}{2}\mathbb{E}e^{-itX} = \mathbb{E}\cos(Xt).$$

Therefore, we get the formulas

$$\mathbb{E}|X|^p = \begin{cases} C_p \int_0^\infty \frac{1-\phi_X(t)}{t^{p+1}} \mathrm{d}t & p \in (0,2) \\ (-C_p) \int_0^\infty \frac{\phi_X(t)-1+\frac{1}{2}\mathbb{E}X^2t^2}{t^{p+1}} \mathrm{d}t & p \in (2,4) \end{cases}, \qquad C_p = \frac{2}{\pi}\sin\left(\frac{p\pi}{2}\right)\Gamma(p+1)$$

Let us now take $X = S = \sum_{k=1}^n a_k\varepsilon_k$ and assume that $\sum_{k=1}^n a_k^2 = 1$. Then by independence

$$\phi_X(t) = \prod_{k=1} \mathbb{E}e^{ita_k\varepsilon_k} = \prod_{k=1}^n \cos(a_kt).$$

By concavity of the logarithm we have

$$x_1^{p_1}\ldots x_n^{p_n} \le p_1x_1 + \ldots + p_nx_n, \qquad x_1,\ldots,x_n,p_1,\ldots,p_n \ge 0, \quad p_1+\ldots+p_n = 1.$$

or taking $y_k = x_k^{p_k}$

$$y_1\ldots y_n \le p_1y_1^{\frac{1}{p_1}} + \ldots + p_ny_n^{\frac{1}{p_n}}, \qquad x_1,\ldots,x_n,p_1,\ldots,p_n \ge 0, \quad p_1+\ldots+p_n = 1.$$

Thus

$$\phi_X(t) \le \prod_{k=1}^n |\cos(a_kt)| \le \sum_{k=1}^n a_k^2|\cos(a_kt)|^{\frac{1}{a_k^2}}.$$

Let us define the following function

$$F_p(s) = \begin{cases} C_p \int_0^\infty \frac{1-|\cos(\frac{t}{\sqrt{s}})|^s}{t^{p+1}} \mathrm{d}t & p \in (0,2) \\ (-C_p) \int_0^\infty \frac{|\cos(\frac{t}{\sqrt{s}})|^s-1+\frac{1}{2}t^2}{t^{p+1}} \mathrm{d}t & p \in (2,4) \end{cases}, \qquad C_p = \frac{2}{\pi}\sin\left(\frac{p\pi}{2}\right)\Gamma(p+1)$$

Note that for $p \in (0,2)$ we get

$$\begin{aligned} \mathbb{E}|X|^p &= C_p\int_0^\infty \frac{1-\phi_X(t)}{t^{p+1}}\mathrm{d}t \ge C_p\int_0^\infty \frac{1-\sum_{k=1}^n a_k^2|\cos(a_kt)|^{\frac{1}{a_k^2}}}{t^{p+1}}\mathrm{d}t \\ &= C_p\int_0^\infty \frac{\sum_{k=1}^n a_k^2\left(1-|\cos(a_kt)|^{\frac{1}{a_k^2}}\right)}{t^{p+1}}\mathrm{d}t = \sum_{k=1}^n a_k^2F(a_k^{-2}). \end{aligned}$$

Since $(-C_p) > 0$ for $p \in (2,4)$, in exactly the same way for $p \in (2,4)$ we get

$$\mathbb{E}|X|^p \le \sum_{k=1}^n a_k^2F_p(a_k^{-2}).$$

1.7.2 Constant $C_{2,1}$

It turns out that the function F_1 is explicitly computable. We have

$$F_1(s) = \frac{1}{\pi\sqrt{s}} \int_{-\infty}^{\infty} \frac{1 - |\cos t|^s}{t^2} \mathrm{d}t = \frac{1}{\pi\sqrt{s}} \sum_{n=-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - (\cos t)^s}{(t + n\pi)^2} \mathrm{d}t.$$

Now, $\frac{1}{\sin^2 t} = \sum_{n=-\infty}^{\infty} \frac{1}{(t+n\pi)^2}$. Therefore

$$F_1(s) = \frac{1}{\pi\sqrt{s}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - (\cos t)^s}{\sin^2 t} \mathrm{d}t = \frac{2}{\pi\sqrt{s}} \int_{0}^{\frac{\pi}{2}} \frac{1 - (\cos t)^s}{\sin^2 t} \mathrm{d}t$$

Note that

$$\int_0^{\frac{\pi}{2}} (1 - (\cos t)^s) \left(-\frac{1}{\operatorname{tg} t} \right)' \mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{s (\cos t)^{s-1} \sin t}{\operatorname{tg} t} \mathrm{d}t = s \int_0^{\frac{\pi}{2}} (\cos t)^s \mathrm{d}t = \sqrt{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)},$$

since the boundary terms vanish by computing an appropriate limit. Hence

$$F_1(s) = \frac{2}{\sqrt{\pi s}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

We now claim that $F_1(s)$ is increasing. Using $\Gamma(x+1) = x\Gamma(x)$ one gets

$$F_1(s+2) = \sqrt{\frac{s}{s+2}} \cdot \frac{s+1}{s} F_1(s) = \left(1 - \frac{1}{(s+1)^2}\right)^{-\frac{1}{2}} F_1(s).$$

and by iterating

$$F(s+2n) = F_1(s) \prod_{k=0}^{n-1} \left(1 - \frac{1}{(s+2k+1)^2}\right)^{-\frac{1}{2}}.$$

Taking the limit $n \to \infty$ the left hand side converges to $\sqrt{2/\pi}$ and therefore

$$F_1(s) = \sqrt{\frac{2}{\pi}} \prod_{k=0}^{\infty} \left(1 - \frac{1}{(s+2k+1)^2}\right)^{\frac{1}{2}}.$$

Suppose now that $|a_k| \leq \frac{1}{\sqrt{2}}$ for all k. Then

$$\mathbb{E}|X| \leq \sum_{k=1}^{n} a_k^2 F_1(a_k^{-2}) \geq \sum_{k=1}^{n} a_k^2 F_1(2) = F_1(2) = \frac{1}{\sqrt{2}}.$$

If for some j we have $|a_j| \geq \frac{1}{\sqrt{2}}$, then

$$\mathbb{E}|X| = \mathbb{E}\left|\sum_{k=1}^{n} a_k \varepsilon_k\right| = \mathbb{E}\left|\sum_{k=1}^{n} a_k \varepsilon_k \varepsilon_j\right| \geq \left|\sum_{k=1}^{n} a_k \mathbb{E}\varepsilon_k \varepsilon_j\right| = |a_j| \geq \frac{1}{\sqrt{2}}.$$

This proves that $C_{2,1} = \sqrt{2}$.

1.7.3 Constants $C_{2,q}$ for $q \in (0, 2)$

Case $q_0 \leq q < 2$ *and* $\forall_k\ |a_k| \leq \frac{1}{\sqrt{2}}$. In this case a technical argument (simplified in [68]) shows that $F_q(s) \geq F_p(\infty)$ for $s \geq 2$. Therefore

$$\mathbb{E}|X|^q \geq \sum_{k=1}^{n} a_k^2 F_q(a_k^{-2}) \geq \sum_{k=1}^{n} a_k^2 F_q(\infty) = F_q(\infty) = \mathbb{E}|G|^q.$$

Case $q_0 \le q < 2$ *and* $\exists_k\ |a_k| \ge \frac{1}{\sqrt{2}}$. Let $A_q = C_{2,q}^{-q}$. By homogeneity we can assume that $a_1 = 1$ and want to prove

$$\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n|^q \ge A_q(1 + a_2^2 + \ldots + a_n^2)^{\frac{q}{2}}.$$

We shall proceed by induction and strengthen induction hypothesis to

$$\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n|^q \ge A_q\Phi_q(a_2^2 + \ldots + a_n^2),$$

where

$$\Phi_q(x) = \begin{cases} \phi_q(x) & x \ge 1 \\ 2\phi_q(1) - \phi_q(2-x) & x \in [0,1] \end{cases}, \qquad \phi_q(x) = (1+x)^{\frac{q}{2}}$$

In other words Φ_q on $[0,1]$ is obtained by reflecting the graph of ϕ_q with respect to $(1, \phi_q(1))$. It is not hard to prove that

$$\Phi_q(x) \ge \phi_q(x), \qquad \text{and} \qquad \Phi_q\left(\frac{x+y}{2}\right) \le \frac{\Phi_q(x) + \Phi_q(y)}{2}, \quad x, y \ge 0, \quad \frac{x+y}{2} \le 1.$$

Let $x = a_2^2 + \ldots + a_n^2$. Suppose that $a_1 = 1$ is not the largest coefficient. Then $x \ge 1$ and thus the inequality reduces to its homogeneous version

$$\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n|^q \ge A_q(1 + a_2^2 + \ldots + a_n^2)^{\frac{q}{2}},$$

This is

$$\mathbb{E}|\varepsilon_1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n|^q \ge A_q(1 + a_2^2 + \ldots + a_n^2)^{\frac{q}{2}},$$

which by dividing by the largest a_k and enumerating becomes an inequality of the form

$$\mathbb{E}|1 + b_2\varepsilon_2 + \ldots + b_n\varepsilon_n|^q \ge A_q(1 + b_2^2 + \ldots + b_n^2)^{\frac{q}{2}}$$

with 1 being the largest coefficient. This is weaker than

$$\mathbb{E}|1 + b_2\varepsilon_2 + \ldots + b_n\varepsilon_n|^q \ge A_q\Phi_q(b_2^2 + \ldots + b_n^2)$$

with all $0 \le b_k \le 1$ and therefore we can assume that $a_1 = 1$ is the largest coefficient.

Under this assumption, if $x \ge 1$, then we are in the case $\max_k a_k^2 \le 1 \le \frac{1+x}{2} = \frac{1}{2}\sum_{k=1}^n a_k^2$ and moreover $\Phi_p = \phi_p$, thus we are in the case established previously.

Finally if $x \le 1$ and $n \ge 3$, then introduce notation $x_\pm = a_2^2 + \ldots a_{n-2}^2 + (a_{n-1} \pm a_n)^2$. Then by induction hypothesis

$$\begin{aligned}\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_n\varepsilon_n|^q &= \frac{1}{2}\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_{n-2}\varepsilon_{n-2} + (a_{n-1} + a_n)\varepsilon_{n-1}|^q \\ &\quad + \frac{1}{2}\mathbb{E}|1 + a_2\varepsilon_2 + \ldots + a_{n-2}\varepsilon_{n-2} + (a_{n-1} - a_n)\varepsilon_{n-1}|^q \\ &\ge \frac{\Phi_q(x_+) + \Phi_q(x_-)}{2} \ge \Phi_q\left(\frac{x_+ + x_-}{2}\right) = \Phi_q(x),\end{aligned}$$

since $\frac{1}{2}(x_+ + x_-) = x \le 1$.

Case $0 < q < q_0$. Let us now observe that by (3) with $q < q_0 < 2$ we get

$$\mathbb{E}|X|^q \ge (\mathbb{E}|X|^{q_0})^{\frac{2-q}{2-q_0}} \ge (\mathbb{E}|G|^{q_0})^{\frac{2-q}{2-q_0}} = \mathbb{E}\left|\frac{r_1 + r_2}{\sqrt{2}}\right|^q,$$

where the second inequality follows from the previous cases.

1.7.4 Constants $C_{p,2}$ for $p \in (2,3)$

This case relies on a very technical proof (see a simplification in [63]) of the fact that $F_p(s) \le F_p(\infty)$ for $s \ge \sqrt{2}$, which shows that

$$\mathbb{E}|X|^p \le \sum_{k=1}^n a_k^2 F_p(a_k^{-2}) \le \sum_{k=1}^n a_k^2 F_p(\infty) = F(\infty) = \mathbb{E}|G|^p,$$

which shows the inequality when $|a_k| \le 2^{-1/4}$ for all k. Now, if for some k one has $|a_k| \ge 2^{-1/4}$, then as we have seen at the very beginning, $\mathbb{E}|X|^4 = 3 - 2\sum_{k=1}^n a_k^4 \le 2$ and therefore

$$(\mathbb{E}|X|^p)^2 \le (\mathbb{E}|X|^2)^{4-p}(\mathbb{E}X^4)^{p-2} \le 2^{p-2} \le (\mathbb{E}|G|^p)^2,$$

where the last inequality is straightforward to check.

1.8 Bibliographical notes

Let us now briefly discuss the state of the art for the above classical Khintchine inequality (1). The inequality was first considered a century ago in [40] by Khintchine in his study of the law of the iterated logarithm and independently by Littlewood [53] in 1930. By monotonicity of moments we easily see that $C_{p,q} = 1$ for $p \le q$. The best constants $C_{p,q}$ are known when one of the numbers p, q equals 2, in which case one of the sides of the inequality has a simple form. The optimal constant $C_{p,2}$ for $p > 2$ equals γ_p/γ_2, where $\gamma_p = (\mathbb{E}|G|^p)^{1/p}$ for $G \sim \mathcal{N}(0,1)$. The equality holds asymptotically when $a_i = n^{-1/2}$ and $n \to \infty$. For the constant $C_{2,q}$ with $q \in (0,2)$ a phase transition occurs, namely there is $q_0 \in (1,2)$ such that for $q_0 \le q \le 2$ one has $C_{2,q} = \gamma_2/\gamma_q$, whereas for $0 < q \le q_0$ we have $C_{2,q} = 2^{\frac{1}{q}-\frac{1}{2}}$, in which case equality holds for $n = 2$ and $a_1 = a_2 = 1$. In fact q_0 is the solution to the equation $\Gamma(\frac{q+1}{2}) = \frac{\sqrt{\pi}}{2}$, $q_0 \approx 1.84742$. The constant $C_{p,2}$ for p even was found by Khintchine himself in [40], whereas the constant $C_{p,2}$ for $p \ge 3$ was established by Whittle in [83] and independently by Young in [84] who was not aware of Whittle's work. Szarek in [80] showed that optimal $C_{2,1}$ equals $\sqrt{2}$, answering the question of Littlewood from [53]. The remaining constants $C_{p,2}$ for $p \in (2,3)$ and $C_{2,q}$ for $q \in (0,2) \setminus \{1\}$ were found by Haagerup in his celebrated work [33] using Fourier methods, see also the article [68] of Nazarov and Podkorytov for a simpler proof for $C_{2,q}$ and the article [63] of Mordhorst for a simpler proof in the case $C_{p,2}$ with $p \in (2,3)$, based on the idea of Nazarov and Podkorytov. In the case of even p, q with p divisible by q the best constants were obtained by Czerwiński in [21]. In [65] the optimal constants $C_{p,q}$ for all even numbers $p > q > 0$ were found, see also a recent work [35] for an alternative proof.

In fact for $p \ge 3$ a much stronger control on the p-th moments is available. If the vector $(a_1^2, \ldots, a_n^2)$ majorizes $(b_1^2, \ldots, b_n^2)$ in the Schur order, then for $p \ge 3$ we have $\mathbb{E}|\sum_{i=1}^n b_i\varepsilon_i|^p \le \mathbb{E}|\sum_{i=1}^n a_i\varepsilon_i|^p$. This was proved by Eaton and Komorowski in [22] and [45] (Komorowski checked Eaton's condition derived for general function Φ in place of $|x|^p$). Pinelis showed in [71] that for even Φ with Φ'' convex one has $\mathbb{E}\Phi(\sum_{i=1}^n a_i\varepsilon_i) \le \mathbb{E}\Phi(\sum_{i=1}^n a_i X_i)$ for any symmetric variance one independent random variables X_i. This was also independently proved in [28]. Taking $\Phi(x) = |x|^p$ for $p \ge 3$ and $X_i \sim \mathcal{N}(0,1)$ gives $C_{p,3} = \gamma_p$.

2 Projections of B_p^n and Khinchine inequalities

Let us define

$$B_p^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ |x_1|^p + \ldots + |x_n|^p \le 1\},$$

which is the unit ball in the ℓ_p^n norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \ge 1$. When $p = \infty$ one defines $\|x\|_\infty = \max_{k=1,\ldots,n} |x_k|$ and thus $B_\infty^n = [-1,1]^n$ is the cube. In this section we shall discuss the problem of finding orthogonal projections of B_p^n onto codimension one hyperplanes having maximal and minimal $(n-1)$-dimensional volume. By H_k we shall denote $(1, \ldots, 1, 0, \ldots, 0)^\perp$ with k ones.

2.1 Projections of convex polytopes

Suppose we are give a convex polytope P in $\mathbb{R}^n$ and we want to project it onto a hyperplane $a^\perp$, where a is some unit vector. It is easy to derive a formula for the volume of such a projection. Let $\mathcal{F}_\mathcal{P}$ be the set of faces of P. If $F \in \mathcal{F}_P$ then $|\operatorname{Proj}_{a^\perp} F| = |F| \cdot |\langle a, n(F)\rangle|$, where $n(F)$ is the unit normal vector to F. Note that in $\operatorname{Proj}_{a^\perp} P$ every point is *covered* two times, so one gets the following expression for the volume of projection

$$|\operatorname{Proj}_{a^\perp} P| = \frac{1}{2}\sum_{F\in\mathcal{F}_P} |F| \cdot |\langle a, n(F)\rangle|.$$

Note that for $p = \infty$ the normal vectors are the standard basis vectors $\pm e_i$ and therefore one gets the formula

$$|\operatorname{Proj}_{a^\perp} P| = 2^{n-1}\sum_{i=1} |a_i|$$

Since by Cauchy-Schwarz

$$1 = \sum_{i=1} a_i^2 \le \sum_{i=1} |a_i| \le \sqrt{n}\left(\sum_{i=1}^n a_i^2\right)^{1/2} = \sqrt{n},$$

the maximal projection is given by H_n and minimal by H_1.

2.2 Projections of B_1^n

We shall consider a more delicate example of $B_1^n = \{|x_1|+\ldots+|x_n| \le 1\}$. It is not hard to see that the boundary ∂B_1^n consists of 2^n faces of equal volume. These faces can be indexed by sequences of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$. The face F_ε is contained in the affine hyperplanes $\varepsilon_1 x_1 + \ldots \varepsilon_n x_n = 1$ and $n(F_\varepsilon) = n^{-1/2}(\varepsilon_1, \ldots, \varepsilon_n)$. Thus one gets

$$|\operatorname{Proj}_{a^\perp} B_1^n| = C_n \sum_{\varepsilon\in\{-1,1\}^n} |\langle \varepsilon, a\rangle| \tag{4}$$

One can determine C_n plugging in $a = e_1$. Let us rewrite our equality in a probabilistic form. Let $r_1, \ldots, r_n$ be i.i.d. symmetric Bernoulli random variables, namely $\mathbb{P}(r_i = \pm 1) = \frac{1}{2}$. Since for $a = (a_1, \ldots, a_n)$ one has $\sum_{\varepsilon\in\{-1,1\}^n} |\langle \varepsilon, a\rangle| = 2^n \mathbb{E}|\sum_{i=1}^n a_i r_i|$, we get

$$|\operatorname{Proj}_{a^\perp} B_1^n| = 2^n C_n \mathbb{E}\left|\sum_{i=1}^n a_i r_i\right|.$$

Recall that we assume here that a is a unit vector. This gives a constraint on the second moment $\mathbb{E}|\sum_{i=1}^n a_i r_i|^2 = 1$. We see that finding extremal projections of B_1^n is equivalent to finding best constants in the corresponding Khinchine inequality. The maximal projection is therefore given by H_1 and minimal by H_2 by Szarek's inequality.

2.3 General formula for projections

Let σ_K be the normalized surface area measure on ∂K and let S be the not normalized surface are measure, that is $S(A) = |\partial K \cap A|$ and $\sigma_K(A) = |\partial K \cap A|/|\partial K|$. Let μ_K be the normalized cone volume measure, that is, for $A \subseteq \partial K$ let $\mu_K(A) = |\operatorname{conv}(\{0\} \cup A)|/|K|$. Let C denote its not normalized version.

We say that K is a convex body if K is convex, compact and has non empty interior. With every symmetric convex body K we can associate a norm $\|x\|_K = \min\{t \ge 0 : x \in tK\}$. We have the following lemma due to Noar and Romik, see [64].

Lemma 15. *If K is a symmetric convex body then σ_K is absolutely continuous with respect to μ_K and for almost all $x \in \partial K$ one has*

$$\frac{d\sigma_K}{d\mu_K}(x) = \frac{n|K|}{|\partial K|}|\nabla(\|\cdot\|_K)(x)|.$$

Sketch of the proof. For points x such that x is perpendicular to the surface of K one has $|x| \cdot \mathrm{d}S(x) = n\mathrm{d}C(x)$. If the angel between the surface and x is α, then $|\cos\alpha| \cdot |x| \cdot \mathrm{d}S(x) = n\mathrm{d}C(x)$. We clearly have $|\cos\alpha| = |\langle n(x), x/|x|\rangle|$. Let $z = \nabla\|\cdot\|_K(x)$. If $x \in \partial K$ then $1+\varepsilon = \|x+\varepsilon x\|_K \approx \|x\|_K + \varepsilon\langle z, x\rangle = 1+\varepsilon\langle z, x\rangle$, which gives $\langle z, x\rangle = 1$. Also, z is a vector perpendicular to ∂K. Thus $n(x) = z/|z|$. We obtain

$$|\cos\alpha| = \frac{1}{|x|} \cdot |\langle n(x), x\rangle| = \frac{|\langle z, x\rangle|}{|x| \cdot |z|}.$$

This gives

$$\frac{|\partial K| \mathrm{d}\sigma_K(x)}{|\nabla\|\cdot\|_K(x)|} = \frac{\mathrm{d}S(x)}{|\nabla\|\cdot\|_K(x)|} = \frac{|\langle z, x\rangle|}{|z|}\mathrm{d}S(x) = n\mathrm{d}C(x) = n|K|\mathrm{d}\mu_K(x).$$

$\square$

The usual Cauchy formula for the volume of projection (explained at the beginning for polytopes) can be written as

$$|\operatorname{Proj}_{a^\perp} K| = \frac{1}{2}|\partial K| \int_{\partial K} |\langle n(x), a\rangle| \mathrm{d}\sigma_K(x).$$

From Lemma 15 we therefore get

$$|\operatorname{Proj}_{a^\perp} K| = \frac{n}{2}|K| \int_{\partial K} |\langle (\nabla\|\cdot\|_K)(x), a\rangle| \mathrm{d}\mu_K(x),$$

since $(\nabla\|\cdot\|_K)(x) = n(x)|(\nabla\|\cdot\|_K)(x)|$.

2.4 Probabilistic formula for projections of B_p^n

According to our formula we get

$$|\operatorname{Proj}_{a^\perp} B_p^n| = C(p,n) \int_{\partial B_p^n} \left| \sum_{i=1}^n a_i |x_i|^{p-1} \operatorname{sgn}(x_i) \right| \mathrm{d}\mu_{B_p^n}(x). \tag{5}$$

The cone volume measure $\mu_{B_p^n}$ enjoys a probabilistic representation in terms of i.i.d. random variables, discovered by Rachev and Rüschendorf in [75] and independently by Schechtman and Zinn in [79]. Let us formulate a generalization of these results discussed in [74].

Lemma 16. *Let K be a symmetric convex body and let Z be any random vector in $\mathbb{R}^n$ with density of the form $f(\|x\|_K)$ for some continuous $f : [0,\infty) \to [0,\infty)$. Let U be a random variable uniform in $[0,1]$, independent of Z. Then*

(a) $\frac{Z}{\|Z\|_K}$ has distribution μ_K and $U^{1/n}\frac{Z}{\|Z\|_K}$ is uniformly distributed on K,

(b) $\frac{Z}{\|Z\|_K}$ and $\|Z\|_K$ are independent.

In particular, for $K = B_p^n$ one can take $Z = (Y_1, \ldots, Y_n)$ where Y_i are i.i.d. random variables having densities $(2\Gamma(1+\frac{1}{p}))^{-1}e^{-|t|^p}$.

Proof. We first claim that for any integrable $h : \mathbb{R}^n \to \mathbb{R}$ the following identity holds

$$\int h = n|K| \int_0^\infty r^{n-1} \int_{\partial K} h(rz)\mathrm{d}\mu_K(z)\mathrm{d}r. \tag{6}$$

To show it one can assume that $h = \mathbf{1}_A$, where $A = [a,b] \cdot A_0$, where $A_0 \subset \partial K$, as these sets generate the sigma algebra of Borel sets in $\mathbb{R}^n$. For $z \in \partial K$ and $r > 0$ we then have $h(rz) = \mathbf{1}_{[a,b]}(r)\mathbf{1}_{A_0}(z)$. Thus (6) reduces to

$$|A| = |K| \left(\int_a^b n r^{n-1} \mathrm{d}r \right) \mu_K(A_0) = |K|(b^n - a^n)\mu_K(A_0) = |[a,b]A_0| \tag{7}$$

and is therefore true. Now, let us notice that for $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\psi(\|Z\|_K)\right] &= \int_{\mathbb{R}^n}\phi\left(\frac{x}{\|x\|_K}\right)\psi(\|x\|_K)f(\|x\|_K)\mathrm{d}x\\ &= n|K|\int_0^\infty \psi(r)f(r)r^{n-1}\mathrm{d}r\int_{\partial K}\phi(z)\mathrm{d}\mu_K(z).\end{aligned}$$

Taking $\phi,\psi \equiv 1$ we learn that $n|K|\int_0^\infty f(r)r^{n-1}\mathrm{d}r = 1$. Thus taking $\psi \equiv$ and next $\phi \equiv 1$ we arrive at

$$\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\right] = \int_{\partial K}\phi(z)\mathrm{d}\mu_K(z), \qquad \mathbb{E}\left[\psi(\|Z\|_K)\right] = n|K|\int_0^\infty \psi(r)f(r)r^{n-1}\mathrm{d}r.$$

The first equation shows that $\frac{Z}{\|Z\|_K}$ has distribution μ_K. Moreover, we get

$$\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\psi(\|Z\|_K)\right] = \mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\right]\mathbb{E}\left[\psi(\|Z\|_K)\right],$$

which shows (b). Finally (7) to gether with the fact that $U^{1/n}$ has density nr^{n-1} on $[0,1]$ shows that

$$\frac{|A|}{|K|} = \mathbb{P}\left(U^{1/n}\in[a,b]\right)\mathbb{P}\left(\frac{Z}{\|Z\|_K}\in A_0\right) = \mathbb{P}\left(U^{1/n}\frac{Z}{\|Z\|_K}\in A\right),$$

which shows the second part of point (a).

□

We can now prove the probabilistic formula for the volume of hyperplane projection of B_p^n.

Lemma 17. *For $p > 1$ and every unit vector $a \in \mathbb{R}^n$, we then have*

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(B_p^n)) = \frac{\mathrm{vol}_{n-1}(B_p^{n-1})}{\mathbb{E}|X_1|}\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|, \tag{8}$$

where here $X_1,\ldots,X_n$ are i.i.d. random variables with density

$$f_p(x) = \frac{p}{2(p-1)\Gamma(1/p)}|x|^{\frac{2-p}{p-1}}e^{-|x|^{\frac{p}{p-1}}}.$$

Proof. By (5) and Lemma 16 (a) for some constant $c_{p,n}$ we have

$$\begin{aligned}\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}B_p^n) &= C(p,n)\mathbb{E}\left|\sum_{i=1}^n a_i\left|\frac{Y_i}{S}\right|^{p-1}\mathrm{sgn}\left(\frac{Y_i}{S}\right)\right| = C(p,n)\cdot\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p-1}}\cdot\mathbb{E}\left|\sum_{i=1}^n a_i\left|\frac{Y_i}{S}\right|^{p-1}\mathrm{sgn}\left(Y_i\right)\right|\\ &= \frac{C(p,n)}{\mathbb{E}S^{p-1}}\cdot\mathbb{E}\left|\sum_{i=1}^n a_i\left|Y_i\right|^{p-1}\mathrm{sgn}\left(Y_i\right)\right|.\end{aligned}$$

It now suffices to observe that $X_i = |Y_i|^{p-1}\,\mathrm{sgn}\,(Y_i)$ for $p > 1$ have densities f_p. We then compute $C_{p,n}$ by taking $a = e_1$. □

2.5 Choquet order

We say that $\mu \prec \nu$ in the (symmetric) Choquet order if for any even convex function $\phi : \mathbb{R}^n \to [0,\infty]$ one has $\int \phi\ \mathrm{d}\mu \le \int \phi\ \mathrm{d}\nu$. We have the following simple lemma.

Lemma 18. *Suppose $\mu \prec \nu$ are symmetric measures on $\mathbb{R}^k$. Then for any even symmetric measure λ on $\mathbb{R}^l$ one has $\mu\otimes\lambda \prec \nu\otimes\lambda$. In particular $\mu^{\otimes n} \prec u^{\otimes n}$.*

Proof. We have to show that

$$\int_{\mathbb{R}^{k+l}} \phi(x,y)\mathrm{d}\mu(x)\mathrm{d}\lambda(y) \le \int_{\mathbb{R}^{k+l}} \phi(x,y)\mathrm{d}\nu(x)\mathrm{d}\lambda(y).$$

In suffices to use the definition of Choquet order for μ and ν with $\tilde{\phi}(x) = \int_{\mathbb{R}^l} \phi(x,y)\mathrm{d}\lambda(y)$. This function is even since

$$\tilde{\phi}(-x) = \int_{\mathbb{R}^l} \phi(-x,y)\mathrm{d}\lambda(y) = \int_{\mathbb{R}^l} \phi(x,-y)\mathrm{d}\lambda(y) = \int_{\mathbb{R}^l} \phi(x,y)\mathrm{d}\lambda(y) = \tilde{\phi}(x),$$

since λ is symmetric. □

We now provide a sufficient condition for μ to be smaller than ν in the Choquet order, on the real line.

Lemma 19. *Suppose $f, g : \mathbb{R} \to [0,\infty)$ are even probability densities satisfying $\int |t|f(t)\mathrm{d}t = \int |t|g(t)\mathrm{d}t < \infty$. Assume moreover that there are $0 < x < y$ such that $\{t \ge 0 : g(t) < f(t)\} = (x,y)$. Then the measures μ, ν with densities f, g satisfy $\mu \prec \nu$.*

Proof. It suffices to prove that for any convex $\phi : [0,\infty) \to [0,\infty]$ one has $\int \phi(t)f(t)\mathrm{d}t \le \int \phi(t)g(t)\mathrm{d}t$. Equivalently

$$\int (\phi(t) - (at+b))(g(t) - f(t))\mathrm{d}t \ge 0,$$

where a, b are arbitrary real numbers. Let $\psi(t) = \phi(t) - (at+b)$. Choose a, b in such a way that $\psi(x) = \psi(y) = 0$. Note that ψ is convex and thus $\psi(t) \le 0$ on $[x,y]$ and $\psi(t) \ge 0$ on $[0,x] \cup [y,\infty)$. In other words $\psi(g-f)$ is non-negative on $[0,\infty)$. □

2.6 Largest projections for $p \in [1,2]$

We shall show the inequality $\mathbb{E}|\sum_{i=1}^n a_i X_i^{(p)}| \le \mathbb{E}|X_i^{(p)}|$, where X_i are i.i.d. with densities f_p. This shows the inequality

$$|\operatorname{Proj}_{a^\perp} B_p^n| \le |\operatorname{Proj}_{(1,0,\dots,0)^\perp} B_p^n|.$$

In fact we shall prove that for $1 \le p \le q \le \infty$ one has

$$\frac{\mathbb{E}|\sum_{i=1}^n a_i X_i^{(p)}|}{\mathbb{E}|X_1^{(p)}|} \le \frac{\mathbb{E}|\sum_{i=1}^n a_i X_i^{(q)}|}{\mathbb{E}|X_1^{(q)}|}$$

and use it with $q = 2$, in which case f_q is Gaussian and the right hand side is 1 (the sum of independent Gaussian random variables is again Gaussian).

Let us introduce $Y_i^{(p)} = X_i^{(p)}/\mathbb{E}|X_i^{(p)}|$. Then $\mathbb{E}|Y_i^{(p)}| = \mathbb{E}|Y_i^{(q)}| = 1$. Our goal is to prove that

$$\mathbb{E}\left|\sum_{i=1}^n a_i Y_i^{(p)}\right| \le \mathbb{E}\left|\sum_{i=1}^n a_i Y_i^{(q)}\right|.$$

Since $\phi(x) = |\sum_{i=1}^n a_i x_i|$ is symmetric and convex, it suffices to show that $\mathcal{L}(Y_i^{(p)}) \prec \mathcal{L}(Y_i^{(q)})$ and use Lemma 18. In order to show that $\mathcal{L}(Y_i^{(p)}) \prec \mathcal{L}(Y_i^{(q)})$ we shall use Lemma 4. To this end one has to check that the densities $\tilde{f}_p$ and $\tilde{f}_q$ of these random variables intersect in exactly two points and that $\tilde{f}_q > \tilde{f}_p$ near the origin (which is clear from the asymptotics). The fact that these two functions intersect in at least two points follows from the next lemma. Below we don't give a precise definition of the number of sign change points as we shall use this notion only in very simple situations, where the meaning of this terms is clear.

Lemma 20. *Let $k, n \ge 1$ be integers and let $g : \mathbb{R} \to \mathbb{R}$ be measurable. Suppose that g changes sign at exactly k points. Assume moreover that $\int_{\mathbb{R}} x^j g(x)\mathrm{d}x = 0$ for all $j = 0, 1, \dots, n-1$. Then $k \ge n$.*

Proof. We prove the lemma by contradiction. Assume that $k \leq n-1$. Let $x_1 < x_2 < \ldots < x_k$ be the sign change points of g. From our assumption, for every polynomial P of degree at most $n-1$ one has $\int Pg = 0$. Let us take $P(x) = (x-x_1)\ldots(x-x_k)$ and $h = Pg$. We have $\int h = 0$. On the other hand, h does not change sign since P changes sign exactly at the same points as g. Since h is not identically zero, we get $\int h \neq 0$, contradiction. $\square$

Using this lemma and the fact that $\int_0^\infty t^i f_p(t)\mathrm{d}t = \int_0^\infty t^i f_q(t)\mathrm{d}t$ for $i = 0, 1$, we get that f_p and f_q intersect in at least 2 points. We have

$$\tilde{f}_p(t) = c_1 t^{\alpha_1} e^{-(t/d_1)^{\beta_1}}, \qquad \tilde{f}_q(t) = c_2 t^{\alpha_2} e^{-(t/d_2)^{\beta_2}},$$

where $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$.

The equation $f_p(t) = f_q(t)$ on $(0,\infty)$ is of the form

$$w(t) := c + (\alpha_1 - \alpha_2)\log t + \left(\frac{t}{d_2}\right)^{\beta_2} - \left(\frac{t}{d_1}\right)^{\beta_1} = 0.$$

We have

$$v(t) := tw'(t) = \alpha_1 - \alpha_2 + \beta_2\left(\frac{t}{d_2}\right)^{\beta_2} - \beta_1\left(\frac{t}{d_1}\right)^{\beta_1}.$$

The inequality $v'(t) \geq 0$ is equivalent to $t^{\beta_2-\beta_1} \geq \beta_1^2 d_2^{\beta_2}\beta_2^{-2} d_1^{-\beta_1}$, which holds on some interval $[0,z]$. Thus v is first increasing and then decreasing. Since $v(0) > 0$, v can have at most one root. By Roll's theorem w can have at most two roots.

In fact we have proved the following corollary.

Corollary 21. *Let $1 \leq p \leq q \leq \infty$ and let a be a unit vector in $\mathbb{R}^n$. Then*

$$\frac{|\operatorname{Proj}_{a^\perp} B_p^n|}{|B_p^{n-1}|} \leq \frac{|\operatorname{Proj}_{a^\perp} B_q^n|}{|B_q^{n-1}|}.$$

2.7 Gaussian mixtures

We shall need the following definition.

Definition 4. A random variable X is called a (centered) Gaussian mixture if there exists a positive random variable R and a standard Gaussian random variable Z, independent of R, such that X has the same distribution as the product RZ.

For example, a random variable X with density of the form

$$f(x) = \sum_{j=1}^m p_j \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{x^2}{2\sigma_j^2}},$$

where $p_j, \sigma_j > 0$ are such that $\sum_{j=1}^m p_j = 1$, is a Gaussian mixture corresponding to the discrete random variable R with $\mathbb{P}(R = \sigma_j) = p_j$.

Recall that an infinitely differentiable function $g : (0,\infty) \to \mathbb{R}$ is called completely monotonic if $(-1)^n g^{(n)}(x) \geq 0$ for all $x > 0$ and $n \geq 0$, where for $n \geq 1$ we denote by $g^{(n)}$ the n-th derivative of g and $g^{(0)} = g$. A classical theorem of Bernstein asserts that g is completely monotonic if and only if it is the Laplace transform of some measure, i.e. there exists a non-negative Borel measure μ on $[0,\infty)$ such that

$$f(x) = \int_0^\infty e^{-tx}\mathrm{d}\mu(t), \qquad x > 0. \tag{9}$$

Bernstein's theorem implies the following equivalence.

Lemma 22. *A symmetric random variable X with density f is a Gaussian mixture if and only if the function $x \mapsto f(\sqrt{x})$ is completely monotonic for $x > 0$.*

Proof. Let X be a Gaussian mixture RZ, where R is positive and Z is an independent standard Gaussian random variable. Denote by ν the law of R. Clearly X is symmetric. Furthermore,

$$\mathbb{P}(X \in A) = \mathbb{P}(RZ \in A) = \int_0^\infty \mathbb{P}(rZ \in A)\mathrm{d}\nu(r) = \int_A \int_0^\infty \frac{1}{\sqrt{2\pi}r} e^{-\frac{x^2}{2r^2}} \mathrm{d}\nu(r)\mathrm{d}x. \tag{10}$$

This implies that X has a density

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2r^2}} \frac{\mathrm{d}\nu(r)}{r}. \tag{11}$$

Thus, $f(\sqrt{x})$ is completely monotonic.

Now, for the converse, let X be a symmetric random variable with density f such that the function $x \mapsto f(\sqrt{x})$ is completely monotonic. By Bernstein's theorem, there exists a non-negative Borel measure μ supported on $[0, \infty)$ such that

$$f(\sqrt{x}) = \int_0^\infty e^{-tx}\mathrm{d}\mu(t), \quad \text{for every } x > 0 \tag{12}$$

or, equivalently, $f(x) = \int_0^\infty e^{-tx^2}\mathrm{d}\mu(t)$ for every $x \in \mathbb{R}$. Notice that $\mu(\{0\}) = 0$, because otherwise f would not be integrable. Now, for a subset $A \subseteq \mathbb{R}$ we have

$$\begin{aligned}\mathbb{P}(X \in A) &= \int_A \int_0^\infty e^{-tx^2}\mathrm{d}\mu(t)\mathrm{d}x = \int_0^\infty \int_A e^{-tx^2}\mathrm{d}x\mathrm{d}\mu(t) \\ &= \int_0^\infty \int_{\sqrt{2t}A} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x \, \sqrt{\frac{\pi}{t}}\mathrm{d}\mu(t) = \int_0^\infty \gamma_n(\sqrt{2t}A)\mathrm{d}\nu(t),\end{aligned}$$

where $\mathrm{d}\nu(t) = \sqrt{\frac{\pi}{t}}\mathrm{d}\mu(t)$. In particular, choosing $A = \mathbb{R}$, we deduce that ν is a probability measure supported on $(0, \infty)$. Let V be a random variable distributed according to ν. Clearly V is positive almost surely. Define $R = \frac{1}{\sqrt{2V}}$ and let Z be a standard Gaussian random variable, independent of R. Then

$$\mathbb{P}(RZ \in A) = \mathbb{P}\left(\frac{1}{\sqrt{2V}} \cdot Z \in A\right) = \int_0^\infty \gamma_n(\sqrt{2t}A)\mathrm{d}\nu(t) = \mathbb{P}(X \in A),$$

that is, X has the same distribution as RZ. □

The following simple lemma allows us to construct completely monotonic function.

Lemma 23. *The following holds true:*

(a) If g is a completely monotonic function on $(0, \infty)$ and h is positive and has a completely monotonic derivative on $(0, \infty)$, then $g \circ h$ is also completely monotonic on $(0, \infty)$.

(b) If f, g are completely monotonic on $(0, \infty)$, then fg is also completely monotonic.

(c) The densities $c_p e^{-|t|^p}$ are Gaussian mixtures for $p \in (0, 2]$. Also f_p is the density of a Gaussian mixture for $p \geq 2$.

2.8 Extremal projections for $p \geq 2$

The following lemma is crucial.

Lemma 24. *Suppose $X_1, \ldots, X_n$ are i.i.d. Gaussian mixtures and let $\Phi : \mathbb{R} \to \mathbb{R}$ be even and such that $\Psi(x) = \Phi(\sqrt{x})$ is convex on $[0, \infty)$. Then*

$$(a_1^2, \ldots, a_n^2) \preceq (b_1^2, \ldots, b_n^2) \implies \mathbb{E}\Phi\left(\sum_{i=1}^n a_i X_i\right) \leq \mathbb{E}\Phi\left(\sum_{i=1}^n b_i X_i\right).$$

If $\Psi(x)$ is concave then the inequality is reversed.

Proof. Observe that $X_i = R_i Z_i$ for some independent $R_i > 0$ and $Z_i \sim \mathcal{N}(0,1)$. Thus

$$\sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i R_i Z_i \sim \left(\sum_{i=1}^n a_i^2 R_i^2 \right)^{1/2} Z_1.$$

We therefore have

$$\mathbb{E}\Phi\left(\sum_{i=1}^n a_i X_i \right) = \mathbb{E}\Psi\left(Z_1^2 \sum_{i=1}^n a_i^2 R_i^2 \right),$$

which is clearly a permutation symmetric and convex function of $(a_1^2, \ldots, a_n^2)$ and thus is Schur convex, which finishes the proof of the lemma. $\square$

Recall that

$$|\operatorname{Proj}_{a^\perp} B_p^n| = C_1(p,n) \cdot \mathbb{E}\left| \sum_{i=1}^n a_i X_i \right|.$$

From the last section we know that X_i are Gaussian mixtures. Thus, using the above lemma with $\Phi(x) = |x|$, we get the following theorem.

Theorem 25. *Fix $p \in (2, \infty]$. For two unit vectors $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ we have*

$$(a_1^2, \ldots, a_n^2) \preceq (b_1^2, \ldots, b_n^2) \implies |\operatorname{Proj}_{a^\perp} B_p^n| \geq |\operatorname{Proj}_{b^\perp} B_p^n|.$$

In particular for any $a \in S^{n-1}$ we have

$$|\operatorname{Proj}_{(1,0,\ldots,0)^\perp} B_p^n| \leq |\operatorname{Proj}_{a^\perp} B_p^n| \leq |\operatorname{Proj}_{n^{-1/2}(1,\ldots,1)^\perp} B_p^n|.$$

3 Extremal sections of B_p^n

We now turn to the dual question of finding extremal values of $|B_p^n \cap a^\perp|$ for $a \neq 0$.

3.1 Formula via negative moments

We are going to follow the idea of Kalton and Koldobsky. We begin with the following simple lemma.

Lemma 26. *Suppose X is a real random variable with continuous bounded density f. Then*

$$f(0) = \lim_{q \to 1^-} \frac{1-q}{2} \cdot \mathbb{E}|X|^{-q}.$$

Proof. Our goal is to show that $f(0) = \lim_{q \to 1^-} \frac{1-q}{2} \cdot \int_{\mathbb{R}} |x|^{-q} f(x) \mathrm{d}x$. Fix $\varepsilon > 0$ and $\delta > 0$ such that for $|x| \leq \delta$ one has $f(0) - \varepsilon \leq f(x) \leq f(0) + \varepsilon$. There exists a constant M such that for $|x| \geq M$ we have $f(x) = 0$. We first observe that

$$\frac{1-q}{2} \cdot \int_{|x|>\delta} |x|^{-q} f(x) \mathrm{d}x \leq \frac{1-q}{2} \delta^{-q} \xrightarrow[q \to 1^-]{} 0.$$

Moreover

$$(f(0) - \varepsilon)\delta^{1-q} \leq \frac{1-q}{2} \cdot \int_{|x| \leq \delta} |x|^{-q} f(x) \mathrm{d}x \leq (f(0) + \varepsilon)\delta^{1-q}$$

Taking $q \to 1^-$ we therefore arrive at

$$f(0) - \varepsilon \leq \liminf_{q \to 1^-} \frac{1-q}{2} \cdot \int_{\mathbb{R}} |x|^{-q} f(x) \mathrm{d}x \leq \limsup_{q \to 1^-} \frac{1-q}{2} \cdot \int_{\mathbb{R}} |x|^{-q} f(x) \mathrm{d}x \leq f(0) + \varepsilon.$$

Taking $\varepsilon \to 0^+$ gives the result. $\square$

Now suppose that X is uniform on some convex body K with $|K| = 1$. For a unit vector a let us consider the section function

$$f_a(t) = |K \cap (a^\perp + ta)|$$

Clearly

$$\int_s^\infty f_a(t) \mathrm{d}t = |K \cap \{\langle x, a\rangle \geq s\}| = \mathbb{P}(\langle X, a\rangle \geq s)$$

and thus f_a is the density of $\langle a, X\rangle$. Therefore

$$|K \cap a^\perp| = f_a(0) = \lim_{q\to 1^-} \frac{1-q}{2} \mathbb{E}\,|\langle X, a\rangle|^{-q}.$$

As for the case of projections, using $X \sim \frac{Y}{S} U^{1/n}$, where $S = \|Y\|_p$ and $Y = (Y_1, \ldots, Y_n)$ with Y_i i.i.d. with densities $c_p e^{-|t|^p}$, we get

$$\begin{aligned}
\mathbb{E}|\,\langle X, a\rangle\,|^{-q} &= \mathbb{E}U^{-\frac{q}{n}} \mathbb{E}|\,\langle Y/S, a\rangle\,|^{-q} = \mathbb{E}U^{-\frac{q}{n}} \cdot \frac{\mathbb{E}S^{-q}}{\mathbb{E}S^{-q}} \mathbb{E}|\,\langle Y/S, a\rangle\,|^{-q} \\
&= \frac{\mathbb{E}U^{-\frac{q}{n}}}{\mathbb{E}S^{-q}} \mathbb{E}|\,\langle Y, a\rangle\,|^{-q} = c_{p,q,n} \mathbb{E}|\,\langle Y, a\rangle\,|^{-q}
\end{aligned}$$

Therefore

$$|B_p^n \cap a^\perp| = f_a(0) = \lim_{q\to 1^-} c_{p,q,n} \frac{1-q}{2}\ \mathbb{E}\left|\sum_{i=1}^n a_i Y_i\right|^{-q}.$$

3.2 Gaussian mixture case

If $p \in (1, 2)$ then Y_i are Gaussian mixtures, $Y_i \sim R_i G_i$. Thus

$$\sum_{i=1}^n a_i Y_i \sim \left(\sum_{i=1}^n a_i^2 R_i^2\right)^{1/2} G_1.$$

Since $\lim_{q\to 1^-} \frac{1-q}{2} \mathbb{E}|G_1|^{-q} = \frac{1}{\sqrt{2\pi}}$ by Lemma 26, we get

$$|B_p^n \cap a^\perp| = \lim_{q\to 1^-} c_{p,q,n} (1-q) \mathbb{E}|G_1|^{-q} \mathbb{E}\left|\sum_{i=1}^n a_i^2 R_i^2\right|^{-\frac{q}{2}} = \sqrt{\frac{2}{\pi}} c_{p,1,n}\ \mathbb{E}\left|\sum_{i=1}^n a_i^2 R_i^2\right|^{-\frac{1}{2}}.$$

According to Lemma 24 we get the following theorem.

Theorem 27. *Let $p \in [1, 2]$ and let $(a_1^2, \ldots, a_n^2) \prec (b_1^2, \ldots, b_n^2)$ for unit vectors a, b. Then $|B_p^n \cap a^\perp| \leq |B_p^n \cap b^\perp|$. In particular*

$$|B_p^n \cap H_n| \leq |B_p^n \cap a^\prec| \leq |B_p^n \cap H_1|.$$

3.3 Case $p > 2$

3.3.1 Minimal sections of the cube

According to Lemma 13 we can always write $Y_i \sim R_i U_i$, where U_i are uniform on $[-1, 1]$ and $R_i \sim c_p x^p e^{-x^p}$. Thus

$$|B_p^n \cap a^\perp| = \lim_{q\to 1^-} c_{p,q,n} (1-q) \mathbb{E}\left|\sum_{i=1}^n a_i R_i U_i\right|^{-q}$$

Let us prove the following lemma.

Lemma 28 (Archimedes-König-Kwapień formula)**.** *Let $U_1, \ldots, U_n$ be i.i.d. uniform on $[-1, 1]$ and let $\xi_1, \ldots, \xi_n$ be i.i.d. uniform on the unit Euclidean ball S^2 in $\mathbb{R}^3$. Then*

$$(1-q)\mathbb{E}\Big|\sum_{i=1}^n x_i U_i\Big|^{-q} = \mathbb{E}\Big|\sum_{i=1}^n x_i \xi_i\Big|^{-q}.$$

Proof. Due to the Archimedes hat-box theorem for any unit vector $\theta \in \mathbb{R}^3$ we have $\langle \xi_i, \theta \rangle \sim U_i$. Moreover for $v \in \mathbb{R}^3$ and θ uniform on S^2 we have

$$\mathbb{E}_\theta | \langle v, \theta \rangle |^{-q} = |v|^{-q} \cdot \frac{1}{2} \int_{-1}^{1} |x|^{-q} \mathrm{d}x = \frac{1}{1-q} |v|^{-q}.$$

We get

$$\frac{1}{1-q} \mathbb{E} \Big| \sum_{i=1}^{n} x_i \xi_i \Big|^{-q} = \mathbb{E}_{\xi,\theta} \Big| \left\langle \sum_{i=1}^{n} x_i \xi_i, \theta \right\rangle \Big|^{-q} = \mathbb{E}_\theta \mathbb{E}_\xi \Big| \sum_{i=1}^{n} x_i \langle \xi_i, \theta \rangle \Big|^{-q} = \mathbb{E} \Big| \sum_{i=1}^{n} x_i U_i \Big|^{-q}.$$

$\square$

Lemma 28 allows us the evaluate the limit

$$|B_p^n \cap a^\perp| = \lim_{q \to 1^-} c_{p,q,n} (1-q) \mathbb{E} \left| \sum_{i=1}^{n} a_i R_i U_i \right|^{-q} = \lim_{q \to 1^-} c_{p,q,n} \mathbb{E} \left| \sum_{i=1}^{n} a_i R_i \xi_i \right|^{-q} = c_{p,1,n} \mathbb{E} \left| \sum_{i=1}^{n} a_i R_i \xi_i \right|^{-1}.$$

This gives

$$\frac{|B_p^n \cap a^\perp|}{|B_p^{n-1}|} = \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E} \left| \sum_{i=1}^{n} a_i R_i \xi_i \right|^{-1}.$$

Case $p = \infty$ is due to König and Koldobsky, [47], namely

$$\frac{|B_\infty^n \cap a^\perp|}{|B_\infty^{n-1}|} = \mathbb{E} \left| \sum_{i=1}^{n} a_i \xi_i \right|^{-1}$$

We show how this formula gives Hadwiger-Hensley theorem on minimal sections of the cube.

Theorem 29. *We have* $|B_\infty^n \cap a^\perp| \geq |B_\infty^{n-1}|$.

Proof. We have

$$\frac{|B_\infty^n \cap a^\perp|}{|B_\infty^{n-1}|} = \mathbb{E} \left| \sum_{i=1}^{n} a_i \xi_i \right|^{-1} = \mathbb{E} \left(\sum_{i,j=1}^{n} a_i a_j \langle \xi_i, \xi_j \rangle \right)^{-\frac{1}{2}} \geq \left(\sum_{i,j=1}^{n} a_i a_j \mathbb{E} \langle \xi_i, \xi_j \rangle \right)^{-\frac{1}{2}} = \left(\sum_{i=1}^{n} a_i^2 \right)^{-\frac{1}{2}} = 1.$$

$\square$

3.3.2 Kanter's lemma

We say that a measure μ is *more peaked* than a measure ν (to be denoted by $\mu \preccurlyeq \nu$) if for every symmetric convex set K one has $\mu(K) \leq \nu(K)$.

We say that f is log-concave if $f = e^{-V}$ for some convex $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We need the following simple lemma.

Lemma 30. *Suppose* $\mu \preccurlyeq \nu$. *Then for every even log concave function* f *one has* $\int f \mathrm{d}\mu \leq \int f \mathrm{d}\nu$.

Proof. We have $\int f \mathrm{d}\mu = \int_0^\infty \mu(\{x : f(x) \geq t\}) \mathrm{d}t$ and the sets $\{x : f(x) \geq t\}$ are symmetric and convex. $\square$

We now formulate Kanter's lemma.

Lemma 31. *Suppose* μ, ν *are symmetric log-concave measures on* $\mathbb{R}^n$ *and* $\mu \preccurlyeq \nu$. *Then for every symmetric log-concave measure* λ *on* $\mathbb{R}^m$ *we have* $\mu \otimes \lambda \preccurlyeq \nu \otimes \lambda$. *Moreover if* $\mu_i \preccurlyeq \nu_i$ *then* $\mu_1 \otimes \ldots \otimes \mu_k \preccurlyeq \nu_1 \otimes \ldots \otimes \nu_k$.

Proof. It suffices to show the first part and for the second part apply induction. Define $K_x = \{y \in \mathbb{R}^m : (x, y) \in K)\}$. We have

$$(\mu \otimes \lambda)(K) = \int \lambda(K_x) \mathrm{d}\mu(x).$$

Since λ is symmetric and $K_{-x} = -K_x$ we get that $f(x) = \lambda(K_x)$ is even. We prove that it is also log-concave. Indeed, by convexity of K we have $K_{px_1+(1-p)x_2} \supset pK_{x_1} + (1-p)K_{x_2}$. Thus by log-concavity of λ we get

$$f(px_1+(1-p)x_2) = \lambda(K_{px_1+(1-p)x_2}) \geq \lambda(pK_{x_1}+(1-p)K_{x_2}) \geq \lambda(K_{x_1})^p \lambda(K_{x_2})^{1-p} = f(x_1)^p f(x_2)^{1-p},$$

which shows that f is log-concave. Thus by the previous lemma we get

$$(\mu \otimes \lambda)(K) = \int f(x) \mathrm{d}\mu(x) \leq \int f(x) \mathrm{d}\nu(x) = (\nu \otimes \lambda)(K).$$

□

Lemma 32. *Suppose symmetric measure on the real line μ, ν have densities f, g satisfying $f(0) = g(0)$. Then if $f - g$ changes sign once from "-" to "+" on $(0, \infty)$, then $\mu \preccurlyeq \nu$.*

Proof. Since μ, ν are symmetric it suffices to show that $\int_0^t g \geq \int_0^t f$. Suppose $f - g$ changes its sign in y_0. In fact we shall show that for every decreasing ψ on $[0, \infty)$ one has $\int \psi f \leq \int \psi g$. Indeed, we have

$$\int \psi(y)(g(y) - f(y)) \mathrm{d}y = \int (\psi(y) - \psi(y_0))(g(y) - f(y)) \mathrm{d}y \geq 0,$$

as both factors change their signs at y_0. □

3.3.3 Minimal sections

Theorem 33. *The function $p \mapsto \frac{|B_p^n \cap a^\perp|}{|B_p^{n-1}|}$ is non-decreasing in p on $(1, \infty)$. In particular*

$$|B_p^n \cap a^\perp| \geq |B_p^{n-1}|, \quad p \geq 2, \qquad \text{and} \qquad |B_p^n \cap a^\perp| \leq |B_p^{n-1}|, \quad p \leq 2.$$

Proof. Let μ_p be the measure with density $v_p(t) = e^{-\beta_p^p |t|^p}$ and let $V_i^{(p)}$ be distributed according to μ_p. We have

$$\frac{|B_p^n \cap a^\perp|}{|B_p^{n-1}|} = \lim_{q \to 1^-} \frac{1-q}{2} \mathbb{E} \left| \sum_{i=1}^n a_i V_i^{(p)} \right|^{-q}.$$

In order to check that the normalization is correct we plug in $a = (1, 0, \ldots, 0)$ and use the fact that

$$\lim_{q \to 1^-} \frac{1-q}{2} \mathbb{E} \left| V_1^{(p)} \right|^{-q} = v_p(0) = 1.$$

Note that

$$\mathbb{E}|X|^{-q} = q\, \mathbb{E} \int_{|X|}^\infty t^{-q-1} \mathrm{d}t = q\, \mathbb{E} \int_0^\infty t^{-q-1} \mathbf{1}_{|X| \leq t} \mathrm{d}t = q \int_0^\infty t^{-q-1} \mathbb{P}\left(|X| \leq t\right) \mathrm{d}t.$$

It is therefore enough to show that

$$p \mapsto \mathbb{P}\left(\left| \sum_{i=1}^n a_i V_i^{(p)} \right| \leq t \right) = \mu_p^{\otimes n}(\{x \in \mathbb{R}^n : |\langle x, a \rangle| \leq t\})$$

is non-decreasing. The set $K_{a,t} = \{x \in \mathbb{R}^n : |\langle x, a \rangle| \leq t\}$ is symmetric and convex. Moreover, for $p_1 \leq p_2$ we have $\mu_{p_1} \preccurlyeq \mu_{p_1}$ since v_{p_1} and v_{p_2} intersect in only one point on $\mathbb{R}_+$. By Kanter's lemma we have $\mu_{p_1}^{\otimes n} \preccurlyeq \mu_{p_2}^{\otimes n}$ and the assertion follows. □

3.4 Maximal section - Ball's theorem

The goal of this section is to prove the following theorem.

Theorem 34. *For any unit vector one has* $|B_\infty^n \cap a^\perp| \leq 2^{n-1}\sqrt{2}$.

We mention that this celebrated fact provides a negative answer to the Busemann-Petty question in high dimensions: for $n \geq 10$ one has

$$\left|\left[-\frac{1}{2}, \frac{1}{2}\right]^n \cap a^\perp\right| \leq \sqrt{2} < \left|r_n B_2^n \cap a^\perp\right|,$$

where r_n is chosen in such a way that $|r_n B_2^n| = 1$. Thus, even though all central sections of the cube are strictly smaller than those of $r_n B_2^n$, the volumes are the same. Such strange examples exist only in dimensions $n \geq 5$.

We now proceed with the proof of Ball's inequality. We shall consider the unit volume cube $C = C_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Without loss of generality we can assume that $a_i > 0$ for all i. Suppose there exists j such that $a_j \geq 1/\sqrt{2}$. In this case consider the section $S = C \cap a^\perp$ and project it onto $e_j^\perp$. The volume of this projection is $|S| \cdot |\langle a, e_j\rangle = |S| a_j$. On the other hand this projection is contained in C_{n-1} and thus has volume at most 1. W get $|S| a_j \leq 1$ and thus $|S| \leq a_j^{-1} \leq \sqrt{2}$ and we are done.

Now, it suffices to assume that $0 < a_j < 1/\sqrt{2}$ for all j. Let X_j be i.i.d. uniform on $[-\frac{1}{2}, \frac{1}{2}]$ and let f_a be the density of $\sum_{j=1}^n a_j X_j$. The idea is to use the Fourier transform. We have

$$\phi_a(t) := \mathbb{E} e^{it\sum_{j=1}^n a_j X_j} = \prod_{j=1}^n \mathbb{E} e^{ita_j X_j} = \prod_{j=1}^n \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{ita_j u} \mathrm{d}u = \prod_{j=1}^n \frac{e^{\frac{1}{2}ia_j t} - e^{-\frac{1}{2}ia_j t}}{ia_j t} = \prod_{j=1}^n \frac{\sin(\frac{1}{2}a_j t)}{\frac{1}{2}a_j t}.$$

We are going to use the Fourier inversion formula (valid when ϕ_a is integrable)

$$f_a(x) = \frac{1}{2\pi} \int e^{-itx} \phi_a(t) \mathrm{d}t.$$

Using this formula with $x = 0$ and changing variables $\frac{1}{2}t = \pi u$ we arrive at

$$|C \cap a^\perp| = f_a(0) = \int \prod_{j=1}^n \frac{\sin(\pi a_j u)}{\pi a_j u} \mathrm{d}u.$$

Applying Hölder inequality (recall that $\sum_{j=1}^n a_j^2 = 1$) we get

$$|C \cap a^\perp| \leq \int \prod_{j=1}^n \left|\frac{\sin(\pi a_j u)}{\pi a_j u}\right| \mathrm{d}u \leq \prod_{j=1}^n \left(\int \left|\frac{\sin(\pi a_j u)}{\pi a_j u}\right|^{\frac{1}{a_j^2}} \mathrm{d}u\right)^{a_j^2}.$$

Therefore, it suffices to prove that

$$\int \left|\frac{\sin(\pi a_j u)}{\pi a_j u}\right|^{\frac{1}{a_j^2}} \mathrm{d}u \leq \sqrt{2}.$$

Let us substitute $x = a_j u$ and introduce $s = a_j^{-2} > 2$. Then we get the equivalent form of the inequality

$$\int \left|\frac{\sin \pi x}{\pi x}\right|^s \mathrm{d}x < \sqrt{\frac{2}{s}}, \qquad s > 2.$$

This is the famous Ball's integral inequality.

3.4.1 Nazarov-Podkorytov lemma

Suppose $f : \mathbb{R} \to [0, \infty)$. The function

$$F(t) = |\{x \in \mathbb{R} : \ f(x) > t\}|$$

is called the distribution function of f. Let $\mathcal{F}_l$ be the space of functions $f : \mathbb{R} \to [0, \infty)$ such that their distribution functions are finite and f^s is integrable for all $s > l$.

Lemma 35. *Suppose $f, g \in \mathcal{F}_l$ have distribution functions F, G such that $F - G$ changes sign from " $-$ " to " $+$ " at y_0. Then*

$$\phi(s) = \frac{1}{s y_0^s} \int (f^s - g^s)$$

is increasing on (l, ∞). In particular,

$$\int f^{s_0} = \int g^{s_0} \qquad \Longrightarrow \qquad \int f^s \geq \int g^s \qquad \text{for all} \ \ s \geq s_0.$$

Proof. We have

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{\mathbb{R}} \int_0^{f(x)} 1\mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}} \int_0^{\infty} \mathbf{1}_{\{t<f(x)\}}\mathrm{d}x\mathrm{d}t = \int_0^{\infty} F(t)\mathrm{d}t.$$

Note that the distribution function of f^s is $F(y^{1/s})$. Thus

$$\int f^s = \int_0^{\infty} F(y^{1/s})\mathrm{d}y = s \int_0^{\infty} u^{s-1} F(u)\mathrm{d}u.$$

We get that

$$\phi(s) = \frac{1}{y_0} \int_0^{\infty} \left(\frac{y}{y_0}\right)^{s-1} (F(y) - G(y))\mathrm{d}y$$

Suppose $s_1 > s_2$. Then

$$\phi(s_1) - \phi(s_2) = \frac{1}{y_0} \int_0^{\infty} \left(\left(\frac{y}{y_0}\right)^{s_1-1} - \left(\frac{y}{y_0}\right)^{s_2-1} \right) (F(y) - G(y))\mathrm{d}y \geq 0,$$

since both factors change their signs in y_0. □

3.4.2 Proof of Ball's integral inequality

Let us define

$$f(x) = e^{-\pi x^2/2}, \qquad g(x) = \left|\frac{\sin \pi x}{\pi x}\right|$$

We note that $\int f^2 = \int g^2 = 1$ and we want to prove the inequality $\int g^s < \int f^s$. We are going to use Nazarov-Podkorytov lemma with $s_0 = 2$. It is enough to check that $F - G$ changes sign from " $-$ " to " $+$ " on $[0, \infty)$. Note that $F(y) = G(y) = 0$ for $y \geq 1$, so we only consider $y \in (0, 1)$. We have $F(y) = \sqrt{\frac{2}{\pi} \ln(\frac{1}{y})}$. The main problem is to estimate $G(y)$.

The function $g(x)$ has zeros for $x \in \mathbb{Z}$. Let $y_m = \max_{[m,m+1]} g$. We clearly have $y_m < \frac{1}{\pi m}$ and $y_m > g(m + \frac{1}{2}) = \frac{1}{\pi(m+\frac{1}{2})}$. Thus $y_m \in (\frac{1}{\pi(m+\frac{1}{2})}, \frac{1}{\pi m})$, which shows that the sequence y_m is decreasing. We have the following claims.

Claim 1. The function $F - G$ changes sign at least once in $(0, 1)$.

To show this we just observe that $\int_0^{\infty} 2y(F(y) - G(y))\mathrm{d}y = \int (f^2 - g^2) = 0$.

Claim 2. The function $F - G$ is positive on $(y_1, 1)$.

Note that if $g(x) > y_1$ then $y \in (0,1)$. Moreover $g(x) \leq f(x)$ for $x \in [0,1]$, since

$$\frac{\sin \pi x}{\pi x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \leq \prod_{k=1}^{\infty} e^{-\frac{x^2}{k^2}} = e^{-\frac{\pi^2}{6}x^2} \leq e^{-\frac{\pi}{2}x^2} = f(x).$$

Thus

$$G(y) = |\{x \in (0,1) : \ g(x) > y\}| < |\{x \in (0,1) : \ f(x) > y\}| \leq F(y).$$

Claim 3. The function $F - G$ is increasing on $(0, y_1)$.

It is enough to show that $|G'(y)| > |F'(y)|$ for $y \in (y_{m+1}, y_m)$. For $y \in (0, y_1)$ such that $y \neq y_j$ we have

$$|G'(y)| = \sum_{x>0: \ g(x)=y} \frac{1}{|g'(x)|}.$$

If $y \in (y_{m+1}, y_m)$ then the equation $g(x) = y$ has:

- one root on $(0,1)$
- two roots on $(k, k+1)$, $k = 1, \ldots, m$
- no roots on $(m+1, \infty)$.

For $x \in (0,1)$ we have

$$|g'(x)| = \frac{\sin(\pi x) - \pi x \cos(\pi x)}{\pi x^2} = \frac{1}{\pi x^2} \int_0^{\pi x} t \sin t \mathrm{d}t \leq \frac{1}{\pi x^2} \int_0^{\pi x} t \mathrm{d}t = \frac{\pi}{2}.$$

For $x \in (k, k+1)$, $k \geq 1$ we have

$$|g'(x)| = \left| \frac{\cos(\pi x)}{x} - \frac{\sin(\pi x)}{\pi x^2} \right| \leq \frac{1}{x} \left(1 + \frac{|\sin(\pi(x-k))|}{\pi x}\right) \leq \frac{1}{x} \left(1 + \frac{\pi(x-k)}{\pi k}\right) = \frac{1}{k}.$$

Putting this together we get that for $y \in (y_{m+1}, y_m)$ we have

$$|G'(y)| \geq \frac{2}{\pi} + 2\sum_{k=1}^{m} k = \frac{2}{\pi} + m + m^2.$$

Since $|F'(y)| = \frac{1}{y\sqrt{2\pi \ln(\frac{1}{y})}}$ we get

$$\frac{|G'(y)|}{|F'(y)|} = |G'(y)| y \sqrt{2\pi \ln\left(\frac{1}{y}\right)} \geq \left(\frac{2}{\pi} + m + m^2\right) y \sqrt{2\pi \ln\left(\frac{1}{y}\right)}$$

We now claim that $y\sqrt{2\pi \ln\left(\frac{1}{y}\right)}$ is increasing on $(0, y_1)$. Note that $y_1 < \frac{1}{\pi} < e^{-1/2}$. For $0 < y < e^{-/12}$ we have

$$\left(y^2 \ln\left(\frac{1}{y}\right)\right)' = 2y \ln\left(\frac{1}{y}\right) - y = y\left(2\ln\left(\frac{1}{y}\right) - 1\right) > 0.$$

For $y > y_{m+1} > \frac{1}{\pi(m+\frac{3}{2})}$ we therefore get

$$\frac{|G'(y)|}{|F'(y)|} \geq \frac{\frac{2}{\pi} + m + m^2}{m + \frac{3}{2}} \cdot \sqrt{\frac{2}{\pi}\pi \ln\left(\pi\left(m + \frac{3}{2}\right)\right)} \geq \sqrt{\frac{2}{\pi} \ln \frac{5\pi}{2}} \geq 1.$$

as $\frac{2}{\pi} + m + m^2 \geq \frac{1}{2} + m + 1 = \frac{3}{2} + m$. The last inequality follows from $\ln 5x \geq x$ for $x \in [1,2]$ (applied to $x = \pi/2$), which can be checked only at the endpoint $x = 1$ ($\ln 5 > 1$) and $x = 2$ ($\ln 10 > 2$) as the left hand side is concave.

Bibliographical notes

Sections of ℓ_p^n balls. The topic of sections of ℓ_p^n balls is widely studied and this development relied on quite a number of interesting and influential ideas. The first result concerning this question has been obtained independently by Hadwiger in [34]) and Hensley in [36], who proved that $|B_\infty^n \cap a^\perp| \geq |B_\infty^n \cap H_1|$, and was motivated by Good's question about solvability of systems of inequalities $|L_i(x)| \leq 1$, where $L_i : \mathbb{Z}^n \to \mathbb{R}$ are linear forms in integer-valued variables. Ball proved in his celebrated paper [2] that H_2 gives the maximal section, providing a simple counterexample to the Busemann-Petty conjecture from [14]. Ball's proof is an application of the Fourier inversion formula together with a brilliant use of Hölder's inequality. In [27] distributional stability of this result for vectors a with absolute values of coordinates at most $1/\sqrt{2}$ is proved.

The case of finite p has been treated for the first time by Meyer and Pajor in [59], who showed that for $p \in [1,2]$ the maximal section is given by H_1, whereas for $p \in [2,\infty)$ this subspace gives the minimal section. The proof relies on the monotonicity in p of the function $p \mapsto |B_p^n \cap H| \cdot |B_p^k|^{-1}$ for any k-dimensional subspace, where the notion of *peakedness* and Kanter's lemma (see [39]) are used. For $p \in [1,2]$ Koldobsky showed in [41], using a Fourier-analytic argument, that the minimal section is given by the diagonal subspace H_n. See also [24] for a simple proofs in the case $p \in [1,2]$ using the Gaussian mixture technique. Recently Eskenazis, Tkocz and the author proved in [26] that H_2 is maximal not only for $p = \infty$, but also for every $p > 10^{15}$. Stability estimates for codimension one sections of B_p^n were established in [15].

The first result concerning lower dimensional sections is due to Vaaler [85], who proved that for any k-dimensional subspace one has $|B_\infty^n \cap H| \geq 2^k$ with equality for subspaces spanned by some k standard basis vectors, see also [1] for an alternative topological proof. As for the upper bound, Ball proved in [3] that for any k-dimensional subspace H one has $|B_\infty^n \cap H| \leq \min(\sqrt{n/k}^k, \sqrt{2}^{n-k})$, which is sharp when k divides n and when $k \geq \frac{n}{2}$. For $k = 2$ maximal sections has recently been found in [37]. Meyer and Pajor in [59] proved that for any k-dimensional subspace H one has $|B_p^n \cap H| \leq |B_p^k|$ for $p \in [1,2]$ and $|B_p^n \cap H| \geq |B_p^k|$ for $p > 2$. For $p > 2$ Barthe proved in [7] the inequality $|B_p^n \cap H| \leq (n/k)^{k(\frac{1}{2}-\frac{1}{p})}|B_p^k|$, which is sharp only when k divides n. For $2 \leq k < n-1$ and $p \in [1,2)$ minimal sections of B_p^n are unknown except for $(k,p) = (2,1)$ in which case Nazarov proved the sharp bound $|B_1^n \cap H| \geq n^2 \operatorname{tg}(\frac{\pi}{2n}) \sin^2(\frac{\pi}{2n})$, see [15].

In the articles of Vaaler [85] and Meyer and Pajor [59] the authors point out that the analogues of their results hold in the complex case, namely for any complex k-dimensional subspace H one has $|B_{p,\mathbb{C}}^n \cap H| \geq |B_{p,\mathbb{C}}^k|$ for $2 \leq p \leq \infty$, whereas reverse inequality holds for $p \in [1,2]$. Minimal complex codimension one sections for $p \in [1,2]$ are given by complex H_n, see the result of Koldobsky and Zymonopoulou [43]. Finally, as we already mentioned, an analogue of Ball's inequality in the complex case is due to Oleszkiewicz and Pełczyński who proved in [70] the inequality $|\mathbb{D}^n \cap a^\perp| \leq |\mathbb{D}^n \cap H_2|$.

Projections of ℓ_p^n balls. The problem of finding $C_{2,1}$ was posed by Littlewood in 1930 in [53], where his famous $\frac{4}{3}$-inequality for bilinear forms was derived. It was Szarek who solved Littlewood's problem in [80], proving that $c_n = 1/\sqrt{2}$. Szarek's result was rephrased in terms of projections of B_1^n by Ball in [4], namely one has $|\operatorname{Proj}_{a^\perp}(B_1^n)| \geq |\operatorname{Proj}_{H_2}(B_1^n)|$. Szarek's inequality has now several simplified proofs, see [33, 51, 81].

Barthe and Naor proved in [9], using the so-called convex ordering of densities, that H_1 gives the maximal projections for $p \in (1,2)$ and the minimal projections for $p > 2$. They also showed that H_n gives the maximal projection for $p > 2$. In the problematic case of minimal projections for $p \in (1,2)$ only the case $p < 1 + 10^{-12}$ is known due to the recent work [26]. In [27] a distributional stability of Szarek's inequality was given in the case when $|a_i| \leq 1/\sqrt{2}$ for all $i = 1, \ldots, n$, which leads to an alternative proof for p close to 1, however with an additional restriction on the sequence (a_i).

Lower dimensional projections of B_p^n are much less understood. The result of Meyer and Pajor from [59] about sections trivially gives $|\operatorname{Proj}_H(B_p^n)| \geq |B_p^n \cap H| \geq |B_p^k|$ for every k-dimensional subspace H and $p > 2$. Barthe in [8] proved for $p \in [1,2]$ the inequality $|\operatorname{Proj}_H(B_p^n)| \geq (\frac{k}{n})^{k(\frac{1}{p}-\frac{1}{2})}|B_p^k|$, which is sharp when k divides n. For the complex case, besides the fact that minimizers for $p \geq 2$ are given by H_1, nothing seems to be known.

For more information about sections and projections of ℓ_p^n balls we refer the reader to the recent survey [67] by Tkocz and the author.

Appendix A

Prékopa-Leindler inequality

We are going to prove the following fundamental theorem.

Theorem 36. *Let f, g, m be nonnegative measerable functions on $\mathbb{R}^n$ and let $\lambda \in [0,1]$. If for all $x, y \in \mathbb{R}^n$ we have*

$$m((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda,$$

then

$$\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^\lambda. \tag{13}$$

We first prove the one-dimensional Brunn-Minkowski inequality.

Lemma 37. *Suppose A, B are non-empty Borel sets. Then $|A+B| \geq |A| + |B|$.*

Sketch of the proof. By simple approximation argument one can assume that A, B are compact. Shifting A and B does not affect the inequality, so one can assume that $\inf A = 0 = \sup B$. Then $A + B \supset A \cup B$ and $A \cap B = \{0\}$. Thus $|A+B| \geq |A| + |B|$. $\square$

Remark 6. The sum of measurable sets is not always measurable. The sum of two Borel sets might not be Borel, but it is always Lebesgue measurable.

We first give two proof of this fact in dimension $n = 1$.

Proof. Let us first justify the Prekopa-Leindler inequality in dimension one. We can assume, considering $f\mathbf{1}_{f \leq M}$ and $g\mathbf{1}_{g \leq M}$ instead of f and g, that f, g are bounded. If we multiply f, g, m by numbers c_f, c_g, c_m satisfying

$$c_m = c_f^{1-\lambda} c_g^\lambda,$$

then the hyphotesis and the thesis do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$ and $c_m = \|f\|_\infty^{-(1-\lambda)} \|g\|_\infty^{-\lambda}$ we can assume (since we are in the situation when f and g are bounded) that $\|f\|_\infty = \|g\|_\infty = 1$. But then

$$\int_{\mathbb{R}} m = \int_0^{+\infty} |\{m \geq s\}| \mathrm{d}s, \qquad \int_{\mathbb{R}} f = \int_0^1 |\{f \geq r\}| \mathrm{d}r, \qquad \int_{\mathbb{R}} g = \int_0^1 |\{g \geq r\}| \mathrm{d}r$$

Note also that if $x \in \{f \geq r\}$ and $y \in \{g \geq r\}$ then by the assumption of the theorem we have $(1-\lambda)x + \lambda y \in \{m \geq r\}$. Hence,

$$(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\} \subset \{m \geq r\}.$$

Moreover, the sets $\{f \geq r\}$ and $\{g \geq r\}$ are non-empty for $r \in [0,1)$. This is very important since we want to use one-dimensional Brunn-Minkowski inequality. We have

$$\begin{aligned}
\int m &= \int_0^{+\infty} |\{m \geq r\}| \mathrm{d}r \geq \int_0^1 |\{m \geq r\}| \mathrm{d}r \geq \int_0^1 |(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\}| \mathrm{d}r \\
&\geq (1-\lambda) \int_0^1 |\{f \geq r\}| \mathrm{d}r + \lambda \int_0^1 |\{g \geq r\}| \mathrm{d}r = (1-\lambda) \int f + \lambda \int g \geq \left(\int f \right)^{1-\lambda} \left(\int g \right)^\lambda.
\end{aligned}$$

Suppose our inequality in true in dimension $n-1$. We will prove it in dimension n. Suppose we have a numbers $y_0, y_1, y_2 \in \mathbb{R}$ satisfying $y_0 = (1-\lambda)y_1 + \lambda y_2$. Define $m_{y_0}, f_{y_1}, g_{y_2} : \mathbb{R}^{n-1} \to \mathbb{R}_+$ by

$$m_{y_0}(x) = m(y_0, x), \quad f_{y_1}(x) = f(y_1, x), \quad g_{y_2}(x) = (y_2, x),$$

where $x \in \mathbb{R}^{n-1}$. Note that since $y_0 = (1-\lambda)y_1 + \lambda y_2$ we have

$$\begin{aligned} m_{y_0}((1-\lambda)x_1 + \lambda x_2) &= m((1-\lambda)y_1 + \lambda y_2, (1-\lambda)x_1 + \lambda x_2) \\ &\geq f(y_1, x_1)^{1-\lambda} g(y_2, x_2)^{\lambda} = f_{y_1}(x_1)^{1-\lambda} g_{y_2}(x_2)^{\lambda}, \end{aligned}$$

hence m_{y_0}, f_{y_1} and g_{y_2} satisfies the assumption of the $(n-1)$-dimensional Prékopa-Leindler inequality. Therefore we have

$$\int_{\mathbb{R}^{n-1}} m_{y_0} \geq \left(\int_{\mathbb{R}^{n-1}} f_{y_1} \right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_{y_2} \right)^{\lambda}.$$

Define new functions $M, F, G : \mathbb{R} \to \mathbb{R}_+$

$$M(y_0) = \int_{\mathbb{R}^{n-1}} m_{y_0}, \quad F(y_1) = \int_{\mathbb{R}^{n-1}} f_{y_1}, \quad G(y_2) = \int_{\mathbb{R}^{n-1}} g_{y_2}.$$

We have seen (the above inequality) that when $y_0 = (1-\lambda)y_1 + \lambda y_2$ then there holds

$$M((1-\lambda)y_1 + \lambda y_2) \geq F(y_1)^{1-\lambda} G(y_2)^{\lambda}.$$

Hence, by one-dimensional Prékopa-Leindler inequality we get

$$\int_{\mathbb{R}} M \geq \left(\int_{\mathbb{R}} F \right)^{1-\lambda} \left(\int_{\mathbb{R}} G \right)^{\lambda}.$$

But

$$\int_{\mathbb{R}} M = \int_{\mathbb{R}^n} m, \quad \int_{\mathbb{R}} F = \int_{\mathbb{R}^n} f, \quad \int_{\mathbb{R}} G = \int_{\mathbb{R}^n} g,$$

so the assertion follows. □

Brunn-Minkowski inequality

Taking $f = \mathbf{1}_A$, $g = \mathbf{1}_B$ and $m = \mathbf{1}_{\lambda A + (1-\lambda)B}$ we get the multiplicative form of the Brunn-Minkowski inequality

$$|\lambda A + (1-\lambda)B| \geq |A|^{\lambda} |B|^{1-\lambda}.$$

If we apply this inequality with $\tilde{K} = K/|K|^{1/n}$, $\tilde{L} = L/|L|^{1/n}$ and $\tilde{\lambda} = \frac{\lambda |K|^{1/n}}{\lambda |K|^{1/n} + (1-\lambda)|L|^{1/n}}$. we get the classical form of the Brunn-Minkowski inequality.

Theorem 38. *If A, B are Borel non-empty sets, then for $\lambda \in [0,1]$ we have*

$$|\lambda A + (1-\lambda)B|^{1/n} \geq \lambda |A|^{1/n} + (1-\lambda)|B|^{1/n}.$$

Remark 7. Note that the above can also be written in the form $|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$.

Brunn's principle

We shall prove the following theorem.

Theorem 39. *Suppose K is a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. Then the function*

$$t \mapsto \mathrm{vol}_{n-1}(K \cap (u^{\perp} + tu))^{1/n-1}$$

is concave on its support.

Proof. We can assume that $u = e_1$. Let $K_t = K \cap (u^{\perp} + tu) = K \cap \{x_1 = t\}$ and consider these as sets in $\mathbb{R}^{n-1}$. We claim that $\lambda K_t + (1-\lambda)K_s \subseteq K_{\lambda t + (1-\lambda)s}$. Indeed, suppose $a \in K_t$ and $b \in K_s$. Then by convexity of K we have $\lambda(t,a) + (1-\lambda)(s,b) = (\lambda t + (1-\lambda)s, \lambda a + (1-\lambda)b) \in K$ and thus $\lambda a + (1-\lambda)b \in K_{\lambda t + (1-\lambda)s}$. Suppose K_s, K_t are non-empty (i.e. we are on the support of our map). By Brunn-Minkowski we get

$$|K_{\lambda t + (1-\lambda)s}|^{\frac{1}{n-1}} \geq |\lambda K_t + (1-\lambda)K_s|^{\frac{1}{n-1}} \geq \lambda |K_t|^{\frac{1}{n-1}} + (1-\lambda)|K_s|^{\frac{1}{n-1}},$$

which proves the desired concavity. □

Corollary 40. *If K is a symmetric convex set the the section having the largest section is always a central section, that is a section passing through the origin.*

Proof. The above section function is even and $\frac{1}{n-1}$-concave, so its maximum is at the origin. □

Isoperimetric inequality

For a compact sets K in $\mathbb{R}^n$ we define $K_t = K + tB_2^n$.

Theorem 41. *Let K be a compact set in $\mathbb{R}^n$ and let B be a ball such that $|K| = |B|$. Then*

(a) $|K_t| \geq |B_t| = \left(\left(\frac{|K|}{|B_2^n|} \right)^{1/n} + t \right)^n |B_2^n|$,

(b) $|\partial K| \geq |\partial B| = n|K|^{\frac{n-1}{n}} |B_2^n|^{\frac{1}{n}}$.

Proof. Suppose $B = rB_2^n$. By the Brunn-Minkowski inequality we have

$$\begin{aligned} |K_t| = |K + tB_2^n| &\geq \left(|K|^{\frac{1}{n}} + t|B_2^n|^{\frac{1}{n}} \right)^n = \left(|B|^{\frac{1}{n}} + t|B_2^n|^{\frac{1}{n}} \right)^n \\ &= (r+t)^n |B_2^n| = |(r+t)B_2^n| = |B + tB_2^n| = |B_t|. \end{aligned}$$

To prove the second part we recall that

$$|\partial K| = \liminf_{\varepsilon \to 0^+} \frac{|K + \varepsilon B_2^n| - |K|}{\varepsilon}.$$

Thus from point (a) we get

$$|\partial K| = \liminf_{\varepsilon \to 0^+} \frac{|K_\varepsilon| - |K|}{\varepsilon} \geq \liminf_{\varepsilon \to 0^+} \frac{|B_\varepsilon| - |B|}{\varepsilon} = |\partial B|.$$

□

Log-concave measures and functions

We say that a measure μ on $\mathbb{R}^n$ is log-concave if $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$ for all Borel sets in $\mathbb{R}^n$. We shall need the description of such measures due to Borel: μ is log-concave if and only if either μ is a Dirac delta, or there exists an affine subspace H of certain dimension $1 \leq d \leq n$ and a convex function $V : H \to \mathbb{R} \cup \{+\infty\}$ such that μ has density e^{-V} on H.

We now show that a measure with log-concave density is log-concave.

Theorem 42. *Suppose μ is a measure with log-concave density. Then*

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Proof. Let A, B be measurable in $\mathbb{R}^n$ and let h be the density of μ. Define $f = \mathbf{1}_A h$, $g = \mathbf{1}_B h$ and $m = \mathbf{1}_{\lambda A + (1-\lambda)B} h$. Then these function clearly satisfy $m(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^\lambda$. Thus

$$|\lambda A + (1-\lambda)B| = \int m \geq \left(\int f \right)^\lambda \left(\int g \right)^{1-\lambda} = |A|^\lambda |B|^{1-\lambda}.$$

□

Fact 1. *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave. Then $F(x) = \int_{\mathbb{R}^m} f(x,y)\mathrm{d}y$ is also log-concave.*

Proof. Define $f_x(y) = f(x,y)$, $f_x : \mathbb{R}^m \to \mathbb{R}$. Take $x_1, x_2 \in \mathbb{R}^n$. The functions $f_{\lambda x_1 + (1-\lambda)x_2}, f_{x_1}, f_{x_2}$ satisfy

$$f_{\lambda x_1 + (1-\lambda)x_2}(\lambda y_1 + (1-\lambda)y_2) \geq f_{x_1}(y_1)^\lambda f_{x_2}(y_2)^{1-\lambda}.$$

Thus by Prékopa-Leindler

$$F(\lambda x_1 + (1-\lambda)x_2) = \int f_{\lambda x_1 + (1-\lambda)x_2} \geq \left(\int f_{x_1} \right)^\lambda \left(\int f_{x_2} \right)^{1-\lambda} = F(x_1)^\lambda F(x_2)^{1-\lambda}.$$

□

Fact 2. *Let f, g be log-concave on $\mathbb{R}^n$. Then $f * g$ is also log-concave.*

Proof. The function $(x, y) \to f(y)g(x - y)$ is clearly log concave. Thus it suffices to integrate it in y and use Fact 1. $\square$

Fact 3. *Let f be log-concave on $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a fixed vector.*

$$\mathbb{R} \ni t \longmapsto \int_{\langle x,v\rangle \geq t} f(x)\mathrm{d}x$$

is also log-concave.

Proof. The function $(x, t) \mapsto f(x)\mathbf{1}_{\langle x,v\rangle \geq t}$ is log-concave (the function $(x, t) \mapsto \mathbf{1}_{\langle x,v\rangle \geq t}$ is log-concave as it is of the form $\mathbf{1}_K$ for a convex K with K being a half-space). It suffices to use Fact 1. $\square$

Appendix B

We are going to prove equivalences of conditions (a)-(d) from Proposition 4. We first show that (b) implies (a).

Lemma 43. *If P is doubly stochastic, then $Py \prec y$ for all $y \in \mathbb{R}^n$ in the sense of definition (a).*

Proof. Let $x = Py$ and let $\mathbf{1} = (1, \ldots, 1)$. We can assume that $x_1 \geq \ldots \geq x_n$ and $y_1 \geq \ldots \geq y_n$ since otherwise just take matrices of permutations Q and R such that $x = Qx'$ and $y = Ry'$, where x', y' have non-increasing sequences of coordinates, and observe that $x \prec y$ if and only if $x' \prec y'$ and $x' = Q^{-1}x = Q^{-1}Py = Q^{-1}PRy'$. Here $Q^{-1}PR$ is again doubly stochastic as a product of three doubly stochastic. To see that a product of two doubly stochastic matrices is doubly stochastic we observe that P is doubly stochastic if and only if $P\mathbf{1} = \mathbf{1}$ and $P^T\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$. Thus if P_1, P_2 are doubly stochastic then $P_2P_1\mathbf{1} = P_2\mathbf{1} = \mathbf{1}$ and $(P_2P_1)^T\mathbf{1} = P_1^TP_2^T\mathbf{1} = P_1^T\mathbf{1} = \mathbf{1}$.

Now, clearly $\mathbf{1}^Tx = \mathbf{1}^TPy = (P^T\mathbf{1})^Ty = \mathbf{1}^Ty$, which means that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Let $t_j = \sum_{i=1}^k p_{ij}$. Note that $t_j \in [0, 1]$ and $\sum_{j=1}^n t_j = k$. We have

$$\sum_{i=1}^k x_i = \sum_{i=1}^k \sum_{j=1}^n p_{ij}y_j = \sum_{j=1}^n \sum_{i=1}^k p_{ij}y_j = \sum_{j=1}^n t_jy_j.$$

Thus

$$\begin{aligned}\sum_{i=1}^k x_i - \sum_{i=1}^k y_i &= \sum_{j=1}^n t_jy_j - \sum_{i=1}^k y_i = \sum_{j=1}^n t_jy_j - \sum_{i=1}^k y_i + y_k\left(k - \sum_{j=1}^n t_j\right) \\ &= \sum_{j=1}^k (y_j - y_k)(t_j - 1) + \sum_{j=k+1}^n t_j(y_j - y_k) \leq 0.\end{aligned}$$

$\square$

We now show that (a) implies (d) which clearly implies (b). Thus we get equivalence of (a), (b) and (d).

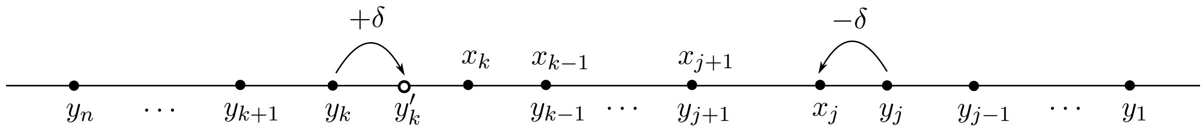
Lemma 44. *If $x \prec y$ in the sense of (a), then x can be obtained from y by applying finitely many T-transformations. In particular, there exists a doubly stochastic matrix P such that $x = Py$.*

Proof. We can assume that $x_1 \geq \ldots \geq x_n$ and $y_1 \geq \ldots \geq y_n$ since permutation matrices are compositions of finite number of T-transformations (transpositions of elements are T-transformation, just take $\lambda = 1$). We can assume $x \neq y$. Let j be the biggest index satisfying $x_j < y_j$ (such j must exist since $x \neq y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$) and then let k be the smallest index greater than j such that

$x_k > y_k$ (such an index must exist since otherwise $\sum_{i=j}^{n} x_i < \sum_{i=j}^{n} y_i$, which gives $\sum_{i=1}^{j-1} x_i > \sum_{i=1}^{j-1} y_i$, contradiction with $x \prec y$). Thus

$$y_k < x_k \leq x_j < y_j, \qquad j < k.$$

Take $\delta = \min(x_k - y_k, y_j - x_j)$ and consider $y_k + \delta$ and $y_j - \delta$ instead of y_k and y_j (this gives a new vector y'). This is a T-transform. Note that after applying this operation the cardinality of the set $I = \{i : x_i = y_i\}$ increased. We shall prove that $x \prec y' \prec y$. Then we can perform induction with respects to $|I|$ to finish the proof.



Note that $y' \prec y$ follows from the previous lemma. We shall show that $x \prec y'$. Let $s_l(x) = \sum_{i=1}^{l} x_i^*$. It is clear from the construction that $(y'_i)^* = y'_i$ (see the above picture). We clearly have $s_l(x) \geq s_l(y')$ for $l \in [1, j-1] \cup [k+1, n]$, since then $s_l(y') = s_l(y)$. Since $s_{j-1}(y') \geq s_{j-1}(x)$, $y'_j \geq x_j$ and $y'_l = y_l = x_l$ for $l \in [j+1, k-1]$ we also have $s_l(y') \geq s_l(x)$ for $l \in [j, k]$. $\square$

Since a convex combination of permutation matrices is doubly stochastic, we get that (c) implies (b). It suffices to show that (b) implies (c). It is enough to show that any doubly stochastic matrix is a convex combination of permutation matrices.

Lemma 45. *If $P = (p_{ij})_{i,j=1}^{n}$ is doubly stochastic, then there exists a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$, such that $p_{1i_1} \cdot \ldots p_{ni_n} > 0$.*

Proof. We shall use Hall's marriage theorem. Let I be the set of rows and J the set of columns of P. We shall build a bipartite graph with parts I and J as follows: for $i \in I$ and $j \in J$ there is an edge between i and j if and only if $p_{ij} > 0$. It is enough to find a perfect matching in this graph. Now we check Hall's condition. Suppose we have a set of rows of cardinality k. Suppose all the non-zero elements in these rows belong to l columns. Thus their sum s is at most l. On the other hand $s = k$. Thus $k \leq l$. $\square$

Lemma 46. *(Birkhoff–von Neumann theorem) Every doubly stochastic matrix is a convex combination of permutation matrices.*

Proof. From the previous lemma there exists a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ such that $p_{1i_1} \cdot \ldots p_{ni_n} > 0$. Let $c = \min\{p_{1i_1}, \ldots, p_{ni_n}\}$ and let P' be a permutation matrix corresponding to $(i_1, \ldots, i_n)$. We can assume $c < 1$ since otherwise P is a permutation matrix. The matrix $R = \frac{P - cP'}{1-c}$ is double stochastic and $P = cP' + (1-c)R$. Note that R has less non-zero elements than P, so an inductive reasoning gives the result. $\square$

References

[1] Akopyan, A., Hubard, A., Karasev, R., Lower and upper bounds for the waists of different spaces. Topol. Methods Nonlinear Anal. 53 (2019), no. 2, 457–490.

[2] Ball, K., Cube slicing in R^n. Proc. Amer. Math. Soc. 97 (1986), no. 3, 465–473.

[3] Ball, K., Volumes of sections of cubes and related problems, Geometric aspects of functional analysis (1987–88), 251–260, Lecture Notes in Math., 1376, Springer, Berlin, 1989.

[4] Ball, K., Mahler's conjecture and wavelets, Discrete Comput. Geom. 13 (1995), no. 3-4, 271–277.

[5] Livne Bar-on, A., The (B) conjecture for uniform measures in the plane, Geometric aspects of functional analysis, 341—353, Lecture Notes in Math., 2116, Springer, Cham, 2014.

[6] Bartczak, M., Nayar, P., Zwara, S., Sharp Variance-Entropy Comparison for Nonnegative Gaussian Quadratic Forms, IEEE Trans. Inform. Theory, vol. 67, no. 12, 2021, 7740–7751.

[7] Barthe, F., Mesures unimodales et sections des boules B_p^n, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 7, 865–868.

[8] Barthe, F., Extremal properties of central half-spaces for product measures. J. Funct. Anal. 182 (2001), no. 1, 81–107.

[9] Barthe, F., Naor, A., Hyperplane projections of the unit ball of ℓ_p^n. Discrete Comput. Geom. 27 (2002), no. 2, 215–226.

[10] Bollobás, B., Leader, I., Products of unconditional bodies, Geometric aspects of functional analysis (Israel, 1992–1994), 13—24, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995.

[11] Böröczky, K. J., Lutwak, E., Yang, D., Zhang, G., The log-Brunn-Minkowski inequality, Adv. Math. 231 (2012), no. 3-4, 1974—1997.

[12] Brunn, H., Über Ovale und Eiflächen, Inaugural Dissertation, München, (1887).

[13] Brunn, H., Über Curven ohne Wendepunkte, Habilitationsschrift, München, (1889).

[14] Busemann, H., Petty, C. M., Problems on convex bodies. Math. Scand. 4 (1956), 88–94.

[15] Chasapis, G., Nayar, P., Tkocz, T., Slicing ℓ_p-balls reloaded: stability, planar sections in ℓ_1, Ann. Probab. 50 (2022), no. 6, 2344-2372.

[16] Chen, S., Huang, Y., Li, Q., Liu, J., L^p-Brunn-Minkowski inequality for $p < 1$, Advances in Mathematics 368 (2020), 107166.

[17] Colesanti, A., Livshyts, G. V., Marsiglietti, A., On the stability of Brunn-Minkowski type inequalities, J. Funct. Anal. 273 (2017), no. 3, 1120—1139.

[18] Cordero-Erausquin, D., Rotem, L., Several Results Regarding the (B)-Conjecture, Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics, vol 2256. Springer, 247–262.

[19] Cordero-Erausquin, D., Rotem, L., Improved log-concavity for rotationally invariant measures of symmetric convex sets, 2021, arXiv:2111.05110

[20] D. Cordero-Erausquin, M. Fradelizi, B.Maurey, The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems, J. Funct. Anal. 214 (2004), no. 2, 410—427.

[21] Czerwiński, W., *Khinchine inequalities* (in Polish), University of Warsaw, Master Thesis, 2008.

[22] Eaton, M., A Note on Symmetric Bernoulli Random Variables, Ann. Math. Statist. 41 (1970), no.4, 1223–1226.

[23] Eitan, Y., The centered convex body whose marginals have the heaviest tails, 2021, arXiv:2110.14382.

[24] Eskenazis, A., Nayar, P., Tkocz, T., Gaussian mixtures: entropy and geometric inequalities, Ann. of Prob. 46 2018, no.5, 2908–2945.

[25] Eskenazis, A., Nayar, P., Tkocz, T., Sharp comparison of moments and the log-concave moment problem, Adv. Math. 334 (2018), 389–416.

[26] Eskenazis, A., Nayar, P., Tkocz, T., Resilience of cube slicing in ℓ_p, 2022, arXiv:2211.01986

[27] Eskenazis, A., Nayar, P., Tkocz, T., Distributional stability of the Szarek and Ball inequalities, 2022, arXiv:2301.09380

[28] Figiel, T., Hitczenko, P., Johnson, W. B., Schechtman, G., Zinn, J., Extremal Properties of Rademacher Functions with Applications to the Khintchine and Rosenthal Inequalities, Transactions of the American Mathematical Society 349 (1997), no.3, 997–1027.

[29] Firey, W. J., p-means of convex bodies, Math. Scand. 10 (1962), 17-–24.

[30] Glover, N., Tkocz, T., Wyczesany, K., Stability of polydisc slicing, 2023, arXiv:2303.16896

[31] Gurvits, L., On multivariate Newton-like inequalities. Advances in combinatorial mathematics, 61–78, Springer, Berlin, 2009.

[32] Gurvits, L.: A short proof, based on mixed volumes, of Liggett's theorem on the convolution of ultra-logconcave sequences, Electron. J. Combin. 16 (2009), Note 5.

[33] Haagerup, U., The best constants in the Khintchine inequality. Studia Math. 70 (1981), no. 3, 231–283.

[34] Hadwiger, H., Gitterperiodische Punktmengen und Isoperimetrie. Monatsh. Math. 76 (1972), 410–418.

[35] Havrilla, A., Nayar, P., Tkocz, T., Khinchin-type inequalities via Hadamard's factorisation, to appear in Int. Math. Res. Not., arXiv:2102.09500

[36] Hensley, D., Slicing the cube in R^n and probability (bounds for the measure of a central cube slice in R^n by probability methods). Proc. Amer. Math. Soc. 73 (1979), no. 1, 95–100.

[37] Ivanov, G., Tsiutsiurupa, I., On the volume of sections of the cube. Anal. Geom. Metr. Spaces 9 (2021), no. 1, 1–18.

[38] Kalton, N. J., Koldobsky, A., Intersection bodies and L_p-spaces. Adv. Math. 196 (2005), no. 2, 257–275.

[39] Kanter, M., Unimodality and dominance for symmetric random vectors. Trans. Amer. Math. Soc. 229 (1977), 65–85.

[40] Khintchine, A., Über dyadische Brüche. *Math. Z.* 18 (1923), no. 1, 109–116.

[41] Koldobsky, A., An application of the Fourier transform to sections of star bodies. Israel J. Math. 106 (1998), 157–164.

[42] Koldobsky, A., Paouris, G., Zymonopoulou, M., Complex intersection bodies. J. Lond. Math. Soc. (2) 88 (2013), no. 2, 538–562.

[43] Koldobsky, A., Zymonopoulou, M., Extremal sections of complex l_p-balls, $0 < p \leq 2$. Studia Math. 159 (2003), no. 2, 185–194.

[44] Kolesnikov, A., Milman, E., Local L^p-Brunn-Minkowski inequalities for $p < 1$, Memoirs of the American Mathematical Society 277 (1360), 2017.

[45] Komorowski, R., On the Best Possible Constants in the Khintchine Inequality for $p \geq 3$, Bulletin of the London Mathematical Society 20 (1988), no.1, 73–75.

[46] König, H., Non-central sections of the simplex, the cross-polytope and the cube. Adv. Math. 376 (2021), Paper No. 107458, 35 pp.

[47] König, H., Koldobsky, A., Volumes of low-dimensional slabs and sections in the cube. Adv. in Appl. Math. 47 (2011), no. 4, 894–907.

[48] König, H., Kwapień, S., Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors. Positivity 5 (2001), no. 2, 115–152.

[49] König, H., Rudelson, M., On the volume of non-central sections of a cube. Adv. Math. 360 (2020), 106929, 30 pp.

[50] Latała, R., On some inequalities for Gaussian measures, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 813–822, Higher Ed. Press, Beijing, 2002.

[51] Latała, R., Oleszkiewicz, K., On the best constant in the Khinchin-Kahane inequality. Studia Math. 109 (1994), no. 1, 101–104.

[52] Liggett, T. M.: Ultra logconcave sequences and negative dependence, J. Combin. Theory Ser. A 79 (1997), 315–325.

[53] Littlewood, J. E., On a certain bilinear form, Quart. J. Math., Oxford Ser. 1 (1930), 164–174.

[54] Lusternik, L. A., Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen, C. R. Acad. Sci. URSS 8 (1935), 55–58.

[55] Lutwak, E., The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993), no. 1, 131–150.

[56] Lutwak, E., The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas, Adv. Math. 118 (1996), no. 2, 244–294.

[57] Ma, L., A new proof of the log-Brunn-Minkowski inequality, Geom. Dedicata 177 (2015), 75–82.

[58] Marsiglietti, A., Melbourne, J., Geometric and functional inequalities for log-concave probability sequences, 2020, arXiv:2004.12005

[59] Meyer, M., Pajor, A., Sections of the unit ball of L_p^n. J. Funct. Anal. 80 (1988), no. 1, 109–123.

[60] Minkowski, H., Volumen und Oberfläche, Math. Ann. 57 (1903), no. 4, 447–495.

[61] Minkowski, H., Geometrie der Zahlen, Teubner, Leipzig, 1910.

[62] Moody, J., Stone, C., Zach, D., Zvavitch, A., A remark on the extremal non-central sections of the unit cube. Asymptotic geometric analysis, 211–228, Fields Inst. Commun., 68, Springer, New York, 2013.

[63] Mordhorst, O., The optimal constants in Khintchine's inequality for the case $2 < p < 3$, Colloquium Mathematicum 147, 2017, 203–216.

[64] Naor, A. Romik, D., Projecting the Surface Measure of the Sphere of l_p^n, Annales de l'Institut Henri Poincaré (B), Probability and Statistics 39 (2003), 241–261.

[65] Nayar P., Oleszkiewicz K., Khinchine type inequalities with optimal constants via ultra log-concavity, Positivity, 16 (2012), 359–371.

[66] Nayar, P., Tkocz, T., On a convexity property of sections of the cross-polytope, Proc. Amer. Math. Soc. 148 (2020), no. 3, 1271–1278.

[67] Nayar, P., Tkocz, T., Extremal sections and projections of certain convex bodies: a survey, arXiv:2210.00885

[68] Nazarov, F. L., Podkorytov, A. N., Ball, Haagerup, and distribution functions. Complex analysis, operators, and related topics, 247–267, Oper. Theory Adv. Appl., 113, Birkhäuser, Basel, 2000.

[69] Oleszkiewicz, K., On p-pseudostable random variables, Rosenthal spaces and l_p^n ball slicing. Geometric aspects of functional analysis, 188–210, Lecture Notes in Math., 1807, Springer, Berlin, 2003.

[70] Oleszkiewicz, K., Pełczyński, A., Polydisc slicing in C^n. Studia Math. 142 (2000), no. 3, 281–294.

[71] Pinelis, I., Extremal Probabilistic Problems and Hotelling's T^2 Test Under a Symmetry Condition, Ann. Statist. 22 (1), 1994, 357–368.

[72] Pournin, L., Shallow sections of the hypercube, Isr. J. Math. (2022), arXiv:2104.08484

[73] Pournin, L., Local extrema for hypercube sections. 2022, arXiv:2203.15054.

[74] Prochno, J., Thäle, C., Turchi, N., Geometry of ℓ_p^n-balls: classical results and recent developments. High dimensional probability VIII – the Oaxaca volume, 121–150, Progr. Probab., 74, Birkhäuser/Springer, Cham, 2019.

[75] Rachev, S. T., Rüschendorf, L., Approximate independence of distributions on spheres and their stability properties. Ann. Probab. 19 (1991), no. 3, 1311–1337.

[76] Rotem, L., A letter: The log-Brunn-Minkowski inequality for complex bodies, arXiv:1412.5321

[77] Saroglou, C., Remarks on the conjectured log-Brunn-Minkowski inequality, Geom. Dedicata 177 (2015), 353–365.

[78] Saroglou, C., More on logarithmic sums of convex bodies, Mathematika 62 (2016), no. 3, 818–841.

[79] Schechtman, G., Zinn, J., On the volume of the intersection of two L_p^n balls. Proc. Amer. Math. Soc. 110 (1990), no. 1, 217–224.

[80] Szarek, S. J., On the best constants in the Khinchin inequality, Studia Math. 58 (1976), no. 2, 197–208.

[81] Tomaszewski, B., A simple and elementary proof of the Kchintchine inequality with the best constant. *Bull. Sci. Math.* (2) 111 (1987), no. 1, 103–109.

[82] Walkup, D. W.: Pólya sequences, binomial convolution and the union of random sets, J. Appl. Probab. 13 (1976), 76–85.

[83] Whittle, P., Bounds for the moments of linear and quadratic forms in independent random variables, Theory Probab. Appl. 5 (1960), 302–305.

[84] Young, R.M.G., On the best possible constants in the Khintchine inequality, J. London Math. Soc. 14 (2), 1976, 496–504.

[85] Vaaler, J. D., A geometric inequality with applications to linear forms. Pacific J. Math. 83 (1979), no. 2, 543–553.

[86] Xi, D., Leng, G., Dar's conjecture and the log-Brunn-Minkowski inequality, J. Differential Geom. 103 (2016), no. 1, 145–189.