Uniformization of regular relations over bi-infinite worlds

M. Skrzypczak, S. Toruńczyk and G. Fabiański

Uniformization

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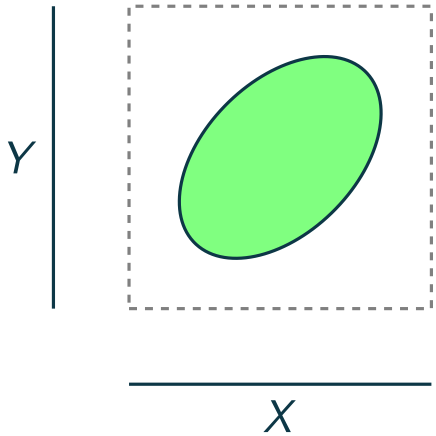

Take a relation $R \subseteq X \times Y$

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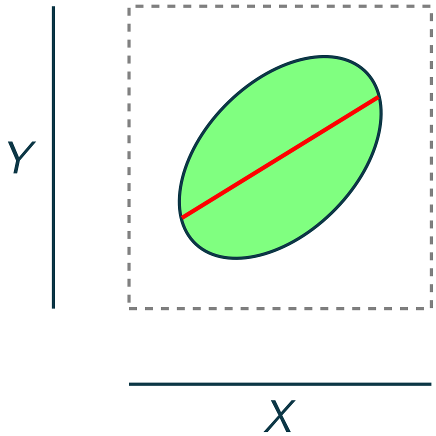

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A uniformization of R is a relation F:

1. $F \subseteq R$

2. F is a function of X

3. F and R have equal projections to X

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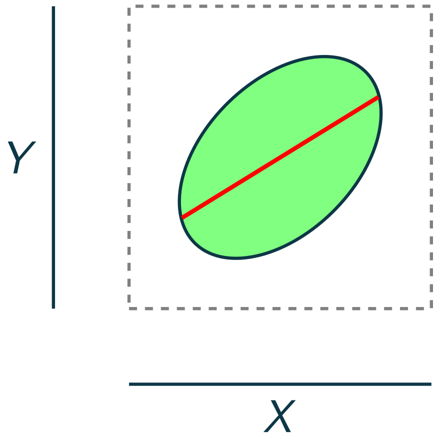

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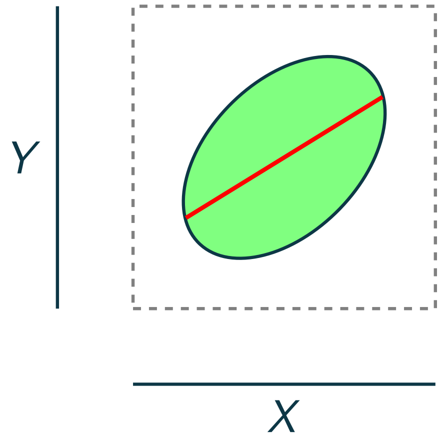

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This is equivalent to axiom of choice.

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Fix a regular relation $R \subseteq \Sigma_X^* \times \Sigma_Y^*$.

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Interpretation: whether axiom of choice is valid, if restricted to regular relations?

In case of *finite* (or *single infinite*) words,

every regular relation admits regular uniformization.

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Problem to decide if given regular relation R admit regular uniformization is decidable.

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Moreover, there is a criterion for it, in terms of syntactic monoid for R.

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R regularly uniformize.

MSO can require that:

- at position where last x is, there will be a
- a and b alternate

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If there is a shift invariant uniformization of R,

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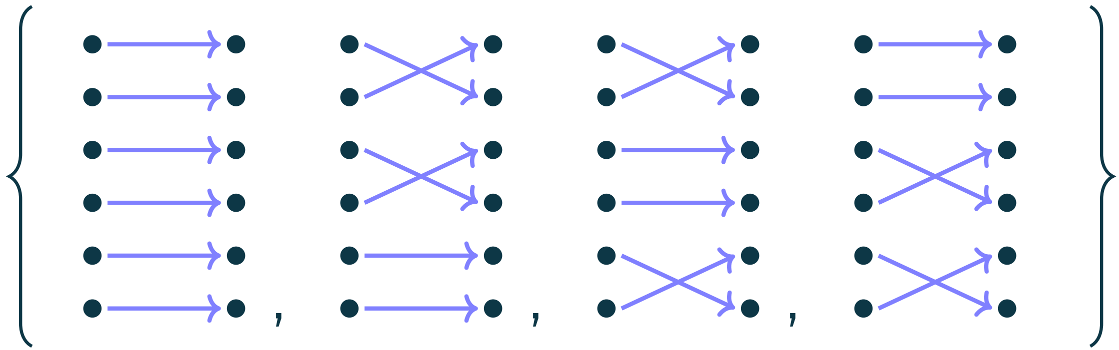

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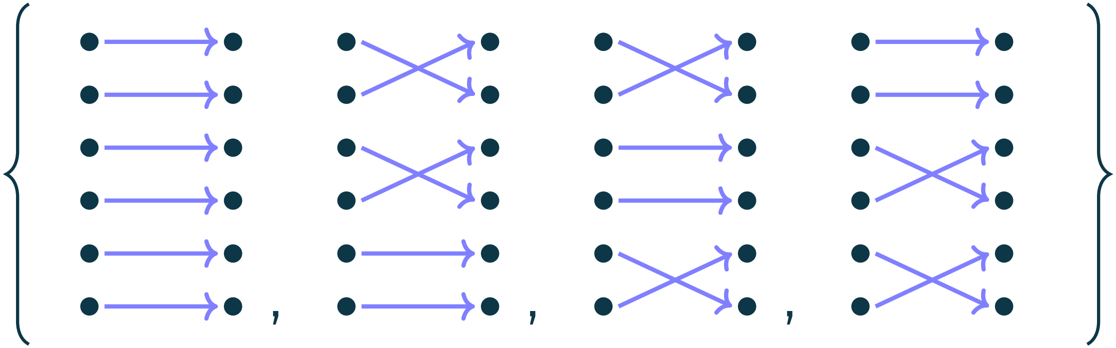
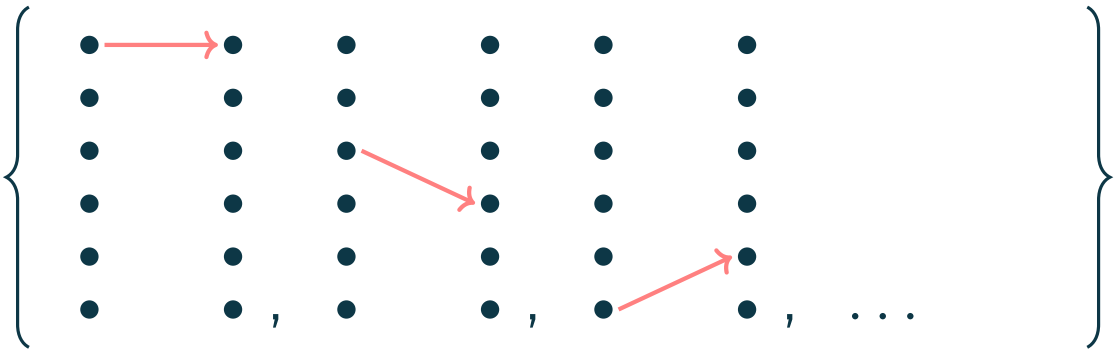


$Y =$



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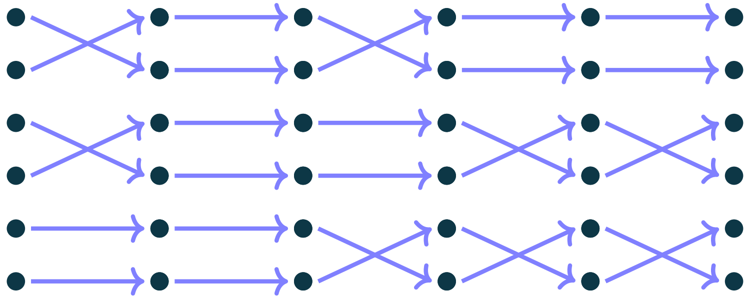

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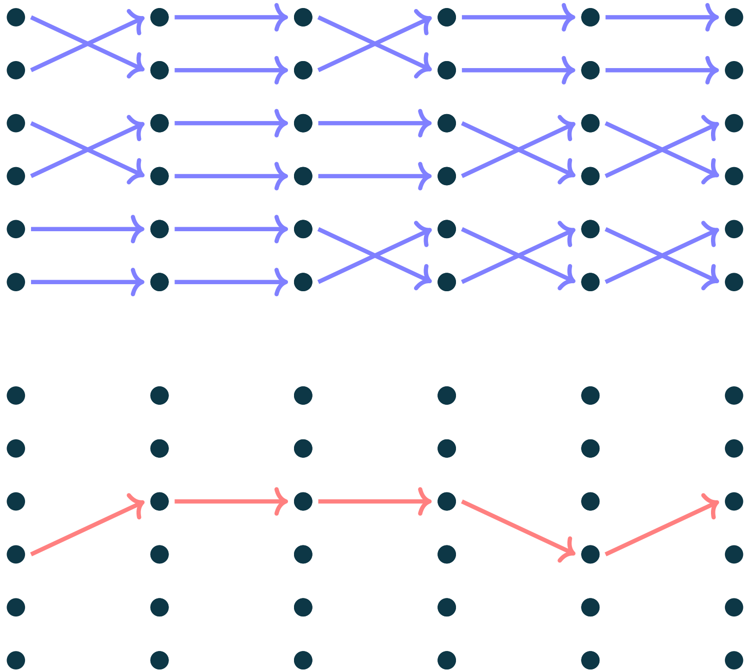

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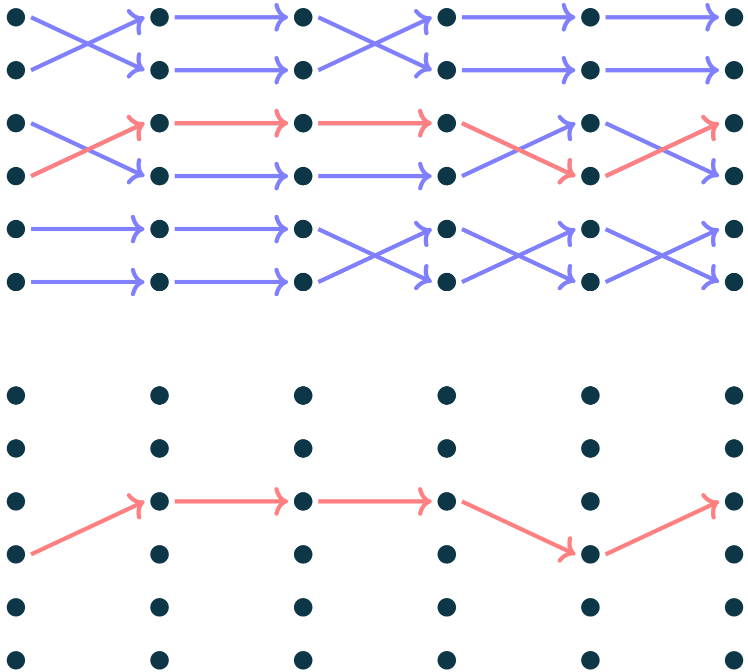

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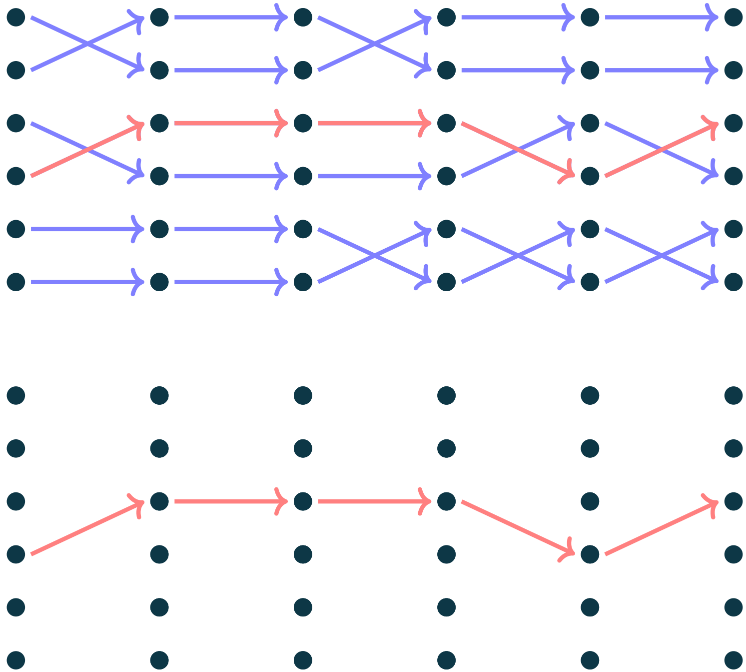

This relation does not admit regular uniformization

and admit shift invariant uniformization

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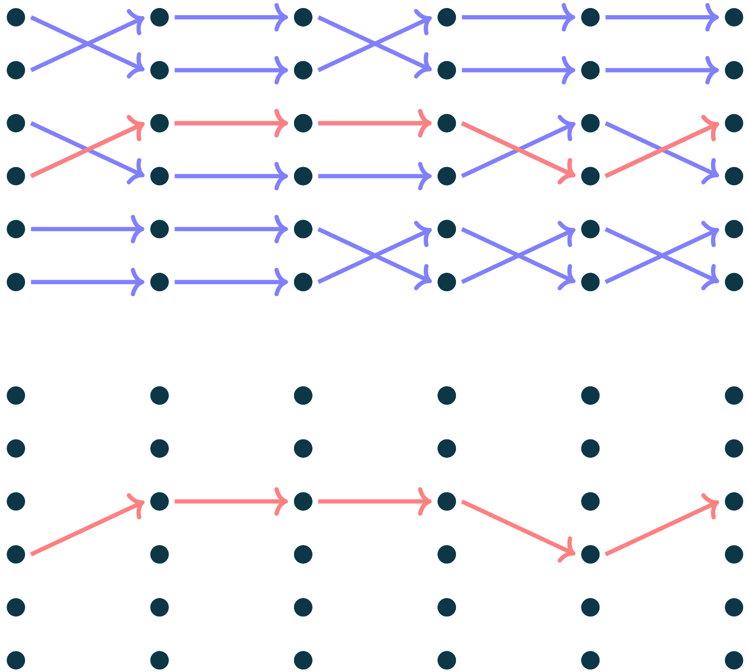

This relation does not admit regular uniformization ← crux of the decidability result

and admit shift invariant uniformization

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⌟ – ⌟ – – – ⌟ – –

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⌟	¦	–	⌟	¦	–	–	–	⌟	¦	–	–	

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Consider the following relation:

xRy if and only if y defines a factorization of x of the form

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$$
\begin{array}{c|cc|cccc|cc}
a & b & d & a & d & d & a & d & c \\
\lrcorner & - & \lrcorner & - & - & - & \lrcorner & - & -
\end{array}
\qquad \in M
$$

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1. Ramsey factorization: $\ldots eeee\ x\ ffff \ldots$, for e, f idempotent

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⌟	⋮	–	⌟	⋮	–	–	–	⌟	⋮	–	–	

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not uniformize, for example for $M = \mathbb{Z}_2$:

$$\ldots(1\ \ 1)(1\ \ 1)(1\ \ 1)\ldots$$

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still not uniformize, for example for group from previous example

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Now R uniformizes.

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- is decidable
- allow algebraic criterion
- there are deeper obstruction for uniformization then just shift invariance.