

The Probabilistic Rabin Tree Theorem

DAMIAN NIWIŃSKI, PAWEŁ PARYS, MICHAŁ SKRZYPczak



Infinite trees

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[“branching time”]

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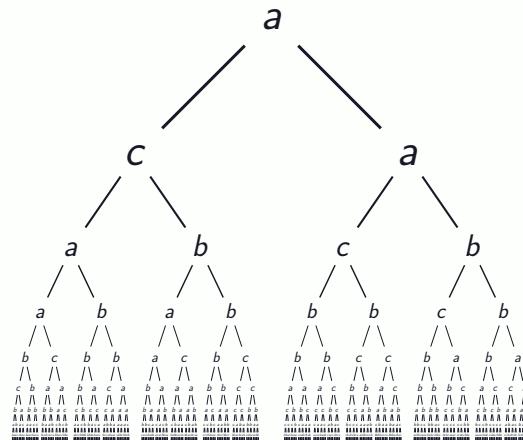
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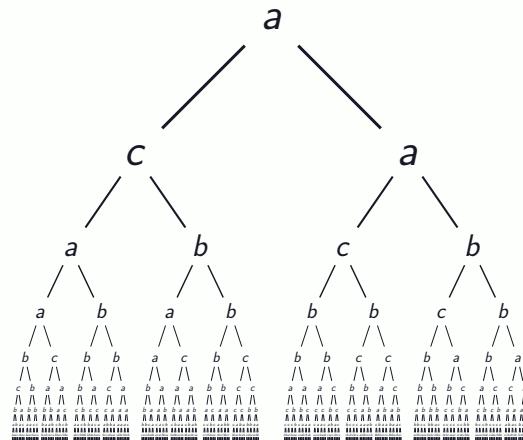
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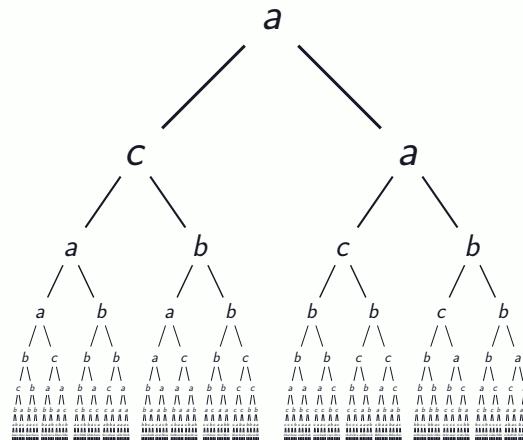


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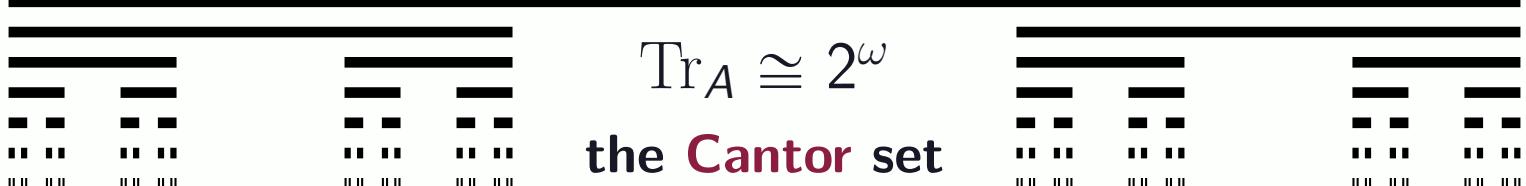
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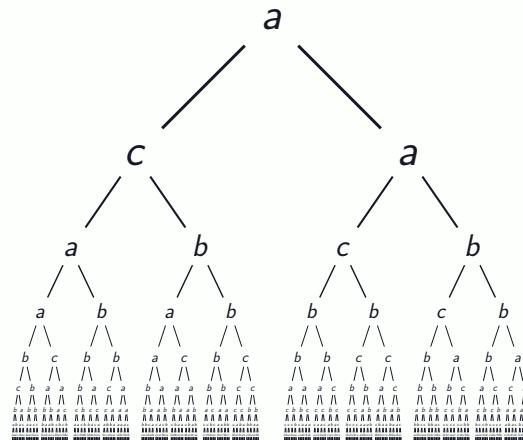
the **Cantor set**



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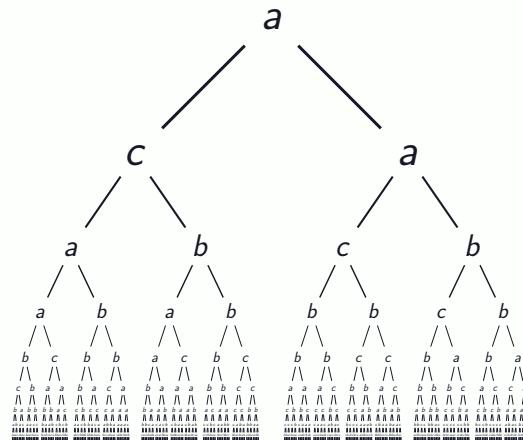
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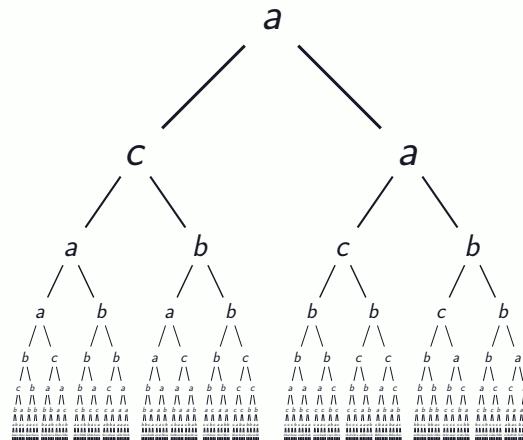


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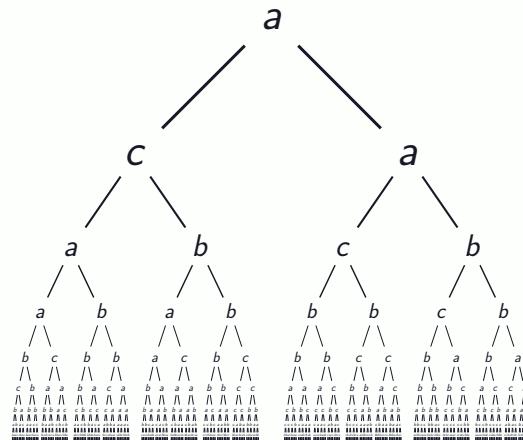
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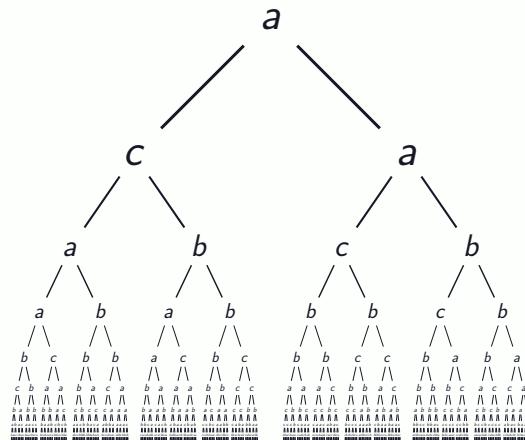
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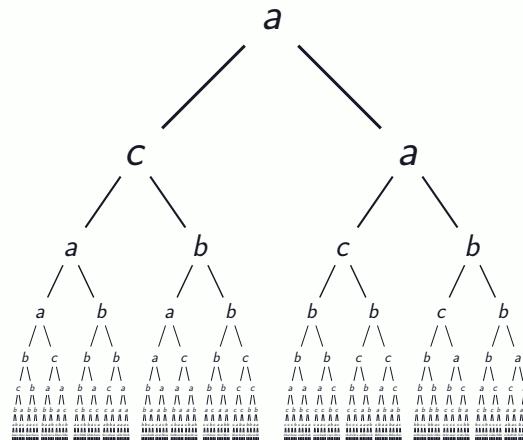
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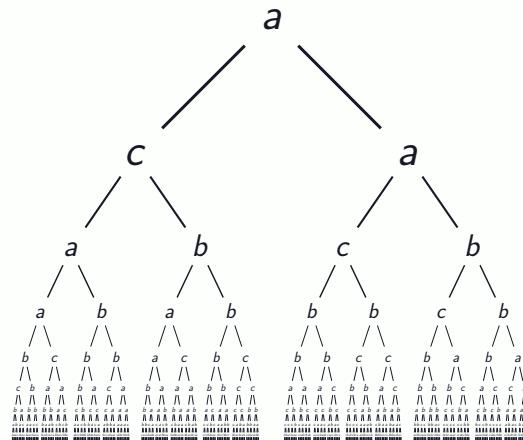
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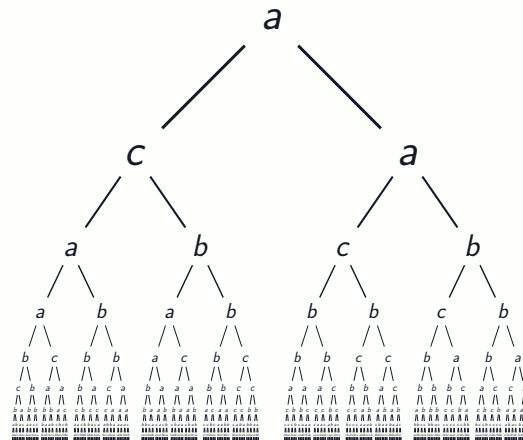
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E.g.  $\text{tp}_{\mathcal{A}} : t \longmapsto \{q \in Q \mid \mathcal{A} \text{ accepts } t \text{ from } q\}$

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“probabilistic powerdomain”

(Saheb-Djahromi [1980])

(Jones, Plotkin [1989])

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$$\bar{\tau} \mapsto \overline{\Delta}(\bar{\tau})$$

Lattice:

$$\overline{\tau_1} \not\sim \overline{\tau_2}, \quad \overline{\tau_1} \not\not\sim \overline{\tau_2}, \quad \dots$$

Types / random variables

$$\tau \in (\text{Tr}_A \rightarrow \mathbb{P}(Q))$$

Complete lattice:

$$\tau_1 \subseteq \tau_2$$

Basic functions:

E.g. $\tau \mapsto \Delta(\tau, \tau)$

Connectives:

$$\tau_1 \wedge \tau_2, \quad \tau_1 \vee \tau_2, \quad \dots$$

~~~  **$\mu$ -calculus**

E.g.  $\nu\tau. \Delta(\tau, \tau)$

|||

$$\{(t, q) \in \text{Tr}_A \times Q \mid \exists \rho. \text{run}(\rho, t) \wedge \rho(\epsilon) = q\}$$

## Distributions

$$\bar{\tau} \in \mathcal{D}(\mathbb{P}(Q))$$

**Partial order:**

$$\tau_1 \leq \tau_2$$

**Liftings:**

$$\bar{\tau} \mapsto \overline{\Delta}(\bar{\tau})$$

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# Unary $\mu$ -calculus

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[ **typing** system ]

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- The functions can be **composed** and the fixpoints can be **nested**:

$$(F_1; F_2 \downarrow) \uparrow (X_0)$$

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➡ may work for other PROBABILITY-like problems