Tutorial from 04.06.2020

I am willing to answer your questions and comments.

You can gain points for indicating non-trivial mistakes in the notes!

Homework deadline: **24:00 on Wednesday 10.06**

1 Solutions of the homework problems

Exercise 1. *Check that the function $f\colon \mathbb{N}^\omega \to X$ defined in Lemma 1 on page 9 of lecture notes WZTM-12 is continuous, open, and surjective.*

The function is constructed using a Souslin scheme $(U_s)_{s\in\mathbb{N}^{<\omega}}$ such that $U_{<>} = X$, U_s is open, non-empty, $\overline{U_{s\hat{}i}} \subseteq U_s$, $U_s = \bigcup_{i\in\mathbb{N}} U_{s\hat{}i}$, $\mathrm{diam}(U_s) \leqslant 2^{-\mathrm{lh}(s)}$ for $s \neq <>$. Then one puts $f(x)$ for $x \in \mathbb{N}^\omega$ as the unique (!) member of the intersection $\bigcap_{n\in\mathbb{N}} U_{x\restriction_n}$.

Please write down all the details, no "hand-waving" arguments :)

It was just an easy check. Everyone who tried managed to get it right. If you have any questions about that, please contact me directly.

Exercise 2 ($\star$)**.** *Construct a set $\mathcal{U} \subseteq [\mathbb{N}]^{\aleph_0}$ that is Ramsey null but comeagre in the standard topology on $[\mathbb{N}]^{\aleph_0}$.*

One idea is to take:

$$\mathcal{U} = \{X \in [\mathbb{N}]^{\aleph_0} \mid \{2n, 2n{+}1\} \subseteq X \text{ for infinitely many } n \in \mathbb{N}\}.$$

This set (if I'm correct) is a dense G_δ that is Ramsey null.

Alternative solution was proposed by Katarzyna Kowalik. Take $j \in \mathbb{N}$ and consider

$$V_j = \bigcup_{s\in\{0,1\}^{<\omega}} \big[s\hat{}(1^j), \mathbb{N}/(|s| + j)\big],$$

where we identify the sequence $s\hat{}(1^j)$ (concatenation of s with j ones) with the corresponding set. I.e. V_j is a union ranging over all finite sequences s of families of sets $X \in [\mathbb{N}]^{\aleph_0}$ that agree with s, then contain j consecutive numbers, and then anything. It is easy to see that each of the sets V_j is open and dense in the standard topology. Therefore, their intersection $A = \bigcap_{j\in\mathbb{N}} V_j$

is a dense G_δ. Notice that this set is essentially a variant of the comeagre set of measure 0.

We claim that A is Ramsey null. Take $t < B$ with $t \in [\mathbb{N}]^{<\aleph_0}$ and $B \in [\mathbb{N}]^{\aleph_0}$. Consider $C \subseteq B$ that is obtained by taking every second element: if $B = \{n_0 < n_1 < \ldots\}$ then $C = \{n_{2i} \mid i \in \mathbb{N}\}$. We claim that $[t, C] \cap A = \emptyset$.

Consider $X \in [t, C]$ and take $j = \max(t) + 55$. We claim that $X \notin V_j \supseteq A$, which concludes the proof. Consider any $s \in \{0,1\}^{<\omega}$, we need to show that $X \notin \big[s\hat{}(1^j), \mathbb{N}/(|s| + j)\big]$. However, by the choice of j and C, we know that for every n there exists a number in $n + [0, j) \stackrel{\text{def}}{=} \{n, n+1, \ldots, n+j-1\}$ that is not in X — it follows from the fact that $n + [0, j)$ contains two consecutive numbers outside t (because $j = \max(t) + 55$) and C contains no two consecutive numbers. On the other hand, every member of $\big[s\hat{}(1^j), \mathbb{N}/(|s| + j)\big]$ contains $|s| + [0, j)$ as a subset. A contradiction.

2 New material from WZTM-13

The next lecture notes (WZTM-13) continue the problems of selection. A new definition is provided, of a *uniformisation* of a relation $P \subseteq X \times Y$: it is a subset $P^* \subseteq P$ that is a graph of a function with the same projection onto X as P.

One of the first results, Proposition on page 2 states that every relation that has a Borel uniformisation must also have Borel projection, see the homework Exercise 16 from tutorials from 7th May (solved in tutorials from 14th May).

The above observation shows that certain relations have no Borel uniformisations because their projections are not Borel (they are analytic ($\mathbf{\Sigma}^1_1$), because the relations themselves are Borel).

The next step is to apply Kuratowski–Ryll–Nardzewski to show that Borel relations with closed sections, such that the map $X \ni x \mapsto P_x \in F(X)$ is Borel admit Borel uniformisations (see page 5). For that, one considers the following subset $P \subseteq F(X) \times X$ for a Polish space X, defined by

$$P \stackrel{\text{def}}{=} \big\{(F, x) \mid x \in F, F \in F(X)\big\}.$$

Exercise 3. *Show that the set P defined above is Borel in the Polish space $F(X) \times X$.*

As it turns out, one can partially get rid of the assumption that the sections are closed, using Parametrisation Lemma from the bottom of page 6: every Borel subset $P \subseteq X \times Y$ can be parametrised by a closed subset $F \subseteq X \times Z$ for some Polish space Z. The parametrisation is witnessed by a continuous bijection $\varphi\colon F \to P$ such that $\varphi(\{x\} \times F_x) = \{x\} \times P_x$ for every $x \in X$.

To achieve the above effect, first one takes a Polish topology τ' on P that extends the one inherited from $X \times Y$ and puts $Z = (P, \tau')$. Then we define

$$\begin{aligned} F &\stackrel{\text{def}}{=} \big\{(x,p) \in X \times P \mid x = \mathrm{proj}_X(p)\big\} \\ &= \bigcup_{x \in X} \big(\{x\} \times (\{x\} \times P_x)\big) \\ \varphi(x,p) &\stackrel{\text{def}}{=} p && \text{for } (x,p) \in F. \end{aligned}$$

Exercise 4. *Check that the set F defined above is closed in $X \times Z$. Check that $\varphi\colon F \to P$ is bijective and continuous (for the domain with the topology inherited from $F \subseteq X \times Z$ and codomain with the topology inherited from $P \subseteq X \times Y$, i.e. τ). Moreover, verify that $\varphi(\{x\} \times F_x) = \{x\} \times P_x$ for every $x \in X$.*

2.1 Large sections uniformisation

The next step is to provide Borel uniformisations of Borel sets with *large* sections, i.e. sections outside a given σ-ideal, see Theorem on page 8.

The first stage of the proof of that theorem is a construction based on Parametrisation Lemma, that allows us to assume that the relation under consideration is closed. For that, one takes $P \subseteq X \times Y$, a Polish space Z, and a closed subset $F \subseteq X \times Z$ for which there exists a continuous bijection $\varphi\colon F \to P$ as above.

Exercise 5. *Verify that* $\mathrm{proj}_X(F) = \mathrm{proj}_X(P)$.

The next stage is to assume that the σ-ideals under consideration behave like σ-ideal of meagre sets: no non-empty open set belongs to them (see Step 2 on page 10).

The simplest applications of the above theorem are, where the σ-ideals under consideration are constantly equal to the known σ-ideals of meagre sets $\mathcal{I}_{\mathrm{MGR}}$ and of μ-measure 0 sets $\mathcal{I}_\mu$. For this purpose, we need to verify that both these σ-ideals $\mathcal{I}$ satisfy the assumption ($\star$) from page 8 of lecture notes, i.e. that for every Borel set $A \subseteq X \times Y$ we have

$$\{x \in X \mid A_x \notin \mathcal{I}\} \in \mathcal{B}(X).$$

Meagre sets We first consider the case of $\mathcal{I} = \mathcal{I}_{\mathrm{MGR}}$. For that, we consider a stronger property (denoted $(\diamond)(A)$) of a set $A \subseteq X \times Y$: for every non-empty open subset $U \subseteq Y$ the set

$$(A)_U \stackrel{\text{def}}{=} \{x \in X \mid (A_x \cap U) \text{ is not meagre in } Y\} \tag{2.1}$$

is Borel in X.

Exercise 6. *Show that the family of Borel sets $A \subseteq X \times Y$ that satisfy property $(\diamond)(A)$ contains basic open sets in $X \times Y$ and is closed under countable unions and complements.*

Then this family is itself a σ-algebra and must therefore coincide with Borel sets.

Measurable sets We now consider the case of $\mathcal{I} = \mathcal{I}_\mu$. Thus, we need to show that for every finite Borel measure on Y and every Borel set $A \subseteq X \times Y$ the set $\{x \in X \mid \mu(A_x) > 0\}$ is Borel in X. It is achieved again by defining certain property of Borel subsets of $X \times Y$: $(\square)(A)$ holds if the map $x \mapsto \mu(A_x)$ is Borel measurable, as a function from X to $\mathbb{R}$.

The approach here is similar as above, and roughly follows Kechris Theorem 17.25, page 113.

First, since every pair of uncountable Polish spaces is Borel isomorphic, without loss of generality we can assume that both X and Y are in fact $\mathbb{N}^\omega$, i.e. $X \times Y$ is zero-dimensional.

Exercise 7. *Prove that then every open set in $\mathbb{N}^\omega \times \mathbb{N}^\omega$ is a countable disjoint union of basic open sets of the form $N_s \times N_r$ for $s, r \in \mathbb{N}^{<\omega}$.*

Exercise 8. *Show that the family of Borel sets $A \subseteq X \times Y$ that satisfy property $(\square)(A)$ contains all basic open sets in $X \times Y$ and is closed under countable disjoint unions and complements.*

Thus, Exercise 7 guarantees that all open sets satisfy $(\square)(A)$. It means that the following exercise concludes the proof.

Exercise 9. *Show that if $\mathcal{C}$ is a family of subsets of a Polish space X such that all open sets belong to $\mathcal{C}$ and the family is closed under countable disjoint unions and complements then $\mathcal{B}(X) \subseteq \mathcal{C}$.*

Hint: consider, for any set $A \subseteq X$, the family of sets $\{B \subseteq X \mid A \cap B \in \mathcal{C}\}$ and prove some properties of these families. You might use the following definition:

Definition 2.1. *We will say that a family of sets $\mathcal{D} \subseteq \mathsf{P}(X)$ is a λ-*algebra *if $\mathcal{D}$ is closed under complements and countable disjoint unions.*

2.2 Novikov Separation Theorem

The next stage is a strengthening of the Lusin Separation Theorem (every two disjoint analytic sets can be separated by a Borel set). Here, a variant of *separation in the limit* is considered, where we ask a countable family of sets to have empty intersection: see Theorem (Novikov) on page 12 of lecture notes. It is then applied to normalise the shape of Borel subsets $X \times Y$ with all sections open (see Item (i)) **or** all sections closed (see Item (ii)), see Theorem (Kunugui, Novikov) on page 15.

We first focus on Item (i): take a Borel subset $A \subseteq X \times Y$ with all the sections A_x open in Y. Fix a countable basis $\{V_n \mid n \in \mathbb{N}\}$ of the Polish topology on Y. We want to present A as $\bigcup_{n\in\mathbb{N}} B_n \times V_n$ for some Borel sets $B_n \subseteq X$ (see the end of page 15).

To achieve the above effect, one first defines $C_n \stackrel{\text{def}}{=} \{x \in X \mid V_n \subseteq A_x\}$ for $n \in \mathbb{N}$.

Exercise 10. *Check directly that the sets C_n are coanalytic.*

Exercise 11. *Check that $A = \bigcup_{n\in\mathbb{N}} C_n \times V_n$.*

We head towards applying Novikov Separation Theorem, so we put $Z_n = C_n \times V_n$.

Exercise 12. *Check that the sets Z_n are coanalytic.*

Exercise 11 implies that $A = \bigcup_{n\in\mathbb{N}} Z_n$.

Exercise 13. *Discuss how to apply Novikov Separation Theorem (page 12 of WZTM-13) to the family $(Z_n)_{n\in\mathbb{N}}$ to obtain Borel sets $E_n \subseteq Z_n$ with $\bigcup_{n\in\mathbb{N}} E_n = A$.*

This allows us to put $A_n \overset{\text{def}}{=} \mathrm{proj}_X(E_n)$ and see that $A_n \in \mathbf{\Sigma}^1_1$.

Exercise 14. *Check that $A = \bigcup_{n\in\mathbb{N}} A_n \times V_n$.*

Finally, one applies the Lusin Separation Theorem (see page 17) and find Borel sets B_n satisfying $A_n \subseteq B_n \subseteq C_n$ which conclude the proof.

2.3 Compact sections uniformisation

The next goal is to show that Borel sets with compact sections admit Borel uniformisations, see 13.6 on page 17. One of the stages of that proof is the observation that a projection of a closed subset $P \subseteq X \times Y$ along a compact space Y onto X is also closed. But we have already seen that, see Exercise 3 from 19th March.

This finally leads to a Theorem by Novikov about Borel maps that have compact fibers (pre-images of points).

Exercise 15. *Parse the proof of Theorem (Novikov) on page 19 of WZTM-13. Consider precisely how the original Novikov theorem is applied and why the rest follows from the Lusin–Souslin Theorem.*

3 New material from WZTM-14

The next piece of material begins with some combinatorial tools used to operate on trees. It partially recalls things that we've seen, mainly in tutorials from 12ve and 19th March.

In particular, for a tree $T \subseteq (A \times B)^{<\omega}$ and $\alpha \in A^\omega$ by $T(\alpha)$ we denote the *section* which is a tree on B defined as

$$T(x) \overset{\text{def}}{=} \{s \in C^{<\omega} \mid \langle \alpha {\restriction} \mathrm{lh}(s), s\rangle \in T\},$$

where we identify $(A \times B)^{<\omega}$ with $(A^{<\omega}) \times (B^{<\omega})$.

The following exercise is left on page 7 of WZTM-14.

Exercise 16. *Consider a fixed tree $T \subseteq (A \times B)^{<\omega}$. Then the function $\alpha \mapsto T(\alpha)$ is a continuous function from A^ω to Tr_B — the space of trees on B.*

Another important combinatorial concept is the Kleene–Brouwer ordering, defined at the end of page 7. For two sequences $s, t \in \mathbb{N}^{<\omega}$ we put $s \leqslant_{\mathrm{KB}} t$ if $t \subseteq s$ or $s \leqslant t$. In other words, descending sequences in the Kleene–Brouwer ordering go **down** the tree or to the left. The crucial property of this order is expressed by the following lemma.

Proposition 3.1 (Proposition on page 8)**.** *The Kleene–Brouwer order on the nodes of a tree T is well-founded if and only if the tree T is well-founded.*

Exercise 17. *Check that the Kleene–Brouwer order is a linear order extending $\subseteq$.*

Based on that order, one can construct a continuous reduction from well-founded trees to well-founded orders. For that, one first fixes a bijection $h\colon \mathbb{N} \to \mathbb{N}^{<\omega}$. The reduction f is defined as follows (see the end of page 10): $(n, m) \in f(T)$ if and only if one of the following conditions hold

- $h(m) \in T$, $h(n) \in T$, and $h(m) \leqslant_{\mathrm{KB}} h(n)$;
- $h(m) \notin T$ and $h(n) \in T$;
- $h(m) \notin T$, $h(n) \notin T$, and $m \leqslant n$.

Exercise 18. *Check that the function f defined above indeed constructs a linear order and that it is continuous.*

4 New homework

This time there are three problems: first two are rather simple, please choose at most one of them to solve; and the third one is harder, you can solve it separately. Thus, the only forbidden thing is to solve both of the first two problems.

Exercise 19. *Carefully solve Exercise 4 — please write down all the details!*

Exercise 20. *Solve Exercise 7 (again with all the details).*

Exercise 21 ($\star$)**.** *Let X be a Polish space and $\mathcal{E}$ be the Effros Borel structure on $F(X)$. Prove that the family of all perfect sets $F \in F(X)$ (sets with no isolated points) is Borel in $F(X)$ (i.e. belongs to $\mathcal{E}$, the σ-algebra generated by sets $[U]_{F(X)}$).*

5 Old homework

In parallel you can still solve the following problem from one of the previous homeworks. Solving it does not count to the restrictions of "at most one problem".

Exercise 22 ($\star$)**.** *Assume that X is a non-empty perfect Polish space and $R \subseteq X^2$ is a comeagre set. Prove that there exists a Cantor set $C \subseteq X$ and a comeagre set $D \subseteq X$ such that $C \times D \subseteq R$.*

As a hint, you may focus on constructing $C \subseteq X$ more than on finding $D \subseteq Y$: begin by finding a Cantor set C such that the set

$$\{y \in Y \mid \forall x \in C.\ (x, y) \in R\}$$

is dense in Y.

Another hint is as follows: if $C \subseteq X$ is a Cantor set then it is compact. We know that if $F \subseteq X \times C$ is closed then the projection $\mathrm{proj}_X(F)$ is closed in X. What about projections as above of sets F that are F_σ?

6 Hints

Hint to Exercise 3 In fact P is G_δ: countable intersection of open sets. Enumerate some fixed countable basis of X as $(U_n)_{n\in\mathbb{N}}$ and for $n \in \mathbb{N}$ define

$$P_n \stackrel{\text{def}}{=} \big([U_n]_{F(X)} \times X\big) \cup \big(F(X) \times (X - U_n)\big),$$

where $[U_n]_{F(X)} = \{F \in F(X) \mid F \cap U_n \neq \varnothing\}$ is a basic open set of $F(X)$.

Notice that each P_n is a G_δ set and therefore their intersection is also G_δ.

We claim that $P = \bigcap_{n\in\mathbb{N}} P_n$. Clearly, if $(F, x) \in P$ (i.e. $x \in F$) then for every $n \in \mathbb{N}$ we have $(F, x) \in P_n$. It remains to see that if for each $n \in \mathbb{N}$ we have $(F, x) \in P_n$ then $x \in F$. Assume contrarily, as witnessed by some basic open set U_m such that $x \in U_m$ while $U_m \cap F = \varnothing$. But then $(F, x) \notin P_m$, because $F \notin [U_m]_{F(X)}$ nor $x \in X - U_m$.

Hint to Exercise 5 Obvious from the definition.

Hint to Exercise 6 See Kechris, Theorem 16.1, page 94.

First take $A = V \times W$ for V open in X and W open in Y. Consider any non-empty open subset $U \subseteq Y$. If $U \cap W = \varnothing$ then the set $(A)_U$ (see Equation (2.1)) equals $\varnothing$ and is Borel in X. Otherwise, $U \cap W$ is an open subset of Y and $A_x \cap U = U \cap W$ for $x \in V$. As Y is a Polish space and $U \cap W$ is non-empty open, it is not meagre in Y and $(A)_U = V$.

Now consider $A = \bigcup_{n\in\mathbb{N}} A_n$ with all A_n having property $(\diamond)(A_n)$. Take U non-empty open in Y. We claim that

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right)_U = \bigcup_{n\in\mathbb{N}} (A_n)_U.$$

Take $x \in (\bigcup_{n\in\mathbb{N}} A_n)_U$. Then $A_x \cap U$ is not meagre in Y, which means that at least one of $(A_n)_x \cap U$ must not be meagre in U. But then $x \in (A_n)_U$. For the opposite implication, if $x \in (A_n)_U$ then clearly also $x \in (A)_U$, because $A_x \supseteq (A_n)_x$.

Now let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of non-empty open sets in Y. We claim that

$$\big((X \times Y) - A\big)_U = X - \bigcap_{U_n \subseteq U} (A)_{U_n},$$

which proves that the family under consideration is closed under complements, because if all the sets $(A)_{U_n}$ are Borel then also the right-hand side of the above equation is Borel.

Take $x \in X$ such that $x \in \big((X \times Y) - A\big)_U$. It means that $(Y - A_x) \cap U$ is not meagre in Y. Since A is Borel, each of its sections has Baire Property. Therefore, there exists a basic open set $U_n \subseteq U$ such that $(Y - A_x) \cap U_n$ is comeagre in U_n. Then $A_x \cap U_n$ is meagre in Y and therefore $x \notin (A)_{U_n}$, which implies that $x \in X - \bigcap_{U_n \subseteq U} (A)_{U_n}$.

Now assume that $x \notin \big((X \times Y) - A\big)_U$. It means that $(Y - A_x) \cap U$ is meagre in Y. But then for each $U_n \subseteq U$ we have that $A_x \cap U_n$ is comeagre in U_n, which implies that $x \in (A)_{U_n}$ and therefore $x \notin X - \bigcap_{U_n \subseteq U} (A)_{U_n}$.

Hint to Exercise 8 The fact is again trivial for basic open sets of the form $N_s \times N_r$. Closure under complement is also obvious, because

$$\mu\big((X \times Y - A)_x\big) = \mu(Y) - \mu(A_x).$$

If $(A_n)_{n\in\mathbb{N}}$ satisfy property $(\square)(A_n)$ and are disjoint then also their sections $(A_n)_x$ are disjoint. Thus, the mapping $x \mapsto \mu\big(\bigcup_{n\in\mathbb{N}}(A_n)_x\big)$ is the arithmetical sum of the mappings $x \mapsto \mu\big((A_n)_x\big)$. It is a standard fact of analysis that arithmetical sum of a countable family of measurable mappings is measurable.

Hint to Exercise 9 See Kechris Theorem 10.1 (iii), page 65.

Recall Definition 2.1.

Without loss of generality we can assume that $\mathcal{C}$ is the smallest λ-algebra containing open sets. We will first prove that $\mathcal{C}$ is closed under finite intersections.

For any set $A \subseteq X$ define $\mathcal{L}(A)$ as the family of sets $B \subseteq X$ such that $A \cap B \in \mathcal{C}$. Clearly $\mathcal{L}(A)$ is closed under countable disjoint unions. It is also closed under complements, because if $B \in \mathcal{L}(A)$ then $A \cap B \in \mathcal{C}$ and therefore (using complements and disjoint unions in $\mathcal{C}$) we have

$$A - B = X - \big((X - A) \cup (A \cap B)\big) \in \mathcal{C}.$$

Therefore, $\mathcal{L}(A)$ is a λ-algebra. Notice that if A is open then all open sets belong to $\mathcal{L}(A)$ and therefore (by minimality) $\mathcal{C} \subseteq \mathcal{L}(A)$. But this means that if $B \in \mathcal{C}$ then every open set A belongs to $\mathcal{L}(B)$. Thus again by minimality $\mathcal{C} \subseteq \mathcal{L}(B)$ (now for any $B \in \mathcal{C}$). Thus, if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.

Now $\mathcal{C}$ must be closed under arbitrary unions, because every union $\bigcup_{n\in\mathbb{N}} A_n$ can be made disjoint by taking $B_n = A_n - \bigcup_{i<n} A_i$ and this takes only finite unions, finite intersections, and complements.

Thus, $\mathcal{C}$ is a σ-algebra that contains open sets, so $\mathcal{B}(X) \subseteq \mathcal{C}$.

Hint to Exercise 10 Each of the sets C_n can be defined as

$$\big\{x \in X \mid \forall y \in U_n.\ (x, y) \in A\big\},$$

where the quantification ranges over a Polish space U_n and the inner condition is Borel in the product $X \times U_n$ (because A is Borel).

Hint to Exercise 11 Clearly if $(x, y) \in \bigcup_{n\in\mathbb{N}} C_n \times V_n$ then $(x, y) \in A$. Consider the opposite implication and take $(x, y) \in A$. By the assumption that the sections A_x are open, we know that there is certain $n \in \mathbb{N}$ such that $y \in V_n \subseteq A_n$. However, then $x \in C_n$ and $y \in V_n$, so $(x, y) \in C_n \times V_n$.

Hint to Exercise 12 In general if $A \subseteq X$ and $B \subseteq Y$ are in some boldfaced pointclass Γ that is closed under finite intersections then $A \times B \subseteq X \times Y$ is also in Γ.

Hint to Exercise 13 The assumption of Novikov Separation Theorem (in the coanalytic case) requires that $\bigcup_{n\in\mathbb{N}} Z_n$ is the whole space. Here we only know that $\bigcup_{n\in\mathbb{N}} Z_n = A$. However, as A itself is Borel, its complement is also Borel, so coanalytic. Thus, we can take $Z_{-1} = (X \times Y) - A$ and then $\bigcap_{n\geqslant -1} Z_n = X \times Y$, with all these sets coanalytic. This gives us Borel sets $E_n \subseteq Z_n$ for $n \geqslant -1$ with $\bigcup_{n\geqslant -1} E_n = X \times Y$. But since for each $n \in \mathbb{N}$ we know that $E_n \subseteq Z_n \subseteq A$, we necessarily have $E_{-1} = Z_{-1}$ and $\bigcup_{n\in\mathbb{N}} E_n = A$.

Hint to Exercise 14 First, as $E_n \subseteq Z_n$ also $A_n = \mathrm{proj}_Z(E_n) \subseteq \mathrm{proj}_Z(Z_n) = C_n$. Thus, $A_n \times V_n \subseteq C_n \times V_n$ and therefore $\bigcup_{n\in\mathbb{N}} A_n \times V_n \subseteq A$, see Exercise 11.

On the other hand, if $(x, y) \in A$ then there exists n such that $(x, y) \in E_n$ and therefore $x \in A_n$, while $y \in V_n$ (because $E_n \subseteq C_n \times V_n$). Therefore, $(x, y) \in A_n \times V_n$.

Hint to Exercise 16 It is enough to check the preimages of single-coordinates, i.e. that the preimage $f^{-1}\big(\{T \in \mathrm{Tr}_B \mid s \in T\}\big)$ is open. However, if $\alpha \in f^{-1}\big(\{T \in \mathrm{Tr}_B \mid s \in T\}\big)$ then in fact $N_{\alpha\restriction_{|s|}} \subseteq f^{-1}\big(\{T \in \mathrm{Tr}_B \mid s \in T\}\big)$.

Hint to Exercise 17 It is just an easy check using the properties of the lexicographic order...

Hint to Exercise 18 First, the domain of $f(T)$ is split into two parts: $h^{-1}(T)$ and the complement. Thus, it is enough to check each of them separately. The complement is trivial (it is the standard order on $\mathbb{N}$ there). And for the preimage it is again simple, because (up to h) it is the Kleene–Brouwer ordering.